The spectrum of a well-generated tensor triangulated category

joint work with Henning Krause

Observation eg $\mathcal{T} = \underset{\substack{\text{comm.}\\ \text{noeth.}}}{D(R)}, \underset{\text{noeth}}{D(X)}, \mathrm{StMod}(kG), \dots$

$$\begin{tikzcd}
\text{frame } \cancel{\text{set}} \text{ of radical thick tensor ideals} = \mathrm{Rad}^c(\mathcal{T}^c) \arrow[r, leftrightarrow, "\text{Stone duality}"] \arrow[d] & \mathrm{Spc}^c(\mathcal{T}^c) \arrow[r, leftrightarrow, "\text{Hochster duality}"] & \text{Balmer spectrum} \\
\text{frame of radical localizing tensor ideals} = \mathrm{Rad}(\mathcal{T}) \arrow[r, leftrightarrow, "\text{Stone duality}"] & \mathrm{Spc}(\mathcal{T}) \arrow[u, "\text{bijection on the underlying sets}"'] &
\end{tikzcd}$$

Question How do the α-localizing ideals of the α-compact objects fit in?

Background on frames

A poset $(L, \leq)$ is a <u>lattice</u> if the join $\underset{\substack{a \vee b\\ \bigvee A}}{}$ and the meet $\underset{\substack{a \wedge b\\ \bigwedge A}}{}$ of any finite set exists

A lattice $(F, \leq)$ is a <u>frame</u> if joins and finite meets exist ($\Rightarrow$ arbitrary meets exist) and

$$a \wedge \bigvee B = \bigvee_{a \in B} (a \wedge b) \qquad \forall a \in F,\ B \subseteq F$$

(infinite-join distributive law)

Frm = category of frames

morphisms: maps that preserve order, join, finite meet

$$X \longmapsto \text{frame of open sets}$$

$$\mathrm{Top} \xrightarrow{\Omega} \mathrm{Frm}^{op}$$

this functor has a right adjoint

A <u>point</u> of F is a frame morphism $x : F \to \{0,1\}$ $\leftrightarrow$ (meet-) prime elt $\bigvee x^{-1}(0) = p_x$

$pt(F)$ is a set of points of F. It is a topological space with open sets:

$$U(a) := \{x \in pt(F) \mid \underbrace{x(a) = 1}^{a \nleq p_x}\}$$

$$\begin{tikzcd} \mathrm{Top} \arrow[r, shift left, "\Omega"] & \mathrm{Frm}^{op} \arrow[l, shift left, "pt"] \\ \left\{\begin{matrix}\text{sober}\\ \text{spaces}\end{matrix}\right\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, shift left, "\Omega"] \arrow[r, phantom, "\equiv"] & \left\{\begin{matrix}\text{spatial}\\ \text{frames}\end{matrix}\right\} \arrow[u, phantom, sloped, "\subseteq"] \arrow[l, shift left, "pt"] \end{tikzcd}$$

— fully faithful

Stone duality

"has enough points"

$\boxed{\text{Definition}}$ $a \in F$ is <u>α-compact</u> if for every set $A \subseteq F$ with $a \leq \bigvee A$ there exists $B \subseteq A$ with

$$a \leq \bigvee B \quad \text{and} \quad |B| < \alpha.$$

$F^\alpha :=$ set of α-compact elements of F.

F is <u>α-coherent</u>, if

1) every element is the join of α-compact elements
2) 1 is α-compact

} can recover F from F^α

3) the α-compact elements are closed under finite meets

$\boxed{\text{Remark}}$ compact $= \aleph_0$-compact
coherent $= \aleph_0$-coherent

coherent $\Rightarrow$ spatial

BUT: α-coherent $\not\Rightarrow$ spatial

Definition Let $\mathcal{K}$ be a essentially small $\otimes$-Δ-category with α-coproducts and α a regular cardinals

A <u>tensor ideal</u> $\mathcal{I} \subseteq \mathcal{K}$ is

- <u>α-localizing</u> if $\mathcal{I}$ is Δ'ed, closed under α-coproducts and $\underbrace{\text{direct summands}}_{\text{for free when } \alpha > \aleph_0}$
- <u>radical</u> if whenever $X^{\otimes n} \in \mathcal{I}$ for some $n \geq 1$ also $X \in \mathcal{I}$

$\operatorname{rad}^\alpha(\mathcal{X})$ = smallest radical α-localizing tensor ideal containing $\mathcal{X}$.

$\operatorname{Rad}^\alpha(\mathcal{K})$ = frame of radical α-localizing tensor ideals of $\mathcal{K}$.

Lemma The α-compact elements of $\operatorname{Rad}^\alpha(\mathcal{K})$ are precisely those of the form $\operatorname{rad}^\alpha(X)$ for some $X \in \mathcal{K}$.

for α-coherent issue is $\exists$? finite meet

Warnings for $\aleph_0 \to \alpha$ For $\aleph_0$:

① for a thick tensor ideal $\mathcal{I}$:

$$\operatorname{rad}^{\aleph_0}(\mathcal{I}) = \{X \in \mathcal{K} \mid X^{\otimes n} \in \mathcal{I}\}$$

② $$\operatorname{rad}^{\aleph_0}(X \otimes Y) = \operatorname{rad}^{\aleph_0}(X) \overset{= \cap}{\wedge} \operatorname{rad}^{\aleph_0}(Y)$$

The same need not hold for $\alpha > \aleph_0$

Definition When $\mathcal{K}$ has α-coproducts we say $\mathcal{K}$ satisfies the <u>α-tensor property</u> if

$$\operatorname{rad}^\alpha(X \otimes Y) = \operatorname{rad}^\alpha(X) \wedge \operatorname{rad}^\alpha(Y)$$

always: $\leq$

Theorem [Krause, L]

Suppose $\mathcal{K}$ satisfies the α-tensor property. Then the frame is α-coherent and the map
$$\operatorname{Ob}(\mathcal{K}) \longrightarrow \operatorname{Rad}^\alpha(\mathcal{K}), \quad X \longmapsto \operatorname{rad}^\alpha(X)$$

is an $\underbrace{\alpha}$-support; it is initial among all α-supports on $\mathcal{K}$.

respects α-$\coprod$'s

Well-generated Δed categories

$\mathcal{T}$: Δ'ed category $\qquad\qquad$ α: regular

$\mathcal{T}^\alpha$: full subcategory of $\underline{\alpha\text{-compact}}$ objects

technical condition, it is not enough to "just" generalize compact $\rightsquigarrow$

$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed] & \coprod_{Y \in \mathcal{Y}} Y \\ \coprod_{Y \in \mathcal{Y}'} Y \arrow[ur] & \end{tikzcd}$$

for some $\mathcal{Y}' \subseteq \mathcal{Y}$ with $|\mathcal{Y}'| < \alpha$

$\rightsquigarrow$ max'l α-perfect class of α-small objects

$\rightsquigarrow$ $\mathcal{T}^\alpha$ is α-localizing

$\mathcal{T}$ is $\underline{\alpha\text{-compactly generated}}$ if

$\mathcal{T} = \mathrm{loc}(\text{set of } \alpha\text{-compact objects})$

$\mathcal{T}$ is $\underline{\text{well-generated}}$ if it is α-compactly generated for some α.

$\boxed{\text{Facts}}$ If $\mathcal{T}$ is α-compactly generated, then

- for $\beta \geq \alpha$: $\mathcal{T}^\beta = \mathrm{loc}^\beta(\mathcal{T}^\alpha)$
- $\mathcal{T}^\alpha$ is essentially small $\Rightarrow$ $\mathrm{Rad}^\alpha(\mathcal{T}^\alpha)$ is a $\underline{\underline{\text{set}}}$
- we have a filtration: $\mathcal{T} = \bigcup_{\beta \geq \alpha} \mathcal{T}^\beta$.

Extension and restriction

$\mathcal{T}$: $\otimes$-Δ-category

$\alpha \le \beta$ regular cardinals

allow: "arbitrary" instead of β
ie $\mathcal{T} \leftrightarrow \mathcal{T}^\beta$
$\mathrm{Rad} \leftrightarrow \mathrm{Rad}^\beta$

$$(-)\uparrow_\alpha^\beta : \mathrm{Rad}^\alpha(\mathcal{T}^\alpha) \rightleftarrows \mathrm{Rad}^\beta(\mathcal{T}^\beta) : (-)\downarrow_\alpha^\beta$$

$$\mathcal{I} \longmapsto \mathrm{rad}^\beta(\mathcal{I})$$

$$\mathcal{J} \cap \mathcal{T}^\alpha \longleftarrow\!\shortmid\ \mathcal{J}$$

in general not morphisms of frames

$\boxed{\text{Facts}}$ - $(-)\uparrow_\alpha^\beta$ preserves joins

- $(-)\downarrow_\alpha^\beta$ preserves meets

- $\mathcal{I} \le (\mathcal{I}\uparrow_\alpha^\beta)\downarrow_\alpha^\beta$ and $\mathcal{J} \ge (\mathcal{J}\downarrow_\alpha^\beta)\uparrow_\alpha^\beta$

Thomason's localization theorem:

$$\mathcal{T}\ \alpha\text{-compactly generated} \Rightarrow \mathcal{I} = (\mathcal{I}\uparrow_\alpha^\beta)\downarrow_\alpha^\beta$$

$\boxed{\text{Lemma}}$ Let $\alpha < \beta$ be regular cardinals and $\mathcal{T}$ α-compactly generated. We assume

$$\mathrm{rad}^\beta(X \otimes Y) = \mathrm{rad}^\beta(X) \otimes \mathrm{rad}^\beta(Y) \quad \forall X, Y \in \mathcal{T}^\alpha$$

Then ① $\mathcal{T}^\alpha$ satisfies the tensor property
② $(-)\uparrow_\alpha^\beta$ is a morphism of frames
③ If $\mathrm{Rad}^\beta(\mathcal{T}^\beta)$ is spatial, then so is $\mathrm{Rad}^\alpha(\mathcal{T}^\alpha)$

If $\mathrm{Rad}^\alpha(\mathcal{T}^\alpha)$ spatial we write $\mathrm{Spc}^\alpha(\mathcal{T}^\alpha)$ for the corresponding topological space.

Stratification

$\boxed{\text{Definition}}$ We call $\mathcal{T}$ <u>α-compactly stratified</u>, if

(CS1) $\mathcal{T}$ is α-compactly generated
(CS2) $\mathrm{Rad}(\mathcal{T})$ is spatial
(CS3) $\mathcal{T}$ satisfies the tensor property } $\rightsquigarrow$ descend
(CS4) the induced map $\mathrm{Spc}(\mathcal{T}) \longrightarrow \mathrm{Spc}^\alpha(\mathcal{T}^\alpha)$ is a bijection
(CS5) every point in $\mathrm{Spc}^\alpha(\mathcal{T}^\alpha)$ is locally closed

from $(-)\Gamma_\alpha^\beta$

$\boxed{\text{Theorem}}$ [Krause, L] $\alpha \leq \beta$ regular

$\mathcal{T}$ α-compactly generated $\Rightarrow$ β-compactly generated

$$\begin{array}{ccc}
\mathrm{Rad}^\alpha(\mathcal{T}^\alpha) & \longleftrightarrow & \mathrm{Spc}^\alpha(\mathcal{T}^\alpha) \\
\downarrow & & \downarrow \\
\mathrm{Rad}^\beta(\mathcal{T}^\beta) & \longleftrightarrow & \mathrm{Spc}^\beta(\mathcal{T}^\beta) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{Rad}(\mathcal{T}) & \longleftrightarrow & \mathrm{Spc}(\mathcal{T})
\end{array}$$

bijections on sets
refining topology

Moreover - $\mathrm{Spc}(\mathcal{T}) = \bigcap_{\beta \leq \alpha} \mathrm{Spc}(\mathcal{T}^\beta)$

- for β big enough: $\mathrm{Spc}^\beta(\mathcal{T}^\beta) = \mathrm{Spc}(\mathcal{T})$

$\boxed{\text{Corollary}}$ $\mathcal{T}$: α-compactly stratified

well-generated cats are closed under Verdier quotients

If $\mathcal{S} \subseteq \mathcal{T}$ a radical localizing tensor ideal, then $\mathcal{T}/\mathcal{S}$ is β-compactly generated for some $\beta \geq \alpha$.

β : $\mathcal{S}$ gen'd by set of β-compact objects.