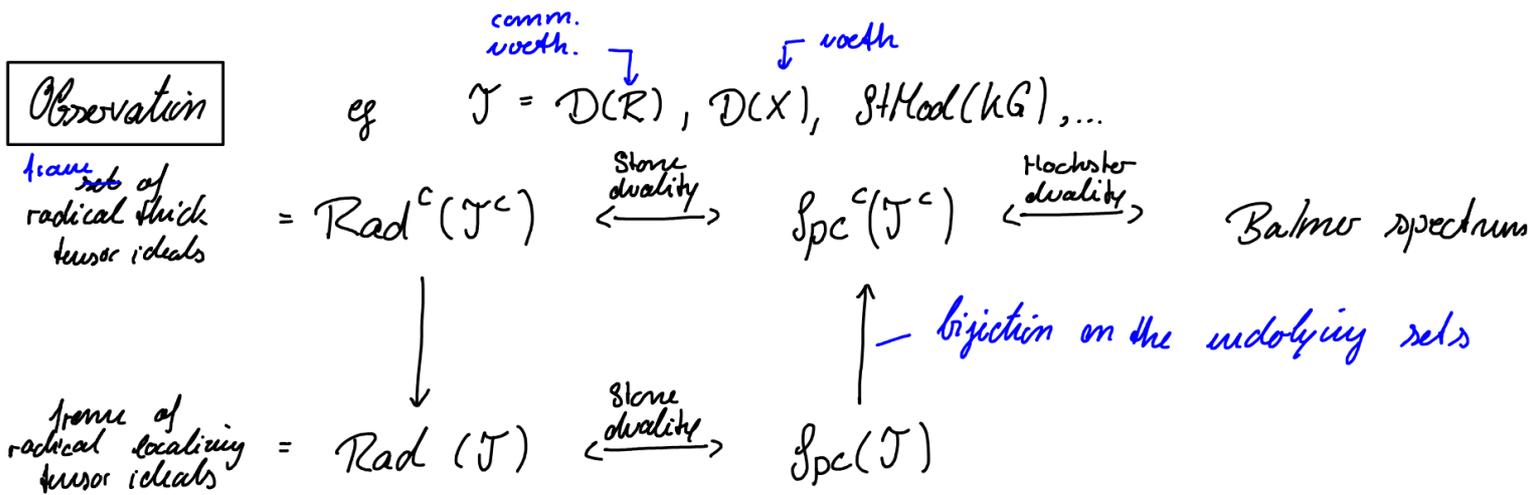


The spectrum of a well-generated tensor triangulated category

joint work with Henning Krause

Observation



Question

How do the α -localizing ideals of the α -compact objects fit in?

Background on frames

A poset (L, \leq) is a lattice if the join and the meet of any finite set exists

a ∨ b
∨ A \downarrow *a ∧ b*
∧ A

A lattice (F, \leq) is a frame if joins and finite meets exist and \Rightarrow arbitrary meets exist

$$a \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} (a \wedge b) \quad \forall a \in F, B \subseteq F$$

(infinite-join distributive law)

$\text{Frm} =$ category of frames

morphisms: maps that preserve order, join, finite meet

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \text{frame of open sets} \\ \text{Top} & \xrightarrow{\Omega} & \text{Frm}^{\text{op}} \end{array}$$

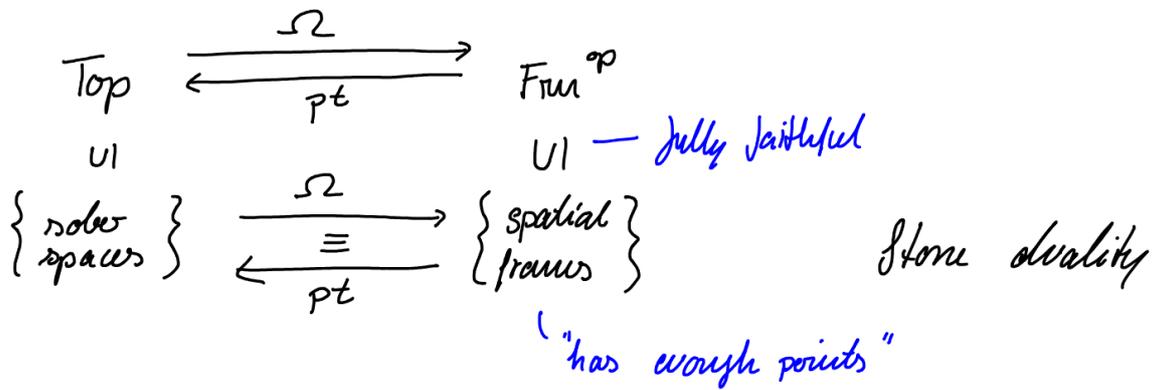
this functor has a right adjoint

A point of F is a frame morphism $x: F \rightarrow \{0,1\}$

\Leftrightarrow (meet-) prime elt
 $\bigvee x^{-1}(0) = p_x$

$pt(F)$ is a set of points of F . It is a topological space with open sets:

$$U(a) := \{x \in pt(F) \mid x(a) = 1\}$$



Definition

$a \in F$ is α -compact if for every set $A \subseteq F$ with $a \leq \bigvee A$ there exists $B \subseteq A$ with

$$a \leq \bigvee B \quad \text{and} \quad |B| < \alpha.$$

$F^\alpha :=$ set of α -compact elements of F .

F is α -coherent, if

- 1) every element is the join of α -compact elements
- 2) 1 is α -compact
- 3) the α -compact elements are closed under finite meets

} can recover F from F^α

Remark

compact = \aleph_0 -compact
 coherent = \aleph_0 -coherent

coherent \Rightarrow spatial

BUT: α -coherent $\not\Rightarrow$ spatial

Definition

Let \mathcal{K} be a essentially small \otimes - Δ -category with α -coproducts and α a regular cardinal

A tensor ideal $\mathcal{I} \subseteq \mathcal{K}$ is

- α -localizing if \mathcal{I} is Δ -ed, closed under α -coproducts and direct summands
for free when $\alpha > \aleph_0$

- radical if whenever $X^{\otimes n} \in \mathcal{I}$ for some $n \geq 1$ also $X \in \mathcal{I}$

$\text{rad}^\alpha(\mathcal{X}) =$ smallest radical α -localizing tensor ideal containing \mathcal{X} .

$\text{Rad}^\alpha(\mathcal{K}) =$ frame of radical α -localizing tensor ideals of \mathcal{K} .

Lemma

The α -compact elements of $\text{Rad}^\alpha(\mathcal{K})$ are precisely those of the form $\text{rad}^\alpha(X)$ for some $X \in \mathcal{K}$.

for α -coherent issue is 37 finite part

Warnings for $\aleph_0 \rightarrow \alpha$

For \aleph_0 :

① for a thick tensor ideal \mathcal{I} :

$$\text{rad}^{\aleph_0}(\mathcal{I}) = \{X \in \mathcal{K} \mid X^{\otimes n} \in \mathcal{I}\}$$

② $\text{rad}^{\aleph_0}(X \otimes Y) = \text{rad}^{\aleph_0}(X) \wedge \text{rad}^{\aleph_0}(Y)$ \leftarrow \cap

The same need not hold for $\alpha > \aleph_0$

Definition

When \mathcal{K} has α -coproducts we say \mathcal{K} satisfies the α -tensor property if

$$\text{rad}^\alpha(X \otimes Y) = \text{rad}^\alpha(X) \wedge \text{rad}^\alpha(Y) \quad \text{always: } \leq$$

Theorem

[Krause, L]

Suppose \mathcal{K} satisfies the α -tensor property. Then the frame is α -coherent and the map

$$\text{Ob}(\mathcal{K}) \rightarrow \text{Rad}^\alpha(\mathcal{K}), \quad X \mapsto \text{rad}^\alpha(X)$$

is an α -support; it is initial among all α -supports on \mathcal{K} .
 respects α -ll's

Well-generated t-Mod categories

\mathcal{T} : t-Mod category α : regular

\mathcal{T}^α : full subcategory of α -compact objects

↑
 technical condition, it is not enough
 to "just" generalize compact to

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{y \in \mathcal{Y}} Y \\ \vdots & & \uparrow \\ \coprod_{y \in \mathcal{Y}'} Y & & \end{array}$$

for some $\mathcal{Y}' \subseteq \mathcal{Y}$
 with $|\mathcal{Y}'| < \alpha$

\leadsto max'l α -perfect class of
 α -small objects

$\leadsto \mathcal{T}^\alpha$ is α -localizing

\mathcal{T} is α -compactly generated if

$$\mathcal{T} = \text{loc}(\text{set of } \alpha\text{-compact objects})$$

\mathcal{T} is well-generated if it is α -compactly generated for some α .

Facts If \mathcal{T} is α -compactly generated, then

- for $\beta \geq \alpha$: $\mathcal{T}^\beta = \text{loc}^\beta(\mathcal{T}^\alpha)$

- \mathcal{T}^α is essentially small $\Rightarrow \text{Rad}^\alpha(\mathcal{T}^\alpha)$ is a set

- we have a filtration: $\mathcal{T} = \bigcup_{\beta \geq \alpha} \mathcal{T}^\beta$.

Extension and restriction

\mathcal{T} : \otimes - Δ -category

allow: "arbitrary" instead of β
 ie $\mathcal{T} \leftrightarrow \mathcal{T}^\beta$
 $\text{Rad} \leftrightarrow \text{Rad}^\beta$

$\alpha \leq \beta$ regular cardinals

$$\begin{array}{ccc}
 (-) \uparrow_\alpha^\beta : \text{Rad}^\alpha(\mathcal{T}^\alpha) & \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} & \text{Rad}^\beta(\mathcal{T}^\beta) : (-) \downarrow_\alpha^\beta \\
 \downarrow & \xrightarrow{\quad} & \text{rad}^\beta(\mathcal{I}) \\
 \mathcal{J} \cap \mathcal{T}^\alpha & \xleftarrow{\quad} & \mathcal{J}
 \end{array}$$

in general not wophisms of frames

Facts

- $(-) \uparrow_\alpha^\beta$ preserves joins
- $(-) \downarrow_\alpha^\beta$ preserves meets
- $\mathcal{I} \leq (\mathcal{I} \uparrow_\alpha^\beta) \downarrow_\alpha^\beta$ and $\mathcal{J} \geq (\mathcal{J} \downarrow_\alpha^\beta) \uparrow_\alpha^\beta$

Thomason's localization theorem:

$$\mathcal{T} \text{ } \alpha\text{-compactly generated} \Rightarrow \mathcal{I} = (\mathcal{I} \uparrow_\alpha^\beta) \downarrow_\alpha^\beta$$

Lemma

Let $\alpha < \beta$ be regular cardinals and \mathcal{T} α -compactly generated. We assume

$$\text{rad}^\beta(X \otimes Y) = \text{rad}^\beta(X) \otimes \text{rad}^\beta(Y) \quad \forall X, Y \in \mathcal{T}^\alpha$$

- Then
- ① \mathcal{T}^α satisfies the tensor property
 - ② $(-) \uparrow_\alpha^\beta$ is a wophism of frames
 - ③ If $\text{Rad}^\beta(\mathcal{T}^\beta)$ is spatial, then so is $\text{Rad}^\alpha(\mathcal{T}^\alpha)$

If $\text{Rad}^\alpha(\mathcal{T}^\alpha)$ spatial we write $\text{Spec}^\alpha(\mathcal{T}^\alpha)$ for the corresponding topological space.

Stratification

Definition

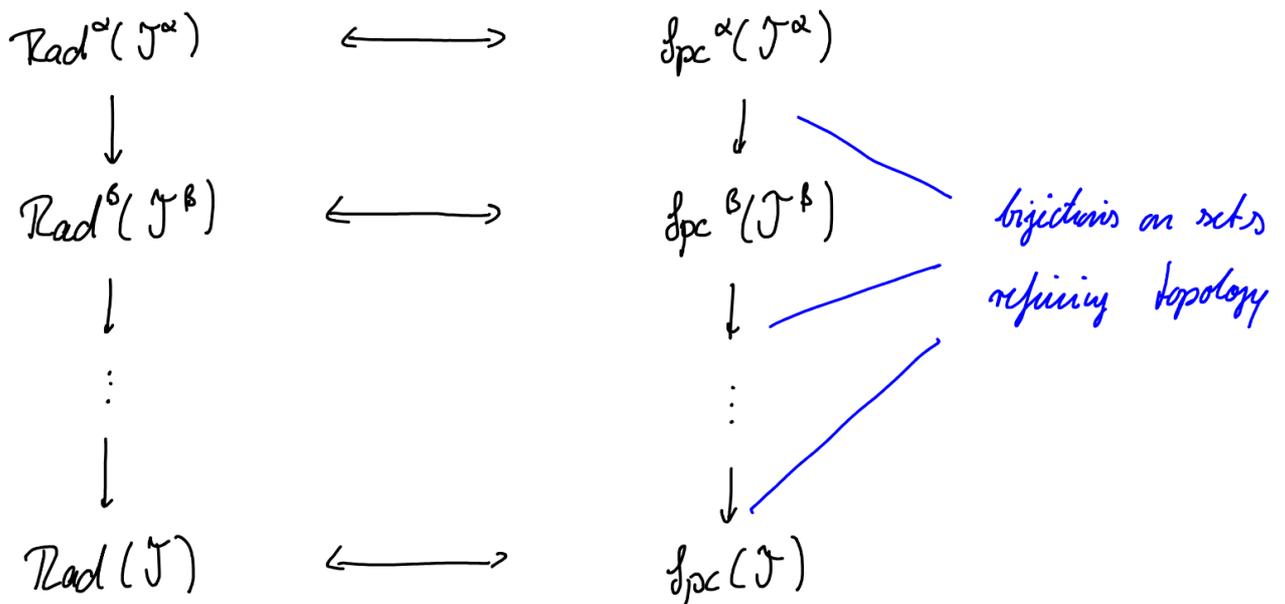
We call \mathcal{T} α -compactly stratified, if

- (CS1) \mathcal{T} is α -compactly generated
 - (CS2) $\text{Rad}(\mathcal{T})$ is spatial
 - (CS3) \mathcal{T} satisfies the tensor property
 - (CS4) The induced map $\text{Spc}(\mathcal{T}) \xrightarrow{\quad} \text{Spc}^\alpha(\mathcal{T}^\alpha)$ is a bijection
 - (CS5) every point in $\text{Spc}^\alpha(\mathcal{T}^\alpha)$ is locally closed
- } \leadsto descend
- from $(-)\uparrow_\alpha^\beta$

Theorem

[Krause, L] $\alpha \leq \beta$ regular

\mathcal{T} α -compactly generated \Rightarrow β -compactly generated



Moreover - $\text{Spc}(\mathcal{T}) = \bigcap_{\beta \leq \alpha} \text{Spc}(\mathcal{T}^\beta)$

- loc β big enough: $\text{Spc}^\beta(\mathcal{T}^\beta) = \text{Spc}(\mathcal{T})$

Corollary

\mathcal{T} : α -compactly stratified

well-generated cabs
are closed under
Verdier quotients

If $\mathcal{I} \subseteq \mathcal{T}$ a radical localizing tensor ideal, then \mathcal{T}/\mathcal{I} is β -compactly generated for some $\beta \geq \alpha$.

β : \mathcal{I} given by set of β -compact objects.