Multilinear tools through filters on groups

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What is intrinsic to a group?

Main question: What structure is intrinsic to a group $G$?

A group $G$ given with the shapes:

$$
\begin{bmatrix}
1 & A \\
0 & I_{12}
\end{bmatrix},
\begin{bmatrix}
I_2 & B \\
0 & I_6
\end{bmatrix},
\begin{bmatrix}
I_3 & C \\
0 & I_4
\end{bmatrix}.
$$
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0 & I_4
\end{bmatrix}.
$$

A group given with the shapes:

$$
\begin{bmatrix}
1 & A & C \\
I_2 & B & I_8
\end{bmatrix},
\begin{bmatrix}
I_2 & X & Z \\
I_3 & Y & I_4
\end{bmatrix}.
$$

What about scalars?
\[ G = \begin{bmatrix} 1 & A & C \\ I_2 & B \\ I_8 \end{bmatrix} \quad H = \begin{bmatrix} I_2 & X & Z \\ I_3 & Y \\ I_4 \end{bmatrix} \]

Verbal subgroups produce a series:

\[ \begin{bmatrix} 1 & A & C \\ I_2 & B \\ I_8 \end{bmatrix} > \begin{bmatrix} 1 \\ I_2 \\ I_8 \end{bmatrix} > \begin{bmatrix} 1 \\ I_2 \\ I_8 \end{bmatrix}, \]

\[ \begin{bmatrix} I_2 & X & Z \\ I_3 & Y \\ I_4 \end{bmatrix} > \begin{bmatrix} I_2 \\ I_3 \\ I_4 \end{bmatrix} > \begin{bmatrix} I_2 \\ I_3 \\ I_4 \end{bmatrix}. \]

In both groups, \( \gamma_1/\gamma_2 \cong K^{16}, \quad \gamma_2 \cong K^8. \)
In some cases, the shape is intrinsic

For a field $K$, let $H_{abc}(K) = \left\{ \begin{bmatrix} I_a & X & Z \\ I_b & Y & \cr I_c & & \end{bmatrix} \mid X \in M_{ab}(K), Y \in M_{bc}(K), Z \in M_{ac}(K) \right\}$.

Theorem (J.B. Wilson 2017)

For groups $H_{abc}(K)$, the integers $a, b, c$ are isomorphism invariants, and they can be computed in polynomial time.

Now: special case of a larger body of work with U. First, J.B. Wilson.

**Idea:** All that information found in algebras associated to

\[
[, ] : \gamma_1 / \gamma_2 \times \gamma_1 / \gamma_2 \hookrightarrow \gamma_2 ,
\]

\[
[, ] : K^{ab + bc} \times K^{ab + bc} \hookrightarrow K^{ac} .
\]
Larger examples are refinable

\[
G = \begin{bmatrix}
1 & * & * & * & * \\
1 & * & * & * & * \\
1 & * & * & * & * \\
1 & * & * & * & * \\
1 & * & * & * & * \\
\end{bmatrix}
= \begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4 \\
\end{bmatrix}
\]

E.g. \( \gamma_0 = \gamma_1 = G \) and \( \gamma_{s+1} = [\gamma_s, \gamma_1] \), for \( s \geq 1 \).
Larger examples are refinable

\[ G = \begin{bmatrix} 1 & * & * & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \end{bmatrix} = \]

E.g. \( \gamma_0 = \gamma_1 = G \) and \( \gamma_{s+1} = [\gamma_s, \gamma_1] \), for \( s \geq 1 \).

\[ L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K \]
Filters produce refinable graded Lie algebras

A filter is a function $\phi : \langle \mathbb{N}^d, \leq \rangle \rightarrow 2^G$ into the normal subgroups with

$$[\phi_s, \phi_t] \leq \phi_{s+t} \quad \text{and} \quad s \preceq t \implies \phi_s \geq \phi_t.$$ 

Theorem (J.B. Wilson 2013)

If $\phi : \mathbb{N}^d \rightarrow 2^G$ is a filter, then

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \langle \phi_{s+t} \mid t \neq 0 \rangle$$

is an $\mathbb{N}^d$-graded Lie ring. Each graded ideal lifts to a filter refinement.
Efficient refinements for filters

**Theorem (M. 2017)**

If \( \phi : \mathbb{N}^d \to 2^G \) is a totally ordered filter and \( H \triangleleft G \) refines \( \phi \), then there exists an efficient algorithm (polynomial time in \( \log |G| \)) that constructs a filter from \( \phi \) including \( H \).

- Provides structure that connects \( \mathbb{N}^d \) to subgroups of \( G \) that can be updated.
- Allows for efficient recursion.
Survey of 500,000,000 groups of order $2^{10}$

Filters uncover new characteristic structure [M.-Wilson].
Refining the algebra to get smaller steps

\[
\phi^{(1)} : \mathbb{N} \to 2^G
\]

\[
L\left(\phi^{(1)}\right) = \bigoplus_{s \neq 0} L_s
\]

\[
\phi^{(2)} : \mathbb{N}^2 \to 2^G
\]

\[
L\left(\phi^{(2)}\right) = \bigoplus_{s \neq 0} L_s
\]

\[
\phi^{(3)} : \mathbb{N}^3 \to 2^G
\]

\[
L\left(\phi^{(3)}\right) = \bigoplus_{s \neq 0} L_s
\]

\[
\phi : \mathbb{N}^d \to 2^G
\]

\[
L(\phi) = \bigoplus_{s \neq 0} L_s
\]
Refinement improves even well-known examples

$L(\gamma) = K_5 \oplus K_4 \oplus K_3 \oplus K_2 \oplus K_1$

$L(\phi) = K_3 \oplus K_2 \oplus K_2 \oplus K_2 \oplus K_2 \oplus K_2 \oplus K_0$
Refinement improves even well-known examples

\[ L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K \]
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\[ L(\gamma) = K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K \]
\[ L(\phi) = K^3 \oplus K^2 \oplus K^2 \oplus K^2 \oplus K \oplus K^2 \oplus K^2 \oplus K \]
Use module and ring theory to start refinement

- Suppose \( \circ : U \times V \rightarrow W \) is a bilinear map of \( K \)-vector spaces.

- Some algebras associated to \( \circ \) are

\[
\begin{align*}
\mathcal{L}_\circ &= \{(X, Z) \mid (Xu) \circ v = Z(u \circ v)\}, \\
\mathcal{M}_\circ &= \{(X, Y) \mid (uX) \circ v = u \circ (Yv)\}, \\
\mathcal{R}_\circ &= \{(Y, Z) \mid u \circ (vY) = (u \circ v)Z\}, \\
\text{Cent}(\circ) &= \{(X, Y, Z) \mid (uX) \circ v = u \circ (vY) = (u \circ v)Z\}, \\
\text{Der}(\circ) &= \{(X, Y, Z) \mid (uX) \circ v + u \circ (vY) = (u \circ v)Z\}.
\end{align*}
\]

- Ongoing work with Brookesbank and Wilson using representation theory of Lie algebras in the context of isomorphism problems.

- Multilinear Algebra package for MAGMA on GitHub [M.-Wilson].
Looking for structure in new places

**Theorem (Brooksbank-M.-Wilson, 2017)**

There exists a polynomial-time algorithm to test isomorphism of groups of exponent $p$ with central commutator subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

- Used by Brooksbank, O’Brien, and Wilson to efficiently search for local structure.
- Implemented in MAGMA.

- Graph showing the relationship between group size $|G|$ and the time in minutes to perform the isomorphism test.
Study the group through $L(\phi)$

Two fundamental problems arise in partially-ordered case:

- Let $G$ be nilpotent, and $\gamma : \mathbb{N} \to 2^G$ the lower central series.

Set $\phi : \mathbb{N}^2 \to 2^G$ such that for $s = (s_1, s_2) \in \mathbb{N}^2$,

$$\phi_s = \gamma_{s_1}.$$

The associated Lie algebra is trivial

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \langle \phi_{s+t} \mid t \neq 0 \rangle = 0.$$
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$$L(\phi) = \bigoplus_{s \neq 0} \phi_s / \langle \phi_{s+t} | t \neq 0 \rangle = 0.$$

Theorem (M. 2018)

If $G$ is nilpotent and $\phi : \mathbb{N}^d \to 2^G$ is a filter, then there exists a filter $\theta : \mathbb{N}^d \to 2^G$ such that

- $\text{im}(\phi) \subseteq \text{im}(\theta)$ and
- there is a surjection $L(\theta) \to G$. 

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The other problem

Let $K$ be a field of order $q$ and $G = \left\{ \begin{bmatrix} 1 & a & c \\ 1 & b & 1 \end{bmatrix} : a, b, c \in K \right\}$.

There are $q + 1$ distinct subgroups $G' < H < G$.

There is a filter $\phi : \mathbb{N}^{q+1} \rightarrow 2^G$, such that

$$\dim L(\phi) = q + 2.$$
A bijection between $G$ and $L(\phi)$ is recovered.

A filter $\phi : \mathbb{N}^d \to 2^G$ is *compatible* if there exists $\mathcal{X} \subset G$:

1. $G = \langle \mathcal{X} \rangle$,
2. for all $s \in \mathbb{N}^d$, $\langle \phi_s \cap \mathcal{X} \rangle = \phi_s$,
3. $H \mapsto H \cap \mathcal{X}$ is a complete lattice embedding from $\text{im}(\phi)$ to $2^\mathcal{X}$,
4. for all $x \in \mathcal{X}$, there exists a unique $s \in \mathbb{N}^d$ such that $x \in \phi_s \setminus \langle \phi_{s+t} \mid t \neq 0 \rangle$.

**Theorem (M. 2018)**

Suppose $G$ is nilpotent and polycyclic. If $\phi : \mathbb{N}^d \to 2^G$ is compatible, then there exists a bijection between the set of bases for $L(\phi)$ and the polycyclic generating sets for $G$. 

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Filters provide different location for structure

- Filters refine many examples of groups: 97% in survey of $2^{10}$.

- Algebras associated to bilinear maps $L_s \times L_t \hookrightarrow L_{s+t}$.

- Structure from entire $\mathbb{N}^d$-graded $L(\phi)$.

- Developed for isomorphism, but are general tools.