# Non-commutative resolutions of singularities using exact substructures

## 1. Synopsis

We introduce (bounded) singularity categories for arbitrary exact categories. An exact category is regular if its singularity category is zero. We recall the known Buchweitz theorem for a Gorenstein exact categories with enough projectives. Then we explore a new concept of a *noncommutative resolution of singularities* (NCR) of a given exact category as an exact substructure which is regular. There exist various alternative versions of non-commutative resolutions in the literature. Our aims here are:

(1) Partially unify and simplify the theory (singularity categories, non-commutative resolutions of singularities and relative singularity categories) for module categories of rings and for coherent sheaves on a quasi-projective variety.
(2) Characterize NCRs corresponding to *cluster tilting* subcategories (as a candidate for a 'minimal' NCR).

**What is new?** The concept to see NCRs as exact substructures and the generality of our approach.

## 2. Definitions and notations

We recall some of the previous definitions. Let $\mathcal{E}$ be an exact category in the sense of Quillen. Recall, for an object $X$ in $\mathcal{E}$, the projective dimension $\operatorname{pd} X$ is defined as the infimum of all $n \in \mathbb{N}_0$ such that $\operatorname{Ext}_{\mathcal{E}}^{\geq n}(X, -) = 0$. Dually $\operatorname{id} X$ is defined as the infimum of all $n \in \mathbb{N}_0$ such that $\operatorname{Ext}_{\mathcal{E}}^{\geq n}(-, X) = 0$. We consider

$$\mathcal{P}^{\leq n} := \{X \in \mathcal{E} \mid \operatorname{pd}_{\mathcal{E}} X \leq n\},$$
$$\mathcal{I}^{\leq n} := \{X \in \mathcal{E} \mid \operatorname{id}_{\mathcal{E}} X \leq n\},$$

and $\mathcal{P}^{<\infty} = \bigcup_{n\geq 0} \mathcal{P}^{\leq n}$, $\mathcal{I}^{<\infty} = \bigcup_{n\geq 0} \mathcal{I}^{\leq n}$. We also use the notation $\mathcal{P}^{<\infty}(\mathcal{E}), \mathcal{P}^{\leq n}(\mathcal{E})$ etc. for these categories.
Using long exact sequences on the Ext-groups it is easy to see that all these full subcategories are extension closed and that $\mathcal{P}^{\leq n}$ is deflation-closed and $\mathcal{I}^{\leq n}$ is inflation-closed, $\mathcal{P}^{<\infty}, \mathcal{I}^{<\infty}$ are thick subcategories. Throughout: All underlying additive categories of exact categories are assumed to be idempotent complete.
The suspension functor in $D^b(\mathcal{E})$ will be denoted by [1] from now on.

## 3. Singularity categories for exact categories

**Summary:** We will define singularity categories for exact categories in a naive manner which leaves the question if it is a derived invariant. There are different ways to address this - one way is to try to generalize Rickard's results for rings, more precisely: Given a two derived equivalent exact categories $\mathcal{E}$ and $\mathcal{E}'$. Is there a triangle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}')$ which restricts to $\operatorname{Thick}_\Delta(\mathcal{P}^{\infty}(\mathcal{E})) \to \operatorname{Thick}_\Delta(\mathcal{P}^{<\infty}(\mathcal{E}'))$? We follow an alternative idea of Orlov and define singularity category for triangulated categories. The question here is to characterize exact categories for which these two coincide.

**Definition 3.1.** We say the $\mathcal{E}$ is **regular** if $\mathcal{E} = \mathcal{P}^{<\infty}(\mathcal{E})$.
We say $X$ is **homologically finite** if for every $Y \in \mathcal{E}$ there exists an $n$ such that $\mathrm{Ext}_{\mathcal{E}}^{\geq n}(X, Y) = 0$. We write $\mathcal{E}^{hf}$ for the full subcategory of homologically finite objects. We say that $\mathcal{E}$ is **Δ-regular** if $\mathcal{E} = \mathcal{E}^{hf}$.

Observe that we always have thick subcategories $\mathcal{P}^{<\infty}(\mathcal{E}) \subseteq \mathcal{E}^{hf} \subseteq \mathcal{E}$ (in the exact category sense, i.e. they fulfill the 2 out of 3 property for short exact sequences and are closed under summands).

**Definition 3.2.** If $\mathcal{T}$ is a triangulated category and $X \in \mathcal{T}$, we say that $X$ is **homological finite** if for every $Y \in \mathcal{T}$ there is a finite subset $I \subseteq \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for all $i \notin I$. We denote by $\mathcal{T}^{hf}$ the full subcategory of homological finite objects.
More generally, given a full additive category $\mathcal{C}$ of $\mathcal{T}$ we say that $X \in \mathcal{C}$ is $\mathcal{C}$-**homological finite** if for every $Y \in \mathcal{C}$ there is a finite subset $I \subseteq \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for all $i \notin I$. We denote by $\mathcal{C}^{hf}$ the full subcategory of homological finite objects.

Observe that $\mathcal{T}^{hf}$ is thick in $\mathcal{T}$ (in the triangulated sense, i.e. it is a triangulated subcategory closed under summands).

We make the following easy observation:

**Lemma 3.3.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ be a full additive category whose extension-closure is $\mathcal{T}$. Then $\mathcal{C} = \mathcal{C}^{hf}$ if and only if $\mathcal{T} = \mathcal{T}^{hf}$.*

PROOF. We assume $\mathcal{C} = \mathcal{C}^{hf}$. Let $X \in \mathcal{C}$. We show that $X \in \mathcal{T}^{hf}$: Let $Y \in \mathcal{T}$. By assumption there is a triangle $Y_1 \to Y \to Y_2 \to Y_1[1]$ with $Y_i \in \mathcal{C}$. Then there exist finite subsets $I_1, I_2$ of $\mathbb{Z}$ with $\mathrm{Hom}(X, Y_i[k]) = 0$ for $k \notin I_i$, $i \in \{1, 2\}$. Then, just take $I = I_1 \cup I_2$ and for $k \notin I$ we conclude that $\mathrm{Hom}(X, Y[k]) = 0$. Therefore $X \in \mathcal{T}^{hf}$. It follows that $\mathcal{T} = \mathrm{Thick}_{\Delta}(\mathcal{C}) \subseteq \mathcal{T}^{hf}$. The other implication is trivially true. □

**Corollary 3.4.** *Let $\mathcal{E}$ be an exact category. Then $\mathcal{E} = \mathcal{E}^{hf}$ if and only if $\mathrm{D}^b(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})^{hf}$.*

**Definition 3.5.** Let $\mathcal{E}$ be an exact category. We define the **singularity category** as the Verdier quotient

$$\mathrm{D}_{sg}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E}) / \mathrm{Thick}_{\Delta}(\mathcal{P}^{<\infty}(\mathcal{E}))$$

For a triangulated category $\mathcal{T}$ we define the **Δ-singularity category** as the Verdier quotient

$$\mathcal{T}_{sg} = \mathcal{T} / \mathcal{T}^{hf}$$

Then every triangle equivalence $\mathcal{T} \to \mathcal{S}$ between triangulated categories induces a triangle equivalence $\mathcal{T}_{sg} \to \mathcal{S}_{sg}$. Clearly, since for an exact category $\mathcal{E}$ we have $\mathrm{Thick}_{\Delta}(\mathcal{P}^{<\infty}(\mathcal{E})) \subseteq \mathrm{Thick}_{\Delta}(\mathcal{E}^{hf}) \subseteq \mathrm{D}^b(\mathcal{E})^{hf}$ (for the second inclusion see next lemma) we get an induced Verdier quotient

$$\mathrm{D}_{sg}(\mathcal{E}) \to (\mathrm{D}^b(\mathcal{E}))_{sg}$$

**Definition 3.6.** We say $\mathcal{E}$ has **Δ-singularities** if this map is an equivalence.

**Open question 3.7.** When are the two singularity categories locally small (i.e. have Hom-sets)? If $\mathcal{E}$ is essentially small, then $D^b(\mathcal{E})$ is also essentially small and hence it holds. And more generally? When are they idempotent complete?

So if $\mathcal{E}$ and $\mathcal{E}'$ have Δ-singularities and are derived equivalent, then their singularity categories are equivalent.

**Example 3.8.** ([40, Example 3.3]) This is an example of two derived equivalent exact categories one is regular and the other one not. Furthermore, one has $\Delta$-singularities and the other one not. Let $R = k[x_0, \dots, x_n]/\langle x_i^2, x_i x_j + x_j x_i\rangle$ be the exterior algebra and $S = k[x_0, \dots, x_n]$ a polynomial ring for $k$ a field, both are graded algebras with $\mathrm{deg x_i} = 1$, $0 \leq i \leq n$.
We consider the categories of graded modules $\mathcal{E} = \mathrm{grR}$ and $\mathcal{E}' = \mathrm{grS}$ with finite-dimensional graded parts. Then BGG-correspondence provides a triangle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}')$.
But $\mathrm{gldim}\,\mathcal{E}' < \infty$ and $\mathrm{gldim}\,\mathcal{E} = \infty$ as $R$ is self-injective. This implies that $\mathrm{D}_{sg}(\mathcal{E}') = 0 = (\mathrm{D}^b(\mathcal{E}'))_{sg}$ but $\mathrm{D}_{sg}(\mathcal{E}) \neq 0 = (\mathrm{D}^b(\mathcal{E}))_{sg}$.

Observe that $\mathcal{E}$ has $\Delta$-singularities if and only if $\mathcal{P}^{<\infty}(\mathcal{E}) = \mathcal{E}^{hf}$ and $\mathrm{Thick}_\Delta(\mathcal{E}^{hf}) = \mathrm{D}^b(\mathcal{E})^{hf}$. We ask if the last equality is always true?
Here is the answer in a special case:

**Lemma 3.9.** *Let $\mathcal{E}$ be an exact category. Then we have:*

(a) $\mathcal{E}^{hf} = \mathrm{D}^b(\mathcal{E})^{hf} \cap \mathcal{E}$ *where we consider $\mathcal{E} \subset \mathrm{D}^b(\mathcal{E})$ as stalk complexes in degree zero.*
(b) *If $\mathcal{E}$ is an exact category with enough projectives then* $\mathrm{Thick}_\Delta(\mathcal{E}^{hf}) = \mathrm{D}^b(\mathcal{E})^{hf}$.

PROOF. (a) It is enough to show: $\mathcal{E}^{hf} \subseteq \mathrm{D}^b(\mathcal{E})^{hf}$. Let $X \in \mathcal{E}^{hf}$ and $Y \in \mathrm{D}^b(\mathcal{E})$. Assume there exists an infinite set $I \subseteq \mathbb{Z}$ such that $\mathrm{Hom}(X, Y[i]) \neq 0$ for $i \in I$. Since $\mathrm{Thick}_\Delta(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$ we may assume $Y \in \mathrm{Thick}^n_\Delta(\bigvee_m \mathcal{E}[m])$ and that there exists $Y_{n-1}, Y'_{n-1} \in \mathrm{Thick}^{n-1}_\Delta(\bigvee_m \mathcal{E}[m])$ and a triangle $Y_{n-1} \to Y \to Y'_{n-1} \xrightarrow{+1}$. Since $\mathrm{Hom}(X, -)$ is a cohomological functor, either for $Y_{n-1}$ or for $Y'_{n-1}$ there exists an infinite subset of $I_{n-1} \subseteq \mathbb{Z}$ with $\mathrm{Hom}(X, ..[i]) \neq 0$ for all $i \in I_{n-1}$. Then inductively, we can produce a $Y_0 \in \mathcal{E}$ such that there exists an infinite set $I_0 \subseteq \mathbb{Z}$ with $\mathrm{Hom}(X, Y_0[i]) \neq 0$ for all $i \in I_0$. Since $\mathrm{Hom}(X, Y[< 0]) = 0$ it follows that $I_0 \subseteq \mathbb{N}$ and therefore a contradiction to $X \in \mathcal{E}^{hf}$.
(b) We need to see $\mathrm{D}^b(\mathcal{E})^{hf} \subseteq \mathrm{Thick}_\Delta(\mathcal{E}^{hf})$. We identify $\mathrm{D}^b(\mathcal{E})$ with $\mathrm{K}^{b,-}(\mathcal{P})$ where $\mathcal{P}$ are the projectives in $\mathcal{E}$. Take $n \in \mathbb{Z}$, then one shows that $X \in \mathrm{K}^{b,-}(\mathcal{P})^{hf}$ is equivalent to $\sigma_{\leq n} X$ and $\sigma_{>n} X$ are homologically finite. Furthermore, we observe $X \in \mathrm{Thick}_\Delta(\sigma_{\leq n} X, \sigma_{>n} X)$. Since for $|n| >> 0$ we have that $\sigma_{\leq n} X$ is quasi-isomorphic to a shifted stalk complex - this has to lie in $\mathrm{Thick}_\Delta(\mathcal{E}^{hf})$. By definition $\sigma_{>n} X \in \mathrm{K}^b(\mathcal{P}) \subseteq \mathrm{Thick}_\Delta(\mathcal{E}^{hf})$ and therefore $X \in \mathrm{Thick}_\Delta(\mathcal{E}^{hf})$. □

Then let us state the obvious:

**Lemma 3.10.** *The following are equivalent:*

(1) *$\mathcal{E}$ is regular.*
(2) $\mathrm{D}_{sg}(\mathcal{E}) = 0$.

*Furthermore, the following are equivalent:*

(a) *$\mathcal{E}$ is $\Delta$-regular.*
(b) $(\mathrm{D}^b(\mathcal{E}))_{sg} = 0$.

PROOF. The implication '(1) implies (2)' is obvious since $\mathrm{Thick}_\Delta(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$.
Assume (2), i.e. we assume $\mathcal{E} \subseteq \mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}(\mathcal{E}))$. We look at

$$\mathrm{D}^{<\infty, \mathcal{E}}(\mathcal{E}) := \{X \in \mathrm{D}^b(\mathcal{E}) \mid \exists m \in \mathbb{Z} \text{ such that } \mathrm{Hom}(X, \mathcal{E}[n]) = 0 \; \forall n \geq m\}$$

this is a thick subcategory of $\mathrm{D}^b(\mathcal{E})$. It contains $\mathcal{P}^{<\infty}$ and so by assumption we have $\mathcal{E} \subseteq \mathrm{Thick}(\mathcal{P}^{<\infty}(\mathcal{E})) \subseteq \mathrm{D}^{<\infty, \mathcal{E}}(\mathcal{E})$. This implies $\mathcal{E} \subseteq \mathcal{P}^{<\infty}$.
The implication '(a) implies (b)' is obvious as in the previous proof. Now assume that (b), i.e. $(\mathrm{D}^b(\mathcal{E}))^{hf} = \mathrm{D}^b(\mathcal{E})$. We intersect with the stalks to get $\mathcal{E} = \mathrm{D}^b(\mathcal{E})^{hf} \cap \mathcal{E} \subseteq \mathcal{E}^{hf}$. □

**Corollary 3.11.** *Let $f\colon \mathcal{E} \to \mathcal{A}$ be an exact functor between exact categories. If the derived functor $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ is a triangle equivalence, then: $\mathcal{E}$ regular if and only if $\mathcal{A}$ is regular.*

Proof. Assume $\mathcal{E}$ is regular. $f\colon \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ is a triangle equivalence which restricts to $\mathcal{E} \to \mathcal{A}$ on stalk complexes. Therefore (using the definition of the previous proof) it restricts to a triangle functor $\mathrm{D}^{<\infty,\mathcal{E}} \to \mathrm{D}^{<\infty,\mathcal{A}}$. Since $\mathcal{E}$ is regular, it follows as in the previous proof that $\mathrm{D}^b(\mathcal{E}) = \mathrm{D}^{<\infty,\mathcal{E}}$. This implies that the essential image $\mathrm{D}^b(\mathcal{A}) = f(\mathrm{D}^b(\mathcal{E})) \subseteq \mathrm{D}^{<\infty,\mathcal{A}}$. In particular, it follows $\mathcal{A} \subseteq \mathrm{D}^{<\infty,\mathcal{A}}$ and this imples $\mathcal{A} = \mathcal{P}^{<\infty}(\mathcal{A})$.
If $\mathcal{A}$ is regular, then since $f$ is homologically exact (cf. Chapter 1), it follows that $\mathcal{E}$ is also regular. $\square$

**Remark 3.12.** If $\mathcal{E}$ is an exact category with enough projectives $\mathcal{P}$ then we have

$$\mathrm{D}_{sg}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})/\mathrm{K}^b(\mathcal{P})$$

**Lemma 3.13.** *If $\mathcal{E} = \mathrm{Filt}(\mathrm{M_1}, \dots, \mathrm{M_n})$ then $\operatorname{pd} X = \max(m_i \mid 1 \leq i \leq n)$ with $m_i := \inf\{m \in \mathbb{N}_{\geq 0} \mid \mathrm{Ext}^{>m}(X, M_i) = 0\} \in \mathbb{N}_{\geq 0} \cup \{\infty\}$.*
*This implies $\mathcal{P}^{<\infty}(\mathcal{E}) = \mathcal{E}^{hf}$. If $\mathcal{E}$ has enough projectives then we get that $\mathcal{E}$ has $\Delta$-singularities.*

**Lemma 3.14.** *Let $\mathcal{E}$ be an exact category with infinite coproducts. Then $\mathcal{P}^{<\infty}(\mathcal{E}) = \mathcal{E}^{hf}$. So if $\mathcal{E}$ has also enough projectives then it has $\Delta$-singularities.*

Proof. Let $\operatorname{pd} X = \infty$. There exists an infinite subset $I \subset \mathbb{N}$ and objects $Y_n, n \in I$ such that $\mathrm{Ext}^n_{\mathcal{E}}(X, Y_n) \neq 0$. So let $Y = \bigoplus_{n \in I} Y_n \in \mathcal{E}$. Then for every $n \in I$ we have $\mathrm{Ext}^n_{\mathcal{E}}(X, Y) = \mathrm{Ext}^n_{\mathcal{E}}(X, Y_n) \oplus \mathrm{Ext}^n_{\mathcal{E}}(X, \bigoplus_{m \in I, m \neq n} Y_m) \neq 0$. Therefore $X$ is not homological finite. $\square$

**Remark 3.15.** Here is an example which is $\Delta$-regular but not regular: Let $\mathcal{E}$ be the abelian category of all representations (over some field) of the following infinite quiver with relations that any two consecutive arrows are zero

$$\begin{tikzcd}
1 \arrow[rr, dotted, no head] \arrow[dr] & & 3 \arrow[rr, dotted, no head] \arrow[dr] & & 5 \arrow[rr, dotted, no head] \arrow[dr] & & 7 & & \cdots \\
& 2 \arrow[ur] \arrow[rr, dotted, no head] & & 4 \arrow[ur] \arrow[rr, dotted, no head] & & 6 \arrow[ur] & & \cdots &
\end{tikzcd}$$

Indecomposables are either projectives or simples, all simples have infinite projective dimension. Nevertheless all indecomposables are homologically finite in $\mathcal{E}$.

**Example 3.16.** ([8, Theorem 4.4.1]) If $R$ is any left and right noetherian ring, then Buchweitz introduced the singularity category for $\mathcal{E} = R\,\mathrm{Mod}$ and showed the theorem for $R\,\mathrm{mod}$ (f.g. $R$-modules).
Of course this can be defined for every ring. If $R$ is left coherent and semiperfect, then $R\,\mathrm{mod}$ has $\Delta$-singularities, cf. [31, Prop. 9.2.14].
If $R$ is any ring then $R\,\mathrm{Mod}$ has $\Delta$-singularities by loc. cit. Lemma 9.2.3.

**Example 3.17.** (Orlov, 2004 in [36]) Now we consider the following geometric situation: Let $X$ be a scheme over a field $K$ which is separated, noetherian, of finite Krull dimension and coh(X) has enough locally frees - following Orlov [36] we will call these properties (ELF). (The last assumption is also called the resolution property cf. [39, Tag 0F85]) Orlov introduced in [36] the singularity category of $X$ as the Verdier quotient

$$\mathrm{D}_{sg}(\mathrm{coh(X)}) = \mathrm{D^b}(\mathrm{coh(X)})/\mathrm{D^{perf}}(\mathrm{X})$$

If $X$ is ELF, then coh(X) has $\Delta$-singularities (Orlov [37], Prop.1.11).

Here are some of my open questions to this subsection:

(3.1) If $\mathcal{E}$ is an exact category, is $\mathcal{P}^{<\infty}(\mathcal{E})$ always regular (as fully exact category of $\mathcal{E}$)? Or stronger: Is $\mathcal{P}^{<\infty}(\mathcal{E}) \subseteq \mathcal{E}$ always homologically exact?
(3.2) If we consider $\mathcal{E}^{hf}$ as homologically finite objects in $\mathcal{E}$. Are all objects in $\mathcal{E}^{hf}$ homologically finite in $\mathcal{E}^{hf}$? Or stronger: Is $\mathcal{E}^{hf} \subseteq \mathcal{E}$ homologically exact?
(3.3) Is $\mathrm{Thick}_\Delta(\mathcal{E}^{hf}) = \mathrm{D}^b(\mathcal{E})^{hf}$? This would imply: $\mathcal{P}^{<\infty} = \mathcal{E}^{hf}$ is equivalent to $\mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) = \mathrm{D}^b(\mathcal{E})^{hf}$?
(3.5) For which exact categories $\mathcal{E}$ do we have $\mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) = \mathrm{D}^b(\mathcal{E})^{hf}$?

## 4. Descriptions as stable categories - Buchweitz theorem

We start with the following definition:

**Definition 4.1.** Let $n \geq 0$. An exact category $\mathcal{E}$ is called **Gorenstein** if $\mathcal{I}^{<\infty} = \mathcal{P}^{<\infty}$. We say it is **$n$-Gorenstein** if we have $\mathcal{I}^{\leq n} = \mathcal{I}^{<\infty} = \mathcal{P}^{<\infty} = \mathcal{P}^{\leq n}$.

This is by definition a *symmetric* condition (it holds for $\mathcal{E}$ if and only if it holds for $\mathcal{E}^{op}$).

**Remark 4.2.** Of course, one can define dually, the *injective* singularity category

$$D_{sg-inj}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E}) / \operatorname{Thick}_\Delta(\mathcal{I}^{<\infty})$$

Then $\mathcal{E}$ is Gorenstein if and only if we have $\mathrm{D}^b_{sg}(\mathcal{E}) = \mathrm{D}^b_{sg-inj}(\mathcal{E})$ (= here means they are Verdier quotients of $\mathrm{D}^b(\mathcal{E})$ with the same kernels). In general, we do not know when these two singularity categories are triangle equivalent.

**Remark 4.3.** One could define *Gorenstein* for triangulated categories as $\mathcal{T}^{hf} = \mathcal{T}^{chf}$ (where cohomologically finite elements are defined dually to homologically finite), so imposing the symmetry condition of the previous remark for Orlov's singularity categories.

We recall from [**38**] the following definition: A full subcategory $\mathcal{P} \subseteq \mathcal{E}$ is called cotilting (resp. $n$-cotilting) if and only if the following hold

(C1) ${}^{\perp}\mathcal{P}$ has enough injectives given by $\mathcal{P}$ itself and
(C2) $\operatorname{Res}({}^{\perp}\mathcal{P}) = \mathcal{E}$.

(resp. (C1) and $(\mathrm{C2})_n$ $\operatorname{Res}_n({}^{\perp}\mathcal{P}) = \mathcal{E}$).

**Lemma 4.4.** *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. If $\mathcal{P}$ is cotilting then ${}^{\perp}\mathcal{P}$ is Frobenius exact with enough injectives given by $\mathcal{P}$ and we have $\mathcal{P}^{<\infty} \cap {}^{\perp}\mathcal{P} = \mathcal{P}$.*

Proof. By definition this category has enough injectives given by $\mathcal{P}$, an easy check shows that ${}^{\perp}\mathcal{P}$ is resolving in $\mathcal{E}$, so it also has enough projectives given by $\mathcal{P}$.
If $X \in \mathcal{P}^{<\infty} \cap {}^{\perp}\mathcal{P}$ then there exists an $n \in \mathbb{N}$ such that $\operatorname{Ext}^{>n}_{\mathcal{E}}(X, -) = 0$. This implies, as ${}^{\perp}\mathcal{P}$ is homologically exact in $\mathcal{E}$, that $\operatorname{Ext}^{>n}_{{}^{\perp}\mathcal{P}}(X, -) = 0$. This implies that $\operatorname{Ext}^1_{{}^{\perp}\mathcal{P}}(X, \Omega^{-n}Y) = 0$ for all $Y \in {}^{\perp}\mathcal{P}$. But every object in ${}^{\perp}\mathcal{P}$ is an $n$-th cosyzygy, so $X$ in $\mathcal{P}({}^{\perp}\mathcal{P}) = \mathcal{P}$. □

**Proposition 4.5.** *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then we have*

(1) *If $\mathcal{P}$ is $n$-cotilting then $\mathcal{E}$ is $n$-Gorenstein*
(2) *If $\mathcal{E}$ is $n$-Gorenstein and ${}^{\perp}\mathcal{P} \subseteq \operatorname{cogen}_{\mathcal{E}}(\mathcal{P})$ then $\mathcal{P}$ is $n$-cotilting*

It would be much nicer if we had an equivalence in (1) but we could not see how to prove that $n$-Gorenstein implies ${}^{\perp}\mathcal{P} \subseteq \operatorname{cogen}(\mathcal{P})$. But in special situations this is fulfilled.

Proof. (1) If $\mathcal{P}$ is $n$-cotilting then $\operatorname{Thick}(\mathcal{P}) = \mathcal{I}^{<\infty}$ follows from [**38**, Lem. 5.8]. But $\operatorname{Thick}(\mathcal{P}) = \mathcal{P}^{<\infty}$ then implies that $\mathcal{E}$ is Gorenstein. Now, we show $\mathcal{P}^{<\infty} \subseteq \mathcal{P}^{\leq n}$. Take $X \in \mathcal{P}^{\infty}$ a projective resolution

$$\Omega^n X \rightarrowtail P_{n-1} \to \cdots \to P_0 \twoheadrightarrow X$$

Then by dimension shift $\operatorname{Ext}^{>0}_{\mathcal{E}}(\Omega^n X, P) = \operatorname{Ext}^{>n}(X, P) = 0$ for all $P \in \mathcal{P}$, therefore $\Omega^n X \in {}^{\perp}\mathcal{P}$ and so $\Omega^n X \in \mathcal{P}^{<\infty} \cap {}^{\perp}\mathcal{P} = \mathcal{P}$ by Lemma 4.4.
Now, we want to see that $\mathcal{I}^{\infty} \subseteq \mathcal{I}^{\leq n}$: We have $\mathcal{I}^{<\infty} = \mathcal{P}^{\leq n}$. Assume $Y \in {}^{\perp}\mathcal{P}$, then we easily verify $\operatorname{Ext}^{>n}_{\mathcal{E}}(Y, X) = 0$ (using the projective resolution of $X$). If we look at an arbitrary $Y$ in $\mathcal{E}$, then clearly $\operatorname{Ext}^{>2n}_{\mathcal{E}}(Y, X) = 0$ (using $\operatorname{pd}_{\mathcal{E}} X \leq n, \operatorname{id}_{\mathcal{E}} \mathcal{P} \leq n$). Assume $\operatorname{Ext}^{m+n}_{\mathcal{E}}(Y, X) \neq 0$ for some

$m \in \{1, \dots n\}$. Then $\Omega^n Y \in {}^\perp\mathcal{P}$ and there exists an $Y' \in {}^\perp\mathcal{P}$ such that $\Omega^n_{\mathcal{E}} Y = \Omega^n_{\mathcal{E}} Y'$ (use the first bit of the injective coresolution of $\Omega^n Y$ to find $Y'$); an easy dimension shift shows

$$\operatorname{Ext}^{m+n}_{\mathcal{E}}(Y, X) \cong \operatorname{Ext}^{m}_{\mathcal{E}}(\Omega^n_{\mathcal{E}} Y, X) \cong \operatorname{Ext}^{m+n}_{\mathcal{E}}(Y', X) \neq 0$$

This contradicts our previous observation.
(2) assume $\operatorname{id}_{\mathcal{E}} \mathcal{P} \leq n$, take $X$ in $\mathcal{E}$ arbitrary and look at the beginning of a projective resolution of $X$:

$$\Omega^n X \rightarrowtail P_{n-1} \to \cdots \to P_0 \twoheadrightarrow X$$

then $\Omega^n X \in {}^\perp\mathcal{P}$ and $X \in \operatorname{Res}_n({}^\perp\mathcal{P})$, so (C2) holds. Condition (C1) is implied by $\mathcal{P}^\perp \subseteq \operatorname{cogen}_{\mathcal{E}} \mathcal{P}$: Clearly $\mathcal{P}$ are injective objects in ${}^\perp\mathcal{P}$ (as ${}^\perp\mathcal{P}$ is a resolving subcategory, it is homologically finite). For $X \in {}^\perp\mathcal{P}$ there is an $\mathcal{E}$-short exact sequence $X \rightarrowtail P \twoheadrightarrow Y$ such that $\operatorname{Hom}(-, P')$ is exact on it for all $P' \in \mathcal{P}$ (this follows by the definition of $\operatorname{cogen}(\mathcal{P})$). But then it follows $Y \in {}^\perp\mathcal{P}$ and this shows that we have enough injectives given by $\mathcal{P}$, so (C1) follows. □

This is the classical result for rings.

**Example 4.6.** Let $R$ be a left and right noetherian ring and $R \operatorname{mod}$ (resp. $\operatorname{mod} R$) the category of finetely generated left (resp. right) $R$-module. In this case, we say $R$ is $n$-**Iwanaga-Gorenstein** if $\operatorname{id}{}_R R \leq n$ and $\operatorname{id} R_R \leq n$. Then the following are equivalent:

(1) $R$ is $n$-Iwanaga-Gorenstein
(2) $R \operatorname{Mod}$ and $\operatorname{Mod} R$ are $n$-Gorenstein
(3) $R \operatorname{mod}$ and $\operatorname{mod} R$ are $n$-Gorenstein

The implication (1) implies (2) and (1) implies (3) are a famous result of Iwanaga [**19**, Theorem 2]. The implication (2) implies (1) is trivial, and (3) implies (1) follows from $\operatorname{id}_{R \operatorname{mod}} R = \operatorname{id}_R R$ and $\operatorname{id}_{\operatorname{mod} R} R = \operatorname{id} R_R$.
Observe that $R \operatorname{mod}$ and $\operatorname{mod} R$ are abelian categories with enough projectives but in general not with enough injectives.

**Lemma 4.7.** *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough projectives and enough injectives. If $\mathcal{E}$ is $n$-Gorenstein and $\mathcal{P}$ is covariantly finite in ${}^\perp\mathcal{P}$, then $\mathcal{P}$ is $n$-cotilting.*

Proof. By Prop. 4.5, (2), it is enough to show ${}^\perp\mathcal{P} \subseteq \operatorname{cogen}_{\mathcal{E}}(\mathcal{P})$. As $\mathcal{P}$ is assumed covariantly finite in ${}^\perp\mathcal{P}$, it is enough to show that ${}^\perp\mathcal{P} \subseteq \operatorname{copres}_{\mathcal{E}}(\mathcal{P})$. For $X \in {}^\perp\mathcal{P}$ take an $\mathcal{E}$-inflation $i \colon X \rightarrowtail I$ with $I$ in $\mathcal{I}(\mathcal{E})$. Then take a deflation $p \colon P \twoheadrightarrow I$ with $P \in \mathcal{P}(\mathcal{E})$. As $\mathcal{E}$ is $n$-Gorenstein, and $I, P \in \mathcal{P}^{<\infty}$, it follows that $L := \ker p \in \mathcal{P}^{<\infty}$. Using a finite projective resolution of $L$, one sees $\operatorname{Ext}^1_{\mathcal{E}}(X, L) = 0$. This implies that $i$ factors as $i = pf$. By the obscure axiom ([**9**, Prop. 7.6]), we conclude that $f \colon X \rightarrowtail P$ is an inflation. □

**Open question 4.8.** Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. We also assume that $\mathcal{P} \subseteq {}^\perp\mathcal{P}$ is covariantly finite and $\mathcal{I}$ contravariantly finite in $\mathcal{I}^\perp$. Then the following are equivalent:

(i) $\mathcal{E}$ is $n$-Gorenstein
(ii) $\operatorname{id}_{\mathcal{E}} \mathcal{P} \leq n$ and $\operatorname{pd}_{\mathcal{E}} \mathcal{I} \leq n$.
(iii) $\mathcal{E}^{op}$ is $n$-Gorenstein.
(iv) There exists a subcategory which is $s$-tilting and $t$-cotilting for some $s, t \geq 0$.
(v) A subcategory is $s$-cotilting for some $s$ if and only if it is $t$-cotilting for some $t$.

Observe that we have already seen that (i),(ii),(iii) are equivalent, and (i) implies (iv). In (ii), do we also have $\operatorname{pd}_{\mathcal{E}} \mathcal{I} = \operatorname{id}_{\mathcal{E}} \mathcal{P}$?

**Definition 4.9.** Given an exact category $\mathcal{E}$ and we define $\mathcal{P} = \mathcal{P}(\mathcal{E})$ be its projectives. The category of **Gorenstein projectives** (denoted by $\mathbf{Gp}(\mathcal{E})$) are the full subcategory of objects $X$ such that there exists an exact complex of projectives

$$\cdots \to P_{-1} \to P_0 \to P_1 \to \cdots$$

such that

(1) $\cdots \to \mathrm{Hom}(P_n, P) \to \mathrm{Hom}(P_{n-1}, P) \to \cdots$ is exact for all $P$ in $\mathcal{P}$
(2) $\mathrm{Im}(P_{-1} \to P_0) = X$

**Proposition 4.10.** *Let $\mathcal{E}$ be an exact category and $\mathcal{P} := \mathcal{P}(\mathcal{E})$.*

(1) *Then $\mathbf{Gp}(\mathcal{E})$ is extension-closed, closed under taking summands and deflation-closed (i.e. closed under taking kernels of deflations) and we have $\mathbf{Gp}(\mathcal{E}) \subseteq {}^{\perp}\mathcal{P}$. With this exact structure it is a Frobenius exact category with projectives $\mathcal{P}$.*
(2) *$\mathbf{Gp}(\mathcal{E})$ is resolving if and only if $\mathcal{E}$ has enough projectives.*
(3) *$\mathbf{Gp}(\mathcal{E})$ is finitely resolving (resp. $n$-resolving) if and only if $\mathcal{E}$ has enough projectives and $\mathcal{P}$ is cotilting (resp. $n$-cotilting). In these cases we have $\mathbf{Gp}(\mathcal{E}) = {}^{\perp}\mathcal{P}$.*

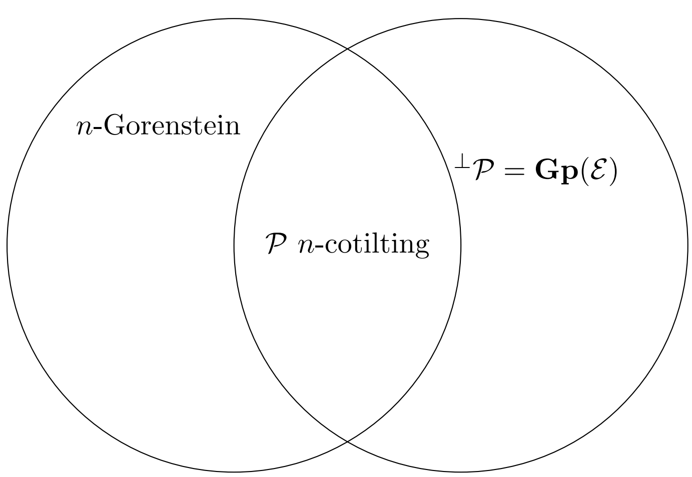
Proof. (1) The proof from [12, Prop. 2.1.7 (1),(2),(3)] can also be used to prove extension-closedness, summand-closedness and deflation-closedness. By definition, $X \in \mathbf{Gp}(\mathcal{E})$ implies all $X_n = \mathrm{Im}(P_{n-1} \to P_n) \in \mathbf{Gp}(\mathcal{E})$ and all short exact sequence $X_n \rightarrowtail P_n \twoheadrightarrow X_{n+1}$ are short exact in $\mathbf{Gp}(\mathcal{E})$, apply $\mathrm{Hom}(-, P)$ with $P \in \mathcal{P}$ to these short exact sequences to conclude $\mathrm{Ext}^{>0}(X, P) = 0$, so $\mathcal{P} \subseteq \mathcal{I}(\mathbf{Gp}(\mathcal{E}))$. Then just use the defining exact sequence in $\mathcal{P}$ to conclude that $\mathbf{Gp}(\mathcal{E})$ is Frobenius.
(2) If $\mathcal{E}$ has enough projectives then these have to be in $\mathbf{Gp}(\mathcal{E})$ and by (1) it is resolving. If $\mathbf{Gp}(\mathcal{E})$ is resolving, we also know it has enough projectives $\mathcal{P} = \mathcal{P}(\mathbf{Gp}(\mathcal{E}))$ (by (1)).
(3) If $\mathcal{P}$ is cotilting then (C1) implies ${}^{\perp}\mathcal{P} \subseteq \mathrm{cogen}_{\infty}(\mathcal{P})$ and the other inclusion follows from the definition of $\mathrm{cogen}_{\infty}(\mathcal{P})$. Then for $X \in {}^{\perp}\mathcal{P} = \mathrm{cogen}_{\infty}(\mathcal{P})$ splice together the projective resolution in $\mathcal{E}$ with the injective coresolution, this shows $X \in \mathbf{Gp}(\mathcal{E})$. By definition $\mathbf{Gp}(\mathcal{E}) \subseteq \mathrm{cogen}_{\infty}(\mathcal{P})$, so we conclude in this case $\mathbf{Gp}(\mathcal{E}) = {}^{\perp}\mathcal{P}$ is finitely resolving ($n$-resolving if $\mathcal{P}$ was $n$-cotilting).
Conversely, if $\mathbf{Gp}(\mathcal{E})$ is finitely (resp. $n$-)resolving, we already know from (1) that $\mathbf{Gp}(\mathcal{E}) \subseteq \mathcal{P}^{\perp}$. We need to see the other inclusion, this follows immediately from the Lemma 4.11. But then all properties for $\mathcal{P}$ being (resp. $n$-)cotilting are fulfilled.

□

**Lemma 4.11.** *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Given an exact sequence $E_1 \rightarrowtail E_0 \twoheadrightarrow X$ with $X \in {}^{\perp}\mathcal{P}, E_0, E_1 \in \mathbf{Gp}(\mathcal{E})$. Then $X \in \mathbf{Gp}(\mathcal{E})$.*

Proof. This can be shown with the same argument as [12, Prop. 2.1.7 (4)]. □

in exact categories wep. $\mathcal{P}$:



**Lemma 4.12.** *Assume that $\mathcal{E}$ is a weakly idempotent complete exact category with enough projectives $\mathcal{P}$ and assume $\mathcal{E} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{I}^{<\infty})$. If $\mathcal{E}$ is $n$-Gorenstein, then $\mathcal{P}$ is $n$-cotilting.*

The proof is very similar to Lemma 4.7.

Proof. We show that ${}^{\perp}\mathcal{P} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{P})$ (the rest follows from Prop. 4.5, (2)).
We assume $\mathcal{E} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{I}^{<\infty})$: So, for $X \in {}^{\perp}\mathcal{P}$, we take a short exact sequence $X \overset{i}{\rightarrowtail} J \twoheadrightarrow Q$ with $J \in \mathcal{I}^{<\infty}$. We choose a short exact sequence $J_1 \rightarrowtail P \overset{p}{\twoheadrightarrow} J$ with $P \in \mathcal{P}$. As $\mathcal{I}^{<\infty} = \mathcal{P}^{<\infty}$ is deflation-closed, we find that $J_1 \in \mathcal{P}^{<\infty}$ and one easily checks $\mathrm{Ext}^{>0}(X, J_1) = 0$ for all $X \in {}^{\perp}\mathcal{P}$ (using the finite projective resolution of $J_1$). This implies $\mathrm{Hom}(X, P) \to \mathrm{Hom}(X, J)$ is surjective, pick a morphism $f \colon X \to P$ that maps to $i$, say $fp = i$ . By the obscure axiom $f$ is an $\mathcal{E}$-inflation. Now, we need to see that $\mathrm{Hom}(f, P)$ is surjective for every $P \in \mathcal{P}$. But since $\mathrm{Hom}(i, P) = \mathrm{Hom}(f, P) \circ \mathrm{Hom}(p, P)$, this follows. Then we see ${}^{\perp}\mathcal{P} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{P})$. □

**Example 4.13.** From [**14**]: If $R$ is an $n$-Iwanaga-Gorenstein ring and $\mathcal{F} = R - \mathrm{Mod}$, then $\mathcal{F} \subseteq \mathrm{cogen}_{\mathcal{F}}(\mathcal{I}^{<\infty})$ and this implies $\mathbf{Gp}(R\,\mathrm{Mod}) = {}^{\perp}(\mathrm{Proj}(R))$.
This implies for $\mathcal{E} = R\,\mathrm{mod}$, i.e. the category of finitely generated left $R$-modules, that $\mathbf{Gp}(\mathcal{E}) = {}^{\perp_{\mathcal{E}}}R =: {}^{\perp}R$, to see this, recall that we only needed to see ${}^{\perp}R \subseteq \mathrm{cogen}_{\mathcal{E}}(R)$. But by the previous result we have ${}^{\perp}R \subseteq \mathrm{cogen}_{R\,\mathrm{Mod}}(\mathrm{ADD}(R))$ and observe that every finitely generated submodule of a free $R$-module is contained in a finitely generated free summand, this implies the claim.

Here we have the following result

Theorem 4.14. *([**26**, Cor 2.2], [**30**, Ex. 2.3]) Let $\mathcal{E}$ be a weakly idempotent complete Frobenius exact category and let $\mathcal{P} = \mathcal{P}(\mathcal{E})$ be the projectives in $\mathcal{E}$. Then the functor $\mathcal{E} \to \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}_{sg}(\mathcal{E})$ induces a triangle equivalence*

$$\underline{\mathcal{E}} \to \mathrm{D}_{sg}(\mathcal{E})$$

As a corollary, follows the following result of Kvamme (in the special case of weakly idempotent complete exact categories). Just take $\mathbf{Gp}(\mathcal{E})$ as Frobenius exact category and use Prop. 4.10 (observe this implies: if $\mathcal{E}$ has enough projectives, then $\mathrm{D}^b(\mathbf{Gp}(\mathcal{E})) \to \mathrm{D}^b(\mathcal{E})$ is fully faithful. If $\mathbf{Gp}(\mathcal{E})$ is finitely resolving, it is a triangle equivalence, cf. [**16**]).

Theorem 4.15. *([**32**] ) Let $\mathcal{E}$ be an exact category with enough projectives. Then $\mathbf{Gp}(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b_{sg}(\mathcal{E})$ induces a fully faithful triangulated functor*

$$\underline{\mathbf{Gp}(\mathcal{E})} \to \mathrm{D}_{sg}(\mathcal{E}).$$

*This is an equivalence if $\mathbf{Gp}(\mathcal{E})$ is finitely resolving in $\mathcal{E}$.*

We prefer to reformulate this last statement to:

Theorem 4.16. *(Buchweitz Theorem)*
*Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and assume that $\mathcal{P}$ is $n$-cotilting. Then, the functor $\mathbf{Gp}(\mathcal{E}) \to \mathrm{D}^b_{sg}(\mathcal{E})$ induces a triangle equivalence*

$$\underline{\mathbf{Gp}(\mathcal{E})} \to \mathrm{D}_{sg}(\mathcal{E})$$

Now, if $\mathcal{E}$ is an exact category with enough projectives, $\mathrm{D}_{sg}(\mathcal{E})$ can be realized as the Heller stabilisation $\mathbb{Z}\underline{\mathcal{E}}$ of the stable category of $\mathcal{E}$ seen as a left triangulated category [**32**, Thm 3.4]. Since the stabilization is functorial, an equivalence of left triangulated stable category $\underline{\mathcal{E}} \cong \underline{\mathcal{E}'}$ implies $\mathcal{E}$ and $\mathcal{E}'$ are singular equivalent (cf. [**32**]), but it also implies that $\mathcal{E}$ and $\mathcal{E}'$ are stable equivalent (investigated in the next chapter). That is the only connection between singular and stable equivalence that we know of.

Here is the my main open questions in this subsection

(4.1) Can we find singular invariants? Can we find classes of singular equivalent exact categories which are not derived equivalent (inspired by Knörrer periodicity)?

## 5. Non-commutative resolutions from exact substructures

Let $\mathcal{A}$ be an exact category. We fix an exact substructure $\mathcal{E}$ of $\mathcal{A}$. We observe that this gives a Verdier localization sequence

$$\mathrm{Ac}(\mathcal{A})/\mathrm{Ac}(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$$

If $\mathcal{E}$ is regular then we want to interpret $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ as a *categorical desingularization* (following Orlov's definition - only that Orlov required that $\mathcal{E}$ is also abelian).

**Definition 5.1.** We fix an exact category $\mathcal{A}$ and an exact substructure $\mathcal{E}$. Let $d \geq 0$ be an integer. We will write **NCR** as a shorthand for *non-commutative resolution* throughout the rest of the chapter.

(*) We call $\mathcal{E}$ a **weak ($d$-)NCR** if $\mathcal{E}$ is regular (resp. $\operatorname{gldim} \mathcal{E} \leq d$).
(*) We call $\mathcal{E}$ a **($d$-)NCR** if $\mathcal{E}$ is regular (resp. $\operatorname{gldim} \mathcal{E} \leq d$) and has enough projectives.
(*) We call $\mathcal{E}$ a **strong ($d$-)NCR** if $\mathcal{E}$ and has enough projectives $\mathcal{Q}$ and $\mathcal{Q}$ has pseudo-kernels and $\operatorname{mod}_\infty \mathcal{Q}$ is regular (resp. $\operatorname{gldim}(\operatorname{mod}_\infty \mathcal{Q}) \leq d$).

Naively, we expect to find weak NCRs in algebraic geometric situations and NCRs when studying certain module or functor categories. For $\mathcal{A}$ Frobenius exact category, what we call a strong $d-$NCR is defined in [**28**] just as an NCR.
The existence of a strong $d$-NCR for a Frobenius category has the consequence that it is equivalent to the Gorenstein-projectives in $\operatorname{mod}_\infty \mathcal{P}$.

Theorem 5.2. *Let $\mathcal{A}$ be an idempotent complete exact category with enough projectives $\mathcal{P}$ and assume it has a strong $d$-NCR, then*

$$\mathbb{P}\colon \mathcal{A} \to \operatorname{mod}_\infty \mathcal{P}, \quad X \mapsto \operatorname{Hom}(-, X)|_{\mathcal{P}}$$

*has a finitely resolving image.*
*Furthermore, if $\mathcal{P}$ is $n$-cotilting in $\mathcal{A}$, then $\mathcal{P}$ is also $(n+d)$-cotilting in $\operatorname{mod}_\infty \mathcal{P}$ and $\mathbb{P}$ restricts to an equivalence of exact categories*

$$\mathbb{GP}\colon \mathbf{Gp}(\mathcal{A}) \to \mathbf{Gp}(\operatorname{mod}_\infty \mathcal{P}).$$

Before we give the proof, let us remark the following corollary which shows that for some exact categories a strong $d$-NCR can not exist (because their bounded derived category does not have a t-structure).

**Corollary 5.3.** *If $\mathcal{A}$ is an idempotent complete exact category with enough projectives and $\mathcal{A}$ admits a strong $d$-NCR, then $\mathcal{A}$ is derived equivalent to an abelian category with enough projectives.*

Since the inclusion of a finitely resolving subcategory induces a triangle equivalence on bounded derived categories, the corollary follows.

Proof. Let $\mathcal{E}$ be a strong $d$-NCR and $\mathcal{Q} = \mathcal{P}(\mathcal{E})$. We have that the restriction functor

$$f = (-)|_{\mathcal{P}}\colon \operatorname{mod}_\infty \mathcal{Q} \to \operatorname{mod}_\infty \mathcal{P}$$

is exact and essentially surjective because $\mathcal{P} \subseteq \mathcal{Q}$ is a full subcategory. Now, we look at the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathbb{P}_{\mathcal{A}}"] \arrow[d, "\mathrm{id}"] & \operatorname{mod}_\infty \mathcal{P} \\ \mathcal{E} \arrow[r, "\mathbb{P}_{\mathcal{E}}"] & \operatorname{mod}_\infty \mathcal{Q} \arrow[u, "f"'] \end{tikzcd}$$

Since $\operatorname{gldim}(\operatorname{mod}_\infty \mathcal{Q}) \leq d$, we have $\operatorname{Im}\mathbb{P}_{\mathcal{E}}$ is $d$-resolving. Since $\mathbb{P}_{\mathcal{A}} = f \circ \mathbb{P}_{\mathcal{E}}$ and $f$ is exact and essentially surjective, it follows that $\operatorname{Im}\mathbb{P}_{\mathcal{A}}$ is also $d$-resolving.
Now assume additionally that $\mathcal{P}$ is $n$-cotilting, clearly the functor $\mathbb{P}_{\mathcal{A}}$ maps complete resolution by projectives into complete resolutions by projectives, therefore it restricts to a functor on Gorenstein projectives (call this $\mathbb{GP}$). Since $\mathbf{Gp}(\mathcal{A})$ is $n$-resolving in $\mathcal{A} \cong \operatorname{Im}\mathbb{P}$ which is $d$-resolving in $\operatorname{mod}_\infty \mathcal{P}$, we conclude $\operatorname{Im}\mathbb{GP}$ is finitely resolving in $\operatorname{mod}_\infty \mathcal{P}$. This implies $\mathbf{Gp}(\operatorname{mod}_\infty \mathcal{P})$ is finitely resolving in $\operatorname{mod}_\infty \mathcal{P}$ and therefore $\operatorname{mod}_\infty \mathcal{P}$ is $\mathcal{P}$ cotilting in $\operatorname{mod}_\infty \mathcal{P}$ by Prop. 4.10. As $\mathcal{A} = \operatorname{Im}\mathbb{P}$ is $d$-resolving in $\operatorname{mod}_\infty \mathcal{P}$ and $\operatorname{id}_{\mathcal{A}} \mathcal{P} \leq n$, it follows easily by dimension shift that $\operatorname{id}_{\operatorname{mod}_\infty \mathcal{P}} \mathcal{P} \leq n + d$ and therefore $\mathcal{P}$ is $(n + d)$-cotilting.
We still need to see that $\mathbb{GP}$ is essentially surjective. But $\mathbb{GP}$ induces an triangle equivalence (since the image is finitely resolving) which induces a triangle equivalence on the singularity categories. But since these are the stable categories we conclude that $\mathbb{GP}$ is essentially surjective. $\square$

**Corollary 5.4.** *([**28**] and an old result by Auslander) If $\mathcal{A}$ is Frobenius exact and has a strong $d$-NCR, then the functor $\mathbb{P}$ induces an equivalence of exact categories $\mathcal{A} \to \mathbf{Gp}(\operatorname{mod}_\infty \mathcal{P})$*

**Example 5.5.** Let $A$ be a left noetherian ring and $\mathcal{A} = A - \operatorname{mod}$ the category of finitely generated left $A$-modules. Take a generator $E = M \oplus A \in \mathcal{A}$ and assume that $\operatorname{add}(E)$ is contravariantly finite in $\mathcal{A}$ and that $\Gamma = \operatorname{End}_A(E)^{op}$ is again left noetherian of finite global dimension. Take the idempotent $e \in \Gamma$ corresponding to the summand $A$ in $E$ and the exact functor $e\colon \Gamma - \operatorname{mod} \to \mathcal{A}, e(X) := eX$. It has a fully faithful left adjoint and a fully faithful left adjoint $\ell$ and a fully faithful right adjoint $r = \operatorname{Hom}_A(E, -)$ (the right adjoint is well-defined since $\operatorname{add}(E)$ is contravariantly finite). So by Chapter 1, we get three exact substructures $\mathcal{S}_\ell, \mathcal{S}_r, \mathcal{S}_c$ where $c = \operatorname{Im}(\ell \to r)$ is the intermediate extension functor.
Now we look at $\mathcal{E} = (A - \operatorname{mod}, \mathcal{S}_r)$, as an exact category this is equivalent to $\operatorname{Im} r$ (seen as extension-closed subcategory of $\Gamma - \operatorname{mod}$. It is deflation-closed and contains $\Gamma = r(E)$. Therefore it is a resolving subcategory. Since we assume $\operatorname{gldim}\Gamma < \infty$ it is finitely resolving and we get that the composition is a traingle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\operatorname{Im} r) \to \mathrm{D}^b(\Gamma - \operatorname{mod})$ such that the composition $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\Gamma - \operatorname{mod}) \xrightarrow{e} \mathrm{D}^b(\mathcal{A})$ equals the natural map $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ induced by the identity on $A - \operatorname{mod}$.

**Example 5.6.** Every exact category has a unique 0-NCR given by the split exact structure. Therefore, we usually look for $d$-NCRs with $d \geq 1$.

If $\mathcal{E}$ is an NCR for $\mathcal{A}$ with enough projectives $\mathcal{P}(\mathcal{E}) =: \mathcal{P}$, then the previous Verdier localization sequence is triangle equivalent to the Verdier localization sequence

$$\mathrm{K}_{ac}(\mathcal{P}) \to \mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{P})/\mathrm{K}_{ac}(\mathcal{P}).$$

where $\mathrm{K}_{ac}(\mathcal{P})$ is the thick subcategory given by complexes which are $\mathcal{A}$-acyclic.

**Example 5.7.** Let $G$ be a finite group and $k$ a field of characteristic $p$ dividing the order of the group. Let $E := \bigoplus_{H \subset G} k(G/H)$ and $\mathcal{P} = \operatorname{add}(E) \subset \operatorname{mod} kG$. Let $\mathcal{A} = \operatorname{mod} kG$ and $\mathcal{E}$ the exact substructure with $\mathcal{P}(\mathcal{E}) = \mathcal{P}$. This is an NCR of $\mathcal{A}$. This way (up to idempotent completion) the triangle equivalence between $\mathrm{D}^b(\mathcal{A})$ and $\mathrm{K}^b(\mathcal{P})/\mathrm{K}^b_{ac}(\mathcal{P})$ from a previous remark is the main result in [**5**].

We also want to keep track on how $\mathcal{A}$-self-orthogonal the projectives $\mathcal{P}(\mathcal{E})$ are, so we define (inspired by the works of [**11**])

**Definition 5.8.** Let $n \in \mathbb{N}_{>0}$. Given a full subcategory $\mathcal{M}$ of $\mathcal{A}$, we say that $\mathcal{M}$ is $n$-**rigid** if $\operatorname{Ext}^{1\sim n}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}) = 0$ (this is a shorthand notation for $\operatorname{Ext}^i_{\mathcal{A}}(M, M') = 0$ for all $M, M' \in \mathcal{M}$, $i \in \{1, \dots, n\}$). We say it is $n\mathbb{Z}$-**rigid** if $\operatorname{Ext}^i_{\mathcal{A}}(M, M') = 0$ for all $M, M' \in \mathcal{M}$, $i \in \mathbb{N}_{>0} \setminus n\mathbb{N}$
Let $\mathcal{E}$ be an exact substructure of $\mathcal{A}$. We define the $\mathcal{A}$-**rigidity** of $\mathcal{E}$ (or more accurately of $\mathcal{P}(\mathcal{E})$) to be

$$\operatorname{rig}_{\mathcal{A}}(\mathcal{E}) = \sup(\{\mathrm{m} \in \mathbb{N}_{>0} \mid \mathcal{P}(\mathcal{E}) \text{ is m-rigid}\} \cup \{0\}) \quad \in \mathbb{N}_{\geq 0} \cup \{\infty\}$$

If $\operatorname{gldim}\mathcal{A} < \infty$ we have $\operatorname{rig}_{\mathcal{A}}(\mathcal{E}) \in \{0, 1, \dots, \operatorname{gldim}\mathcal{A} - 1\} \cup \{\infty\}$.

We define the **(projective) rigidity dimension** of $\mathcal{A}$ to be

$$\mathrm{rdim}(\mathcal{A}) = \sup\{\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \mid \mathcal{E} \text{ NCR }\}$$

If $\mathcal{A}$ is regular with enough projectives, it follows $\mathrm{rdim}\mathcal{A} = \infty$ since we may take $\mathcal{E} = \mathcal{A}$.

We have the following (version of the Auslander-Reiten formulation of the Nakayama conjecture)

**Proposition 5.9.** *(Nakayama Conjecture for NCRs) If $\mathcal{A}$ is exact category and $\mathcal{E}$ is an NCR with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) = \infty$ then $\mathcal{A} = \mathcal{E}$ (in particular $\mathcal{A}$ is also regular with enough projectives).*

Proof. We have $\mathcal{P} := \mathcal{P}(\mathcal{E})$ is homologically exact in $\mathcal{A}$ since it is self-orthogonal. This implies that $\mathrm{D}^b(\mathcal{E}) \cong \mathrm{K}^b(\mathcal{P}) \to \mathrm{D}^b(\mathcal{A})$ is fully faithful. Therefore, the inclusion of the exact substructure $\mathcal{E} \to \mathcal{A}$ is homologically exact implying it is the identity, cf. Chapter 1. □

**Corollary 5.10.** *If $\mathcal{A}$ is not regular and $\mathcal{E}$ NCR, then $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) < \infty$.*

**Corollary 5.11.** *If $\mathcal{A}$ is hereditary and Krull Schmidt and $\mathcal{E}$ an NCR which is not equal to $\mathcal{A}$ then $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) = 0$ (i.e. $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{P}(\mathcal{E}), \mathcal{P}(\mathcal{E})) \neq 0$)*

**Remark 5.12.** For modules over rings: Via generator correspondence and Müller correspondence this is very much related to the so-called dominant dimension, cf. correspondences explained in [**35**]. We added the adjective *projective* since the rigidity dimension of a finite-dimensional algebra is defined using generator-cogenerators and not just generators.

The main reason to introduce $\mathcal{A}$-rigidity for $\mathcal{E}$ is the following easy observation:

**Lemma 5.13.** *Let $\mathcal{A}$ be an exact category. If $\mathcal{E}$ is an exact substructure with enough projectives $\mathcal{P}(\mathcal{E})$. If we have $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{n}$, then we have $\mathrm{Res}^{\mathcal{E}}_n(\mathcal{P}(\mathcal{E})) = \mathrm{Res}^{\mathcal{A}}_n(\mathcal{P}(\mathcal{E}))$.*

Proof. Let $\mathcal{P} := \mathcal{P}(\mathcal{E})$ be $n$-rigid (in $\mathcal{A}$). The inclusion $\mathrm{Res}^{\mathcal{E}}_n(\mathcal{P}) \subseteq \mathrm{Res}^{\mathcal{A}}_n(\mathcal{P})$ is trivial. Let $X \in \mathrm{Res}^{\mathcal{A}}_n(\mathcal{P})$. By definition we have an $\mathcal{A}$-exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

with $P_i \in \mathcal{P}$. To see that it is exact in $\mathcal{E}$, we split it in short exact sequences and show that $\mathrm{Hom}(\mathcal{P}, -)$ is exact on it: Set $X = P_{-1}$, let $Q_i = \ker(P_i \to P_{i-1})$, $i = 0, \ldots, n-1$, observe $P_n = Q_{n-1}$. By dimension shift we have $\mathrm{Ext}^1_{\mathcal{A}}(P, Q_i) \cong \mathrm{Ext}^2_{\mathcal{A}}(P, Q_{i+1}) \cong \cdots \cong \mathrm{Ext}^{n-i}_{\mathcal{A}}(P, P_n) = 0$ for all $P \in \mathcal{P}$, $i \in \{0, \ldots, n-1\}$. □

Theorem 5.14. *Let $\mathcal{A}$ be an exact category. The assignment $\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ gives a bijection between:*

1. *$d$-NCRs $\mathcal{E}$ with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$.*
2. *$d$-rigid subcategories $\mathcal{P}$ of $\mathcal{A}$ with $\mathrm{Res}^{\mathcal{A}}_d(\mathcal{P}) = \mathcal{A}$.*

**Remark 5.15.** Let $\mathcal{P}$ be a subcategory as in (2).

1. If $\mathcal{P}$ is also $(d+1)$-rigid, then $\mathcal{P}$ is deflation closed and a $d$-resolving subcategory of $\mathcal{A}$, this implies $K^b(\mathcal{P}) = D^b(\mathcal{P}) \to D^b(\mathcal{A})$ is a triangle equivalence. If $\mathcal{E}$ is the $d$-NCR with $\mathcal{P}(\mathcal{E}) = \mathcal{P}$, then Prop.5.9 implies $\mathcal{E} = \mathcal{A}$. In particular, $\mathcal{P} = \mathcal{P}(\mathcal{A})$ is then even self-orthogonal in $\mathcal{A}$.
2. If $\mathcal{P}$ is not $(d+1)$-rigid, then it can not be deflation-closed (because else, we can apply the same argument as in (1) to deduce that $\mathcal{P}$ has to be even selforthogonal).

Proof. The assignment $\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ is one to one between exact substructures with enough projectives and admissibly contravariantly finite subcategories. If we now additionally assume $d$-NCR with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$ then we clearly get a subcategory as in (2). For the converse we just use the previous lemma to see that the exact structure has gldim $\leq d$. □

**Definition 5.16.** ([**33**, Def. 3.1]) A $d$-rigid subcategory $\mathcal{P}$ in an exact category $\mathcal{A}$ is called

(1) **left maximal $d$-rigid** if $\mathrm{Res}_d^{\mathcal{A}}(\mathcal{P}) = \mathcal{A}$ and
(2) **right maximal $d$-rigid** if $\mathrm{Cores}_d^{\mathcal{A}}(\mathcal{P}) = \mathcal{A}$

Observe, given an exact category $\mathcal{A}$ with enough projectives $\mathcal{P} = \mathcal{P}(\mathcal{A})$, then $\mathcal{P}$ is left maximal $d$-rigid iff $\operatorname{gldim} \mathcal{A} \leq d$. But $\mathcal{P}$ is right maximal rigid implies $\mathcal{P} = \mathrm{Cores}_d^{\mathcal{A}}(\mathcal{P}) = \mathcal{A}$ because $\mathcal{P}$ is projective.

**Proposition 5.17.** *If $\mathcal{A}$ has enough projectives $\mathcal{P}$ and let $\mathcal{A}' = \mathbf{Gp}(\mathcal{A})$. Then, restricting exact substructures from $\mathcal{A}$ to $\mathcal{A}'$ gives an injective map from (a) to (b) where*

(a) *$d$-NCRs $\mathcal{E}$ with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$ and $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{A}'$*
(b) *$d$-NCRs $\mathcal{E}'$ with $\mathrm{rig}_{\mathcal{A}'}(\mathcal{E}') \geq \mathrm{d}$*

*If additionally $\mathcal{P}$ is $n$-cotilting with $d \geq n$, then the inclusion $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{A}'$ in (a) is always true.*

Proof. The first part is more generally true: Let $\mathcal{A}$ be an exact category with enough projectives. Let $\mathcal{A}' \subseteq \mathcal{A}$ be a resolving subcategory. Let $\mathcal{E}$ be as in (a). The homologically exactness implies that $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{A}'$ is $d$-rigid and since $\mathcal{A}'$ is deflation-closed we have $\mathrm{Res}_d^{\mathcal{A}'}(\mathcal{P}(\mathcal{E})) = \mathcal{A}'$. Clearly the map is injective since an exact substructure with enough projectives is determined by its category of projectives.
Now assume also that $\mathrm{id}_{\mathcal{A}} \mathcal{P} \leq n$ and $\mathbf{Gp}(\mathcal{A}) = {}^{\perp}\mathcal{P}$. If $d \geq n$ and $\mathcal{P}(\mathcal{E})$ is $d$-rigid, we have that $\mathcal{P} \subseteq \mathcal{P}(\mathcal{E})$, so $\mathrm{Ext}_{\mathcal{A}}^{1\sim d}(\mathcal{P}(\mathcal{E}), \mathcal{P}) = 0$. Since $\mathrm{id}_{\mathcal{A}} \mathcal{P} \leq n$, it follows that $\mathcal{P}(\mathcal{E}) \subseteq {}^{\perp}\mathcal{P} = \mathbf{Gp}(\mathcal{A})$. $\square$

Here are my open questions in this section:

(5.1) Given an exact category, does there always exist a non-trivial weak NCR?
(5.2) Exact substructures of an exact category form a complete lattice - do *maximal* elements exist in the subposet of regular exact substructures (maybe plus some rigidity...)?
(5.3) If $\mathcal{A}$ is an exact category with enough projectives. Is $\mathcal{A}$ regular if and only if $\mathrm{rdim}(\mathcal{A}) = \infty$?

## 6. NCRs from cluster tilting subcategories

**Definition 6.1.** Given two exact substructures $\mathcal{E}$ and $\mathcal{F}$ (with the same underlying additive category), we say that $\mathcal{F}$ is the **translate** of $\mathcal{E}$ (or $(\mathcal{E}, \mathcal{F})$ a **translated pair**) if $\mathcal{E}$ has enough projectives, $\mathcal{F}$ has enough injectives and $\mathcal{P}(\mathcal{E}) = \mathcal{I}(\mathcal{F})$.

Cf. Chapter 2, assume that the underlying additive category is weakly idempotent complete, then translated pairs are (via $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{P}(\mathcal{E})$) in bijection to functorially finite generator-cogenerators.

**Example 6.2.** We have that an exact substructure $\mathcal{E}$ is a Frobenius exact structure if and only if $(\mathcal{E}, \mathcal{E})$ is a translated pair.

The following example is the reason for the naming (translated stands for Auslander-Reiten translated).

**Example 6.3.** Let $\mathcal{A}$ be the category of finite-dimensional modules over a finite-dimensional algebra $\Lambda$. Let $G = \Lambda \oplus X$. We consider $\mathcal{E} = (\Lambda - \mathrm{mod}, F_G)$ the exact substructure with enough projectives given by $\mathrm{add}(G)$. By [3] we have that $\mathcal{E} = (\Lambda - \mathrm{mod}, F^H)$ is equal to the exact substructure with enough injectives given by $\mathrm{add}(H)$ with $H = \tau^- X \oplus \mathrm{D}\,\Lambda$. So, with short-hand notation $(F_{\tau^- G}, F_G)$ is a translated pair iff $\mathrm{add}(\tau^- G \oplus \Lambda) = \mathrm{add}(\tau^- G \oplus \mathrm{D}\,\Lambda)$ (i.e. $\Lambda$ has to be self-injective or $\Lambda$ of finite global dimension and $G$ the Auslander generator)

Let us recall the following definition from e.g. [32]

**Definition 6.4.** Let $\mathcal{A}$ be an exact category. Let $\mathcal{M}$ be a full additively closed subcategory and $d \geq 0$ an integer. We $\mathcal{M}$ is $(d+1)$**-cluster tilting** if it is a functorially finite generator-cogenerator with

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{E} \mid \operatorname{Ext}^{1\sim d}(X, M) = 0 \, \forall M \in \mathcal{M}\} \\ &= \{X \in \mathcal{E} \mid \operatorname{Ext}^{1\sim d}(M, X) = 0 \, \forall M \in \mathcal{M}\}\end{aligned}$$

**Lemma 6.5.** [**13**, Prop. 2.9] *Let $\mathcal{A}$ be an exact category and $\mathcal{M}$ a $d$-rigid, generating-cogenerating covariantly functorially finite subcategory. The following are equivalent*

(1) *$\mathcal{M}$ is $(d+1)$ cluster tilting*
(2) $\operatorname{Res}_d^{\mathcal{A}}(\mathcal{M}) = \mathcal{A}$

This has the the following corollary.

**Corollary 6.6.** *Let $\mathcal{A}$ be an exact category and $\mathcal{M}$ a full additively closed subcategory and $d \geq 0$ an integer. The following are equivalent*

(1) *$\mathcal{M}$ is $(d+1)$-cluster tilting in $\mathcal{A}$*
(2) *$\mathcal{M}$ is $d$-rigid and* $\operatorname{Res}_d^{\mathcal{A}}(\mathcal{M}) = \mathcal{A} = \operatorname{Cores}_d^{\mathcal{A}}(\mathcal{M})$

From this, we directly get:

**Proposition 6.7.** *Let $d \geq 1$ and $\mathcal{A}$ an exact category. The assignment $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{P}(\mathcal{E})$ gives a bijection between*

(1) *translated pairs $(\mathcal{E}, \mathcal{F})$ with* $\operatorname{gldim} \mathcal{E} \leq d$, $\operatorname{gldim} \mathcal{F} \leq d$ *and* $\operatorname{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$
(2) *$(d+1)$-cluster tilting subcategories in $\mathcal{A}$.*

In other words $(d+1)$-cluster tilting subcategories in $\mathcal{A}$ are the projectives in a $d$-NCR of $\mathcal{A}$ with $\mathcal{A}$-rigidity $\geq d$ and in a $d$-NCR of $\mathcal{A}^{op}$ with $\mathcal{A}^{op}$-rigidity $\geq d$ .

**Example 6.8.** For $d = 0$, the split exact substructure $\mathcal{A}_0$ is the unique 0-NCR and it is a Frobenius exact category, so $(\mathcal{A}_0, \mathcal{A}_0)$ is a translated pair. It corresponds to the unique 1-cluster tilting subcategory in $\mathcal{A}$ given by $\mathcal{A}$ itself.

Then there are *geometrically* inspired examples.

**Example 6.9.** The first instances of noncommutative resolutions where found as algebraic analogues of algebraic geometric resolutions of very easy types of singularities (this is the reason for calling this concept 'noncommutative resolution').
The connection between cluster tilting and noncommutative resolutions of singularities is apparent in the following:

(1) For simple singularities: algebraic McKay correspondence (using an Auslander generator of an exact category of finite type, i.e. a 1-cluster tilting subcategory. This exact category is the Cohen-Macaulay modules of the local ring) [**4**], [**34**]
(2) For some non-isolated singularities in [**25**]

Furthermore, there are many more cluster tilting subcategories in Cohen-Macaulay modules over commutative noetherian local rings which are isolated singularities (i.e. geometrical examples) and more generally over (non-commutative) orders over isolated singularities found: [**20**],[**27**], [**21**], [**10**], [**15**], [**2**] (here: use [**32**] to pass from cluster tilting in the stable category to cluster tilting in the Frobenius exact category), also in graded Cohen-Macaulay module categories [**23**].

**Corollary 6.10.** *(also of* [**13**, Prop. 2.9]*) For $\mathcal{A}$ a Frobenius exact category, the projectives of a $d$-NCR with* $\operatorname{rig} \geq d$ *of $\mathcal{A}$ are already $(d+1)$-cluster tilting subcategory if and only if they are covariantly finite cogenerator (in $\mathcal{A}$).*

Using Prop. 5.17, the previous corollary amounts to: if $\mathcal{A}$ has enough projectives $\mathcal{P}$ and $\mathcal{P}$ is $n$-cotilting, then we have a bijection between

(1) $d$-NCRs $\mathcal{E}$ with rig $\geq d$ and the category $\mathcal{P}(\mathcal{E})$ is a covariantly finite cogenerator in ${}^{\perp}\mathcal{P}$
(2) $(d+1)$-cluster tilting subcategories in $\mathbf{Gp}(\mathcal{A})$

We conclude with: Let $d \geq 0$, by now, there is a remarkable list of examples of $d$-cluster tilting subcategories in exact categories already found, apart from geometrically inspired examples Ex. 6.9 we also have:

(1) Higher Auslander-Reiten theory is developed in cluster tilting subcategories (with many examples for artin algebras) [**24**], [**18**], [**17**],
(2) most instances of *cluster categories* are algebraic - this means their cluster tilting subcategories lift to cluster tilting subcategories in a Frobenius exact enhancement [**7**], [**6**], [**1**], [**29**], [**22**],...

We also have the following structural result of S. Kvamme.

THEOREM 6.11. *(*[**33**, Theorem A]*) Every weakly idempotent complete $d$-exact category is equivalent (as $d$-exact category) to a $d$-cluster tilting subcategory in a weakly idempotent complete exact category. Furthermore, the ambient exact category is unique up to exact equivalence.*

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