Seminar: ℓ^2-Invariants

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The talks should have the duration of 1.5 hours. They will take place on Wednesdays at 14:15, but if you are speaking, it is a good idea to come earlier (say, at 14:00) to try out the technical setup. We will use the Zoom room with ID **844 4651 5607**. The password is the name given to the sum of the diagonal entries in a square matrix (all lowercase).

If you wish to make a slideshow presentation, Prof. Clara Löh has a collection of useful advice on preparing such talks on her webpage [4]. Alternatively, you can use a virtual whiteboard, especially if you have a writing tablet. Zoom has basic whiteboard functionality, but there is plenty of other software freely available. Here are two examples:
Openboard: `https://openboard.ch/index.en.html`
Xournal: `http://xournal.sourceforge.net/`

Each speaker is expected to produce a written account of what was done during their presentation, to be shared with the other participants. This can be a short handout or the slides / virtual whiteboards used during the presentation. Students who wish to get a grade, however, must hand in a report of about 10 pages. Please send this material to the organizers at most one week after your talk.

If you have questions about the preparation of your talk, you are welcome to contact either of the organizers, ideally about one week before the presentation.

1 Hilbert spaces and the von Neumann algebra

Date: November 4
Main reference: [2, Sections 2.1 and 2.2]
Speaker: Simon Lang

Recall the definition and some basic properties of Hilbert spaces, including the classification of separable Hilbert spaces (Theorem 2.12) and the foundational example of the Fourier isomorphism $\ell^2\mathbb{Z} \cong L^2[-\pi, \pi]$ (Example 2.14). Also mention the Riesz Lemma (Theorem 2.18) and define the adjoint of a bounded map of Hilbert spaces.

The group von Neumann algebra $\mathcal{R}G$ of a discrete countable group G (Definition 2.23) will be one of the central characters throughout the seminar, and it should be carefully defined. This requires in particular discussing the weak topology on operator spaces. State the characterization of $\mathcal{R}G$ as the algebra of bounded operators on a Hilbert space that commute with the left regular representation λ (Theorem 2.24), and use it to prove that $\mathcal{R}\mathbb{Z} \cong L^\infty[-\pi, \pi]$ (Example 2.26).

2 The von Neumann trace and dimension

Date: November 11
Main reference: [2, Sections 2.2 and 2.3]
Speaker: Lars Munser

The existence of a reasonable notion of trace for operators in $\mathcal{R}G$ (and, more generally, in the amplified group von Neumann algebras $\mathcal{M}_n(\mathcal{R}G)$) is the main ingredient that allows one to define the dimension of (finitely generated) Hilbert $\mathcal{L}G$-modules. This lies at the core of the theory of ℓ^2-invariants.

Define the von Neumann trace functional on $\mathcal{R}G$ (Definition 2.31), and discuss the case $G = \mathbb{Z}$ (Example 2.32). Extend this definition to (finitely generated) Hilbert $\mathcal{L}G$-modules (Proposition 2.33, Definition 2.34) and present its main properties (Theorems 2.35 and 2.36). Introduce the von Neumann dimension of a Hilbert $\mathcal{L}G$-module (Definition 2.37) and its properties (Theorem 2.44), and illustrate with some/all of the Examples 2.38 through 2.41.

You are free to choose which properties of trace and dimension you wish to prove (if any).

3 G-CW-complexes and their ℓ^2-chain complexes

Date: November 18
Main reference: [2, Sections 3.1 and 3.2]
Speaker: Jan-Philipp Zwank

This talk sets up the connection between topology and the theory of Hilbert $\mathcal{L}G$-modules developed so far. The goal is to introduce G-CW-complexes (Definition 3.3) and their ℓ^2-chain complexes (Definition 3.10), and show that "under favorable circumstances" these are chain complexes of Hilbert $\mathcal{L}G$-modules (Theorem 3.11). Some details of the construction are rather technical and can be left out, but the relevance of the "proper" and "finite type" hypotheses of Theorem 3.11 should be made very explicit.

4 ℓ^2-Betti numbers

Date: November 25
Main reference: [2, Sections 3.3 and 3.4]
Speaker: Giacomo Bertizzolo

In this talk we reach one of the main goals of the seminar: introducing ℓ^2-homology and ℓ^2-Betti numbers (Definition 3.13), and establishing important properties (Example 3.14, Theorems 3.18, 3.19), including (without proof) ℓ^2-Poincaré Duality (Theorem 3.25). A large portion of this talk should be spent on computing ℓ^2-Betti numbers on concrete examples, highlighting how these computations strongly rely on the good properties of the von Neumann dimension established earlier.

Additionally, explain how Hopf's Conjecture (Conjecture 1.4), which is a priori unrelated to ℓ^2-invariants, would follow from Singer's Conjecture (Conjecture 1.5). If you have time, you may also state as a teaser Lück's Approximation Theorem (Theorem 1.6), which relates ℓ^2-Betti numbers to classical Betti-numbers.

5 Applications to algebra and topology

Date: December 2
Main reference: [2, Sections 3.5 and 3.6]
Speaker: Kevin Li

We will see examples of how the theory of ℓ^2-invariants developed so far feeds back into independent questions in algebra and topology. On the algebraic side, explain how Atiyah's conjecture implies Kaplansky's (purely algebraic) conjecture (Theorem 3.32). As for topology, discuss how ℓ^2-Betti numbers obstruct the existence of non-trivial finite-sheeted self-coverings on connected CW-complexes of finite type (Corollary 3.35), mapping torus structures on manifolds (Theorem 3.38), and S^1-CW-structures on topological spaces (3.40). If you have time, you may also mention some of the consequences of these results in the setting of 3-manifold topology.

6 On a celebrated question by Gromov

Date: December 9
Main reference: [6, Chapter 14]

Speaker: Marco Moraschini

In this lecture I will give a gentle and expository introduction to a well-known question by Gromov asking whether the vanishing of the simplicial volume of an oriented closed connected aspherical manifold implies the vanishing of all its ℓ^2-Betti numbers. To this end, we will discuss before some historical motivations behind the question and then we will introduce the notion of simplicial volume and its basic properties.

According with the remaining time, we will discuss some strategies for getting a positive answer to Gromov's question and we will enlighten the difficulties we could find along the path.

7 The extended von Neumann dimension and ℓ^2-Betti numbers of general G-spaces

Date: December 16
Main reference: [2, Sections 4.2 and4.3]
Speaker: Daniel Echtler

We wish to construct from a group G a G-CW-complex whose ℓ^2-Betti numbers are then invariants of G. This will however require broadening the definition of the von Neumann dimension past our original setting – in particular, we will then be able to assign ℓ^2-Betti numbers to entirely general G-CW-complexes, and even to G-spaces without a CW-structure.

In this talk, we will see the construction of an equivalence between the category of finitely generated projective left $\mathcal{R}G$-modules and the category of finitely generated Hilbert $\mathcal{L}G$-modules (Theorem 4.5), which motivates the definition of the extended von Neumann dimension of left $\mathcal{R}G$-modules (Definition 4.6). Briefly present the main features of the extended von Neumann dimension (Theorem 4.7) and the semiheredity properties of the ring $\mathcal{R}G$ (Proposition 4.8).

With these tools in hand, define the ℓ^2-Betti numbers of a G-space (Definition 4.9) and show that this definition generalizes the previous one (Theorem 4.10). State the main properties of these more general ℓ^2-Betti numbers (Theorem 4.11).

8 ℓ^2-Betti numbers of groups

Date: January 13
Main reference: [2, Sections 4.1, 4.4, and 4.5]
Speaker: Matthias Uschold

Given a group G, define classifying spaces for G (Theorem 4.2 / Definition 4.3) and sketch how one can construct such a space EG (Theorem 4.4). Kammeyer discusses a more general construction for families of subgroups of G, so you should streamline the discussion by focusing on the family $\mathcal{TRIV}$ comprised only of the trivial group.

Armed with the definition of ℓ^2-Betti numbers for general G-spaces from the preceding talk, and with the construction of classifying spaces, define the ℓ^2-Betti numbers for a group G (Definition 4.12). Present your favorite examples from Table 4.1, and two of the three applications in subsections 4.5.1 to 4.5.3, where the theory of ℓ^2-invariants yields group-theoretical results.

9 An excursion through functional calculus

Date: January 20
Main reference: [2, Section 5.2]
Speaker: José Pedro Quintanilha

Our next goal is to prove Lück's Approximation Theorem (Theorem 5.2), but this will require introducing some more results on functional analysis.

In this talk, we follow Kammeyer's exposition of various flavors of functional calculus, whose most general setting, Borel Functional Calculus (Theorem 5.10), will ultimately allow us to translate the statement of Lück's Approximation Theorem into one about convergence of measures.

The presentation in the book is often not fully detailed, so this talk should be more about systematizing the various results than providing detailed proofs (for instance, as is done in Table 5.1). It would however be nice to see the statement (and maybe proof) of Proposition 5.15, as it will be directly quoted in the next talk.

10 Lück's Approximation Theorem

Date: January 27
Main reference: [2, Sections 5.1 and 5.3]
Speaker: José Pedro Quintanilha

State Lück's Approximation Theorem (Theorem 5.2), as well as its group-theoretical incarnation (Theorem 5.3), and present its proof (Section 5.3). Please highlight the points where the machinery developed in the previous talk comes into use. The ideal presentation answers Exercise 5.3.1.

It may be necessary to quickly explain the identification between ℓ^2-homology and the kernel of the ℓ^2-Laplacian (Proposition 3.23), in case it has not been presented before in the seminar.

11 ℓ^2-Betti numbers of equivalence relations

Date: February 3
Main reference: [5, Sections 4.1 and 4.2]
Speaker: Clara Löh

We connect the tools from ergodic theory to the theory of ℓ^2-invariants. This talk fits neatly with the program of the course on ergodic theory of groups from Summer Semester 2020 (whose notes are available online [3]). We will enhance our algebraic/functional-analytic machinery to one that incorporates the action of our group on a standard probability space, and then use an algebraic replacement of classifying spaces (namely, Tor functors) to construct ℓ^2-Betti numbers of measured equivalence relations (Definition 4.2.3).

We will see that the familiar ℓ^2-Betti numbers of groups can be computed via essentially free measure-preserving actions on probability spaces (Theorem 4.2.5), and use this to conclude that classical notions of equivalence in ergodic group theory (namely measure equivalence and the existence of orbit equivalent for essentially free actions) are reflected in the ℓ^2-Betti numbers of groups (Corollary 4.2.8). We will also deduce a proportionality principle relating lattices in a topological group (Corollary 4.2.11).

12 Outlook on the Atiyah Conjecture and related results

Date: February 10
Main reference: [2, Section 5.6]
Speaker: Lars Munser

This talk is intended as a survey of results and questions related to the Atiyah Conjecture. The exposition in Kammeyer's book is rather loose, so the speaker is free to focus on whichever points they deem interesting, or even to present related topics from a different source (for example Lück's book [6, Chapter 10]).

13 An ℓ^2-invariant obstructing knot sliceness

Date: February 17
Main reference: [1]
Speaker: Stefan Friedl

The main reference for this talk is a long paper describing a filtration on the (topological) knot concordance group, and a certain ℓ^2-signature invariant that was used to show that certain knots are not slice, where all previously known invariants had failed. In this talk one would like to see the (sketch of the) construction of this ℓ^2-signature and get an idea of why it obstructs sliceness.

References

[1] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and L^2-signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.

[2] Holger Kammeyer. *Introduction to ℓ^2-invariants*, volume 2247 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.

[3] Clara Löh. Lecture notes for the course *Ergodic Theory of Groups* taught at the University of Regensburg in the summer semester 2020. Draft pdf available at `http://www.mathematik.uni-regensburg.de/loeh/teaching/all_lecture_notes.html`.

[4] Clara Löh. *Slippery Slides*. Available at `http://www.mathematik.uni-regensburg.de/loeh/seminars/slippery.pdf`.

[5] Clara Löh. *Ergodic theoretic methods in group homology: a minicourse on L^2-Betti numbers in group theory*. Springer Nature, 2020. Draft pdf available at `http://www.mathematik.uni-regensburg.de/loeh/teaching/all_lecture_notes.html`.

[6] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 2002.