# Fragments of derived Morita theory for exact categories with enough projectives

## 1. Synopsis

Given $\mathcal{T}$ a tilting subcategory $\mathcal{E}$, then as $\mathcal{T}^{\perp}$ is an exact category with enough projectives, one always finds a bounded derived equivalence to $\mathcal{E}' = \operatorname{mod}_S \mathcal{T}$ where $S$ are $\mathcal{T}^{\perp}$-admissible morphisms. This suggest that for a derived equivalence between two exact categories with enough projectives, we need to find a selforthogonal subcategory $\mathcal{T}$ which is generating ($\operatorname{Hom}(\mathcal{T}, \Sigma^n X) = 0\ \forall n \in \mathbb{Z}$ implies $X = 0$) and a class of morphisms which we call $\Delta$-suitable morphisms (cf. Def. 1.8). The definition is still suboptimal as it is difficult to verify, but $\Delta$-suitability is used to show that we can find $\mathcal{C} = \operatorname{mod}_S \mathcal{T} \subseteq \Delta$ as h-admissible exact subcategory (i.e. admissible exact and $\operatorname{Ext}^n(X, Y) \cong \operatorname{Hom}_\Delta(X, \Sigma^n Y)$ for all $X, Y$ in $\mathcal{C}$, $n > 1$). Fully $\Delta$-suitable means additionally $\operatorname{Thick}_\Delta(\mathcal{C}) = \Delta$. In this case, if we assume $\Delta$ algebraic, then the realization functor for $\mathcal{C}$ is a triangle equivalence, cf. [**2**].
Using this, the following characterization of the triangulated categories triangle equivalent to $D^b(\operatorname{mod}_S \mathcal{T})$ is then immediate.

Theorem 1.1. *Let $\Delta$ be an algebraic triangulated category and assume that there exists $\mathcal{T} \subseteq \Delta$ selforthogonal, generating and $S \subseteq \operatorname{Mor} - \mathcal{T}$ fully $\Delta$-suitable. For every other algebraic triangulated category $\Delta'$ the following are equivalent.*

(a) *$\Delta$ and $\Delta'$ are triangle equivalent.*
(b) *There exists $\mathcal{T}' \subseteq \Delta'$ selforthogonal, generating together with $S' \subseteq \operatorname{Mor} - \mathcal{T}'$ fully $\Delta$-suitable and an additive equivalence $F\colon \mathcal{T} \to \mathcal{T}'$ with $F(S) = S'$.*

We also conclude that a triangulated category is triangle equivalent to a bounded derived category of an exact category with enough projectives if and only if it contains a selforthogonal generating subcategory which admits fully $\Delta$-suitable morphisms.

**1.1. Exact categories with enough projectives revisited.** We recall the following notion from chapter 4.

**Definition 1.2.** Let $\mathcal{C}$ be an idempotent complete additive category. We call a class of morphisms $S \subseteq \operatorname{Mor} - \mathcal{C}$ **homotopy-closed** if $s \in S$ and $\operatorname{coker} \operatorname{Hom}_{\mathcal{C}}(-, s) \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{C}}(-, t)$ in $\operatorname{Mod} \mathcal{C}$ implies $t \in S$.
We say that $S$ is **suitable** if it is homotopy closed and $\operatorname{mod}_S \mathcal{C} = \{F\colon \operatorname{coker} \operatorname{Hom}(-, s) \mid s \in S\}$ is a resolving subcategory of $\operatorname{mod}_\infty \mathcal{C}$.

**Lemma 1.3.** *We fix an essentially small idempotent complete additive category $\mathcal{P}$ and look at the following sets:*

(1) *Exact categories $\mathcal{E}$ with enough projectives $\mathcal{P}(\mathcal{E}) = \mathcal{P}$.*
(2) *Suitable classes of morphisms $S \subseteq \operatorname{Mor} - \mathcal{P}$.*

*The assignements $\mathbb{S}(\mathcal{E}) = \{\mathcal{E} - \text{admissible morphisms in } \mathcal{P}\}$ and $\mathbb{M}(S) := \operatorname{mod}_S \mathcal{P}$ are mutually inverse bijections.*

Proof. For $\mathcal{E}$ with enough projectives $\mathcal{P}$, the functor $\mathcal{E} \to \operatorname{mod}_S \mathcal{P}$, $X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{P}}$ with $S = S_{adm}$ the $\mathcal{E}$-admissible morphisms in $\mathcal{P}$ is an equivalence of exact categories (cf. Chapter 4, Cor. **??**). This and (b) in Chapter 4, Lemma **??** imply the bijection. $\square$

**Definition 1.4.** We consider classes of morphisms in $\mathcal{P}$ with respect to inclusion, then for suitable morphsims $S, S'$ we have

$$S' \subseteq S \Leftrightarrow \operatorname{mod}_{S'} \mathcal{P} \subseteq \operatorname{mod}_S \mathcal{P}.$$

We consider exact categories with enough projectives given by $\mathcal{P}$ with the partial order $\mathcal{E}' \leq \mathcal{E}$ if and only if $\mathcal{E}'$ is equivalent to a fully exact subcategory of $\mathcal{E}$. When we consider the sets in the previous Lemma with these poset structures, it is straightforward to see that the bijection in the previous lemma becomes an isomorphism of posets.

**Remark 1.5.** The (unique) maximal suitable morphisms are all morphisms $S$ which admit weak kernels because then $\operatorname{mod}_S \mathcal{P} = \operatorname{mod}_\infty \mathcal{P}$.
The (unique) minimal suitable morphisms are the ones admissible in the split exact structure on $\mathcal{P}$. Then $\operatorname{mod}_S \mathcal{P} = \mathcal{P} \subseteq \operatorname{mod}_\infty \mathcal{P}$.

**1.2. Selforthogonal subcategories in triangulated categories with suitable morphisms.** Let $\Delta$ be an algebraic triangulated category. We look at the full subcategory of *non-negative* objects in $\Delta$

$$\Delta^{nn} := \{X \in \Delta \mid \operatorname{Hom}_\Delta(X, \Sigma^{<0} X) = 0\}.$$

This is closed under taking direct summands but not under direct sums (it can be just $\{0\}$). We have the following easy observation: Subcategories of $\Delta^{nn}$ which are closed under direct sums are precisely the same as additive subcategories of $\Delta$ which are non-negative. Recall from [2], a full additive subcategory $\mathcal{C}$ of $\Delta$ is **admissible exact** if it is non-negative ($\operatorname{Hom}(C, \Sigma^{<0}C') = 0$ for all $C, C'$ in $\mathcal{C}$) and extension-closed.
If $\mathcal{X}$ is a non-negative subcategory of $\Delta$, then its extension-closure is also non-negative and an admissible exact subcategory. This follows from the easy observation: If $X \to Y \to Z \to \Sigma X$ is a triangle in $\Delta$ with $X, Z, X \oplus Z \in \Delta^{nn}$ then $Y \in \Delta^{nn}$.

**Remark 1.6.** If $\mathcal{C}$ is an extension-closed subcategory in $\Delta$ and $\mathcal{C} \cap \Delta^{nn}$ is additively closed, then it is the unique largest admissible exact subcategory in $\mathcal{C}$.
(By the previous discussion, we have that $\mathcal{C} \cap \Delta^{nn}$ is extension-closed, the rest is obvious.)

Now, we assume that $\mathcal{T} \subseteq \Delta$ is an essentially small full additively closed subcategory which is **selforthogonal** (i.e $\operatorname{Hom}_\Delta(T, \Sigma^n T') = 0$ for all $n \neq 0$, $T, T'$ in $\mathcal{T}$) and **generating** (this means: $\operatorname{Hom}_\Delta(T, \Sigma^n X) = 0$ for all $n \in \mathbb{Z}$, $T$ in $\mathcal{T}$ implies $X = 0$ in $\Delta$).

**Remark 1.7.** There are many conditions called *generating* in a triangulated category, our definition is from [1, Def. 5.2.1], in the stacks project this is called *weakly generates*. The main example for us is the following: If $\Delta = \mathrm{D}^b(\mathcal{E})$ is the the bounded derived category of an exact category with enough projectives $\mathcal{P}$, then $\mathcal{P}$ generates $\Delta$. It will be a necessary condition for us to assume on $\mathcal{T}$.

Thirdly, we take a class of suitable morphisms $S \subseteq \operatorname{Mor}\mathcal{T}$. We start with

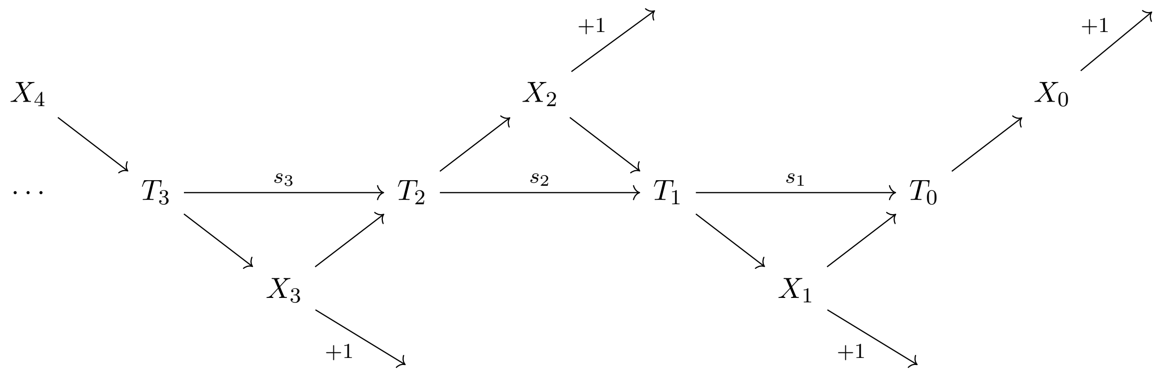
$$\mathcal{T}^\perp := \{X \in \Delta \mid \operatorname{Hom}_\Delta(T, \Sigma^n X) = 0 \text{ for all } n \neq 0, T \in \mathcal{T}\}$$

This is an extension-closed subcategory of $\Delta$ so we can see this as an extriangulated category. We consider the functor

$$\Phi\colon \mathcal{T}^\perp \to \operatorname{Mod}\mathcal{T}, \quad X \mapsto \operatorname{Hom}_\Delta(-,X)|_{\mathcal{T}} =: (-,X)|_{\mathcal{T}}$$

The functor $\Phi$ maps triangles to short exact sequences by definition of $\mathcal{T}^\perp$.

**Definition 1.8.** Given a suitable class of morphisms $S$ on an additive subcategory $\mathcal{T}$ in a triangulated category $\Delta$. We call $S$ **$\Delta$-suitable** if for every sequence $(s_n)_{n\in\mathbb{N}}$ in $S$ with $s_{n+1}$ is a weak kernel of $s_n$ for every $n$ there exists triangles




with $X_n \in \Delta^{nn} \cap \mathcal{T}^{\perp}$ for all $n \in \mathbb{N}$ and $s_n$ factors over $X_n$ for all $n \in \mathbb{N}$.

Given a suitable class of morphisms $S \subseteq \mathrm{Mor} - \mathcal{T}$. Can we extend $\mathcal{T} \subseteq \Delta$ to an admissible exact category $\mathrm{mod}_S\, \mathcal{T} \subseteq \Delta$?
In general not. It follows easily from the definition:

**Lemma 1.9.** *Let $\mathcal{T}$ be selforthogonal in a triangulated category $\Delta$ and $S \subseteq \mathrm{Mor} - \mathcal{T}$ a suitable class of morphisms. If $\mathcal{C} := \mathrm{mod}_S\, \mathcal{T}$ is an admissible exact subcategory of $\Delta$ extending the inclusion $\mathcal{T} \subseteq \Delta$, then*

(1) *$S$ is $\Delta$-suitable.*
(2) *$\mathrm{mod}_S\, \mathcal{T}$ is h-admissible exact (i.e. $\mathrm{Ext}^n_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\Delta}(X, \Sigma^n Y)$ are isomorphism for all $X, Y$ in $\mathcal{C}$).*

Proof. (1) As $\mathcal{C}$ is a resolving category, every $X$ has a projective resolution. Split it into short exact sequences, call the syzygies $X_n$, $n \in \mathbb{N}$. As $\mathcal{C}$ is admissible exact, these short exact sequence are part of distinguished triangles and all $X_n \in \mathcal{C}$ are non-negative. Now, every consecutive weak kernel sequence arises in this way, so $S$ is $\Delta$-suitable.
(2) As $\mathcal{C}$ is admissible exact, we have an isomorphism $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\Delta}(X, \Sigma Y)$ for all $X, Y$ in $\mathcal{C}$. Now, choose a projective resolution of $X$, call the syzygies $X_n$, $n \in \mathbb{N}$. Now, we can use dimension shift twice, once to see $\mathrm{Ext}^{n+1}_{\mathcal{C}}(X, Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(X_n, Y) \cong \mathrm{Hom}_{\Delta}(X_n, \Sigma Y)$ and a second time apply $\mathrm{Hom}(-, \Sigma Y)$ to the triangles $X_n \to T_{n-1} \to X_{n-1} \xrightarrow{+1}$ to conclude $\mathrm{Hom}(X_n, \Sigma Y) \cong \mathrm{Hom}(\Sigma^{-n} X, \Sigma Y) \cong \mathrm{Hom}(X, \Sigma^{n+1} Y)$.

□

Now, this is the main thing to prove:

**Lemma 1.10.** *(Extension-Lemma) If $S$ is a $\Delta$-suitable class of morphisms in a selforthogonal subcategory $\mathcal{T}$ which generates a triangulated category $\Delta$.*
*We look at the subcategory $\mathcal{C} := \{X \in \Delta \mid \exists (s_n)_n, X_n$ as above such that $X = X_0\}$. Then we claim: $\mathcal{C}$ is an admissible exact subcategory equivalent to $\mathrm{mod}_S\, \mathcal{T}$ and $\mathcal{C} \subseteq \Delta$ extends $\mathcal{T} \subseteq \Delta$.*

Before we give the proof in the next section, let us state the consequence.

**Definition 1.11.** If $(\mathcal{T}, S)$ with $\mathcal{T}$ selforthogonal in $\Delta$ and $S$ $\Delta$-suitable. Then we say that $S$ is **fully** $\Delta$-suitable if $\mathrm{Thick}_{\Delta}(\mathrm{mod}_S\, \mathcal{T}) = \Delta$.

Clearly triangle equivalences map fully $\Delta$-suitable morphisms to fully $\Delta$-suitable morphisms, so Theorem 1.1 follows trivially.

**Example 1.12.** Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ - we see $\mathcal{P}$ as stalk complexes in $\Delta = D^b(\mathcal{E})$, then this is a selforthogonal subcategory. We take $S$ to be the $\mathcal{E}$-admissible morphisms (they are suitable) in $\mathcal{P}$ and also $\Delta$-suitable.

**Example 1.13.** If $\mathcal{E}$ is an exact category and $\mathcal{T} \subseteq \mathcal{E}$ is a tilting subcategory of $\mathcal{E}$ (cf. [3]), this means

$$\mathcal{T}^{\perp_\mathcal{E}} := \mathcal{T}^\perp \cap \mathcal{E} = \{X \in \mathcal{E} \mid \mathrm{Ext}^{>0}_\mathcal{E}(T, X) = 0 \quad \forall T \in \mathcal{T}\}$$

has enough projective given by $\mathcal{T}$ and every object in $\mathcal{E}$ has a finite coresolution by objects in $\mathcal{T}^{\perp_\mathcal{E}}$. Let $S$ be the class of $\mathcal{T}^{\perp_\mathcal{E}}$-admissible morphisms then $S$ is fully $\Delta$-suitable in $\mathcal{T} \subseteq D^b(\mathcal{E})$ and $\mathcal{T}^{\perp_\mathcal{E}} = \mathrm{mod}_S\, \mathcal{T}$

**Example 1.14.** Let $\mathcal{T}$ be an additive category. We consider it as self-orthogonal subcategory inside $K^b(\mathcal{T})$. Then, all $\Delta$-suitable morphisms in $\mathcal{T}$ are fully $\Delta$-suitable and they are precisely the suitable morphisms $S$ such that $\mathrm{mod}_S\, \mathcal{T} = \mathcal{P}^{<\infty}(\mathrm{mod}_S\, \mathcal{T})$. There is a maximal $\Delta$-suitable class of morphisms given by $\mathrm{mod}\ _S\mathcal{T} = \mathcal{P}^{<\infty}(\mathrm{mod}_\infty\, \mathcal{T}) = \mathrm{Res}(\mathcal{T}) \subseteq \mathrm{mod}_\infty\, \mathcal{T}$.
More generally, if $\Delta$ is triangulated, $\mathcal{T} \subseteq \Delta$ self-orthogonal and $\mathrm{Thick}_\Delta(\mathcal{T}) = \Delta$ (i.e. $\mathcal{T}$ a tilting subcategory in a triangulated category in the sense of Keller), the same statement holds true.

**Example 1.15.** Let $\mathcal{T}$ be an additive category. We consider it as self-orthogonal subcategory inside $K^+(\mathcal{T})$. Then all suitable morphisms in $\mathcal{T}$ are $\Delta$-suitable but none are fully $\Delta$-suitable:
Given fully $\Delta$-admissible morphisms $S$ in $\mathcal{T} \subseteq \Delta$, since $\Delta \cong D^b(\mathrm{mod}_S\, \mathcal{T}) \cong K^{+,b}(\mathcal{T}) \subseteq K^+(\mathcal{T})$, we necessarily have a full triangulated subcategory of $K^+(\mathcal{T})$ with $K^{+,b}(\mathcal{T}) \neq K^+(\mathcal{T})$.

**1.3. Proof of the Extension-Lemma.** Recall, $\mathcal{T}$ selforthogonal and generating in $\Delta$, $S \subseteq \mathrm{Mor}\mathcal{T}$ is $\Delta$-suitable and $\mathcal{C} := \{X \in \Delta \mid \exists (s_n)_n, X_n$ as above such that $X = X_0\}$.
Claim: $\mathcal{C}$ is an admissible exact subcategory and with the admissible exact structure equivalent as exact category to $\mathrm{mod}_S\, \mathcal{T}$.
So, divide and conquer, we show one property after the other, in this order

(i) $\mathcal{C}$ is closed under direct sums
(ii) $\mathcal{C}$ is non-negative
(iii) additive equivalence to $\mathrm{mod}_S\, \mathcal{T}$
(iv) $\mathcal{C}$ is extension-closed (and so admissible exact in $\Delta$)
(v) exact equivalence to $\mathrm{mod}_S\, \mathcal{T}$

We look at the composition $\varphi\colon \mathcal{C} \subseteq \mathcal{T}^\perp \xrightarrow{\Phi} \mathrm{Mod}\, \mathcal{T}$ defined by $X \mapsto \mathrm{Hom}_\Delta(-, X)|_\mathcal{T}$. As $\mathcal{T}$ is generating, the functor $\varphi$ reflects isomorphism, this can be used to see that

(i) $\mathcal{C}$ is closed under direct sums:
Assume $X, Y$ in $\mathcal{C}$, pick the first morphisms $s^X, s^Y \in S$ in the definition of $\mathcal{C}$. As $\mathrm{mod}_S\, \mathcal{T}$ is resolving and $S$ homotopy closed, we have that $s := s_X \oplus s_Y \in S$. We extend $s$ to a sequence of consecutive weak kernels in $S$ and as $S$ is $\Delta$-suitable, we can factor $s$ as $T_1 \to Z_1 \to T_0$ such that we have a distinguished triangle Let $Z_1 \to T_0 \to Z \xrightarrow{+1}$ with $Z \in \mathcal{C}$. Now, by definition $\mathrm{Hom}_\Delta(-, Z)|_\mathcal{T} \cong \mathrm{coker}\, \mathrm{Hom}_\mathcal{T}(-, s) \cong \mathrm{Hom}_\Delta(-, X \oplus Y)|_\mathcal{T}$ and as $\varphi$ reflects isomorphism we conclude $X \oplus Y \cong Z$ is non-negative.
(ii) Also, it implies that $\mathcal{C}$ is a non-negative subcategory (as $C \oplus C'$ non-negative implies $\mathrm{Hom}(C, \Sigma^{<0} C') = 0$).
(iii) Now as $\mathcal{C}$ is a non-negative subcategory we can easily deduce that $\varphi$ is a fully faithful functor: For $X, Y$ in $\mathcal{C}$ we choose again $s^X : T_1^X \to T_0^X$ and $s^Y : T_1^Y \to T_0^Y$ from the definition of $\mathcal{C}$.
First observe that $\varphi$ gives an isomorphism whenever both objects are in $\mathcal{T}$ (by Yoneda) and also if the first object is in $\mathcal{T}$, because $\mathrm{Hom}_\Delta(T, Y) = (\mathrm{coker}\, \mathrm{Hom}(-, s^Y))(T) = \mathrm{Hom}_{\mathrm{Mod}\, \mathcal{T}}(\varphi(T), \mathrm{coker}\, \mathrm{Hom}(-, s^Y)) = \mathrm{Hom}(\varphi(T), \Phi(Y))$ for every $T \in \mathcal{T}$. Now apply $\mathrm{Hom}_\Delta(-, Y)$ to the triangles for $X$, we find a left exact sequence (as $\mathcal{C}$ is non-neg.)

$$0 \to \mathrm{Hom}_\Delta(X, Y) \to \mathrm{Hom}_\Delta(T_0^X, Y) \to \mathrm{Hom}(T_1^X, Y)$$

Now, to see that $\varphi$ induces an isomorphism on $\mathrm{Hom}(X,Y)$ it suffices to see that

$$0 \to \mathrm{Hom}_\Delta(\varphi(X),\varphi(Y)) \to \mathrm{Hom}_\Delta(\varphi(T_0^X),\varphi(Y)) \to \mathrm{Hom}(\varphi(T_1^X),\varphi(Y))$$

is also exact, but $\varphi$ maps triangles to exact sequences, so this claim follows.

We observe that for $X \in \mathcal{C}$ (defined by $(s_n)$ in $S$)

$$\varphi(X) = \mathrm{Hom}_\Delta(-,X)|_{\mathcal{T}} \cong \operatorname{coker} \mathrm{Hom}_{\mathcal{T}}(-,s_1)$$

so, $\varphi$ induces an equivalence of additive categories $\varphi\colon \mathcal{C} \to \mathrm{mod}_S\,\mathcal{T}$ which maps triangles to exact sequences.

(iv) Next, we claim $\mathcal{C}$ is extension-closed: For this we first observe that for every short exact sequence $\sigma\colon \varphi(X) \rightarrowtail \varphi(Y) \twoheadrightarrow \varphi(Z)$, $X,Y,Z$ in $\mathcal{C}$ in $\mathrm{mod}_S\,\mathcal{T}$ there exists a triangle $\delta\colon X \to Y \to Z \xrightarrow{+1}$ with $\varphi(\delta) \cong \sigma$. Just take $C := \mathrm{cone}(X \to Y)$ and look at the standard triangle $X \to Y \to C \xrightarrow{+1}$ applying $\mathrm{Hom}(T,-)$ with $T \in \mathcal{T}$ implies that $\mathrm{Hom}_\Delta(-,C)|_{\mathcal{T}} \cong \mathrm{Hom}_\Delta(-,Z)|_{\mathcal{T}}$ but as $\mathcal{T}$ is generating this implies $C \cong Z$.

Now, this easily implies $\mathcal{C}$ is extension-closed, take a triangle $X \to Y \to Z \xrightarrow{+1}$ with $X,Z$ in $\mathcal{C}$. As $X, Z, X \oplus Z$ are non-negative, this implies $Y$ is non-negative. We apply $\Phi$ implies that $\Phi(Y) = \varphi(Y) \in \mathrm{mod}_S\,\mathcal{T}$. Now, we take the short exact sequences from a projective resolution of $\varphi(Y)$, by the first consideration there exist the triangles as required for $Y \in \mathcal{C}$.

(v) To the that $\varphi$ is an equivalence of exact categories, it is enough to show that it induces a surjection on $\mathrm{Ext}^1$'s. But that had just been discussed in (iv).

# Bibliography


[1] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006. MR2182076

[2] J. C. Letz and J. Sauter, *Realization functors in algebraic triangulated categories*, 2024.

[3] J. Sauter, *Tilting theory in exact categories*, 2022.