Algebras associated to bilinear maps

Suppose \(\phi : U \times V \to W\) is a bilinear map of \(K\)-vector spaces. With \(X, Y, Z\) as linear operators, some algebras associated to \(\phi\) are:

- \(C(\phi) = \{ (X, Y, Z) \mid u \circ v = u \circ (vY) = (u \circ v)Z \}\),
- \(L(\phi) = \{ (X, Z) \mid u \circ v = (u \circ v)Z \}\),
- \(M(\phi) = \{ (X, Y') \mid u \circ v = u \circ (vY) \}\),
- \(R(\phi) = \{ (Y, Z) \mid u \circ (vY) = (u \circ v)Z \}\),
- \(\text{Der}(\phi) = \{ (X, Z) \mid u \circ v + u \circ (vY) = (u \circ v)Z \}\).

All algebras above are efficiently computed using linear algebra. If \(\phi\) comes from a group, like \([,]_{s,t}\), then \(\text{Aut}(G)\) acts on all the above algebras.

The algebra \(\text{Der}(\phi)\) is a Lie algebra, while others are associative.

Multilinear algebra MAGMA software available on GitHub and https://thetensor.space

Structure in nilpotent groups

We cannot easily find characteristic subgroups of nilpotent groups, but we know they exist in abundance. Consider the lower central series of upper unitriangular matrices, \(\gamma_0 = G, \gamma_1 = \text{Fit}(G), \quad (\forall s \geq 1 \quad \gamma_{s+1} = [\gamma_s, \gamma_1]).\)

Set \(L_0(\gamma) = 0\) for \(s \neq 0\), \(L_s(\gamma) = \gamma_s/\gamma_{s+1}\).

There is an \(N\)-graded \(K\)-algebra and \(K[\gamma_0/\gamma_1]\)-module:

\[L(\gamma) = \bigoplus_{s=0} L_s(\gamma) \cong K^5 \oplus K^4 \oplus K^3 \oplus K^2 \oplus K.\]

The grading of \(L(\gamma)\) has \(K\)-bilinear maps from the commutator:

\[ [\cdot, \cdot] : L_s(\gamma) \times L_t(\gamma) \to L_{s+t}(\gamma) \]

Geometry explains exact sequences

Associate ideals of \(K[x, y, z]\) from definitions of algebras, see [FMW]:

\[ \mathcal{J}(\phi, C(\phi)) = (x - y, y - z), \quad \mathcal{J}(\phi, R(\phi)) = (y - z), \quad \mathcal{J}(\phi, L(\phi)) = (x - z), \quad \mathcal{J}(\phi, \text{Der}(\phi)) = (x + y - z). \]

Theorem 2 ([BMW1]). Suppose \(\phi : U \times V \to W\) is non-degenerate. For \(LMR = L(\phi) \oplus M(\phi) \oplus R(\phi)\), the following are exact sequences of Lie algebras and groups, respectively:

\[ 0 \to C(\phi) \to LMR(\phi) \to \text{Der}(\phi), \]
\[ 0 \to C(\phi)^{\times} \to LMR(\phi)^{\times} \to \text{Aut}(\phi). \]

Sequences in Theorem 2 hold for multilinear maps, based on geometry.