Brown representability for triangulated categories with a linear action by a graded ring

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necessary and sufficient conditions for a functor to be representable

A contravariant functor $f: \mathcal{C}^{op} \to \mathrm{Set}$ is representable if $f \cong \mathrm{Hom}_{\mathcal{C}}(-, X)$

Brown representability for triangulated categories with a linear action by a graded ring

- [NEEMAN; 1996]: For triangulated categories with small coproducts
- [BONDAL, VAN DEN BERGH; 2003], [ROUQUIER; 2008] For Ext-finite triangulated categories with a strong generator

S : $\mathbb{Z}$-graded ring
$\mathrm{Ext}(X,Y) = \coprod_{d \in \mathbb{Z}} \mathrm{Hom}(X, \Sigma^d Y)$ is a graded S-module and composition is a linear map

Why Brown representability?

→ existence of right adjoint functors

Given $f : \mathcal{S} \to \mathcal{T}$ consider $\mathrm{Hom}_{\mathcal{T}}(f(-), Y)$.

Why a linear action by a graded ring?

→ weaken the Ext-finite condition

eg : A : noetherian ring
$\mathcal{T} = D_b(\mathrm{mod}\, A)$

$\mathrm{Ext}^*_{\mathcal{T}}(X, Y)$ in general not bounded
⇝ not finitely generated over a ring

$\boxed{\text{Theorem}}$ [L; 2022]

there exists $G \in \mathcal{T}$ that builds any other object using finitely many cones and suspensions

S : $\mathbb{Z}$-graded graded-commutative noetherian ring

$\mathcal{T}$: S-linear triangulated category such that
- $\mathcal{T}$ has a $\boxed{\text{strong generator}}$
- $\mathcal{T}$ is $\boxed{\text{Ext-finite}}$
- $\mathcal{T}$ is idempotent complete

$\mathrm{Ext}_{\mathcal{T}}(X,Y) = \coprod_{d \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^d Y)$ is a finitely generated S-module

$f : \mathcal{T}^{op} \longrightarrow \mathrm{grMod}(S)$ a $\boxed{\text{graded } S\text{-linear}}$ cohomological functor

$\mathrm{Ext}_{\mathcal{T}}(X,Y) \longrightarrow \mathrm{Ext}_S(f(X), f(Y))$ is S-linear and $f(\Sigma^n X) = f(X)[-n]$

Then f $\boxed{\text{graded representable}}$ $\iff$ f is $\boxed{\text{locally finite}}$

$f \cong \mathrm{Ext}_{\mathcal{T}}(-, X)$

$f(X)$ is a finitely generated S-module for any $X \in \mathcal{T}$

Example: $D_b(\mathrm{mod}\ RG)$

G: finite group
R: commutative noetherian ring

$H^*(G,R) := \mathrm{Ext}^*_{RG}(R,R)$ a $\mathbb{Z}$-graded, graded-commutative $\boxed{\text{noetherian}}$ ring

[VENKOV; 1959]
[EVENS; 1961]

$\mathcal{T} = D_b(\mathrm{mod}\ RG)$

Need:
- $H^*(G,R)$-linear ✓
- Ext-finite ✓
- $\boxed{\text{strongly generated}}$?
- idempotent complete ✓

If $D_b(\mathrm{mod}\ RG)$ has a strong generator, then a graded $H^*(R,G)$-linear functor is graded representable if and only if it is locally finite.

Example: $D_b(\mathrm{mod}\, R)$ for R complete intersection

$R = Q/(f_1, \dots, f_c)$ a quotient of a regular local ring Q by a regular sequence $f_1, \dots, f_c$

$$HH^*(R/Q) := \mathrm{Ext}^*_{R \otimes^L_Q R}(R, R) \cong R[\chi_1, \dots, \chi_c], \quad |\chi_i| = 2$$

$\mathcal{T} = D_b(\mathrm{mod}\, R)$

Need:
- $HH^*(R/Q)$ - linear ✓
- Ext - finite ✓
- $\boxed{\text{strongly generated}}$?
- idempotent complete ✓

Yes in the following cases
- R artinian
- essentially of finite type over a field
- equicharacteristic excellent local ring

If $D_b(\mathrm{mod}\, R)$ has a strong generator, then a graded $HH^*(R/Q)$-linear functor is graded representable if and only if it is locally finite.