The Yoneda category and effaceable functors

1. Synopsis

For an exact category we introduce its Yoneda category and the category of Yoneda effaceables. The category of Yoneda effaceables is a Frobenius category. We show that there is a triangle equivalence between the bounded derived category of the effaceable functors and the stable Yoneda effaceables. As an application, we show that the 2-functor assigning to an exact category its effaceable functors is preserving homological exactness.
What is new? The main result is new in this generality but known for finite-dimensional modules over finite-dimensional algebras.

2. Introduction

The category of effaceable functors is an abelian category which we can assign to every exact category. It is always an extension-closed subcategory in the category of all additive functors on $\mathcal{E}$. Similar to Auslander's correspondence for exact categories, cf. [**14**], the exact structure of $\mathcal{E}$ corresponds (by taking effaceable functors $\operatorname{eff}(\mathcal{E})$) to certain Serre subcategories in $\mathcal{P}^2(\mathcal{E})$, cf. [**10**] and Chapter 2. But unlike Auslander correspondence, many (non-equivalent) exact categories can have equivalent effaceable functor categories. In this case we say they are **stable equivalence** to each other.

Opposite to the Auslander categories effaceable functors still contain some residue of homological properties of an exact category. This was presumably also a motivation for Auslander and Reiten's series of papers [**1**], [**2**], [**3**],[**4**],[**5**] on stable equivalence of dualizing R-varieties (the category of finitely presented functors on the stable module category is the category of effaceable functors).

We show as a corollary of the second theorem:

Theorem 2.1. *(cf. Theorem 8.2) If $\mathcal{E} \to \mathcal{F}$ is a homologically exact functor between exact categories, then $\operatorname{eff}(\mathcal{E}) \to \operatorname{eff}(\mathcal{F})$ is also homologically exact.*

Furthermore, we proved the following: If an exact category $\mathcal{E}$ has enough projectives (resp. enough injectives) then so has $\operatorname{eff}(\mathcal{E})$. If $\mathcal{E}$ has enough injectives then $\operatorname{gldim} \operatorname{eff}(\mathcal{E}) \leq 3 \operatorname{gldim} \mathcal{E} - 1$, cf. Cor. 4.10. If $\mathcal{E}$ is a Frobenius category then so is $\operatorname{eff}(\mathcal{E})$.

Theorem 2.2. *(cf. Theorem 8.1) Let $\mathcal{E}$ be a weakly idempotent complete exact category. Then there is a triangle equivalence*

$$\mathrm{D}^b(\operatorname{eff}(\mathcal{E})) \to \underline{\mathcal{Y}\operatorname{eff}(\mathcal{E})}.$$

This result has been proven in [**16**] for $\mathcal{E} = kQ \operatorname{mod}$ with Q Dynkin quiver and k a field, for an arbitrary finite acyclic quiver Q in [**18**] and for $\mathcal{E} = \Lambda \operatorname{mod}$ with Λ a finite-dimensional algebra in [**12**].

What about the hereditary case? Using Neeman's result we would have a Verdier localization sequence

$$\mathrm{D}^b(\operatorname{eff}(\mathcal{E})) \to \mathrm{K}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E})$$

Given two hereditary exact categories, when are they derived equivalent? In this case $\mathcal{Y}\text{eff} = \text{mod}_1 \, \mathrm{D}^b(\mathcal{E})$ and derived equivalence implies derived stable equivalence.

3. The Yoneda category

Definition 3.1. Let $\mathcal{E}$ be the full subcategory of $\mathrm{D}^b(\mathcal{E})$ given by the essential image of stalk complexes in degree 0. We define the following full subcategory of $\mathrm{D}^b(\mathcal{E})$

$$\mathcal{Y}(\mathcal{E}) := \text{add}\{E[n] \mid n \in \mathbb{Z}, E \in \mathcal{E}\}$$

as the **Yoneda category** of $\mathcal{E}$. More generally, for every admissible exact subcategory $\mathcal{C}$ in a triangulated category $\mathcal{T}$, we define $\mathcal{Y}_{\mathcal{T}}(\mathcal{C}) = \text{add}\{C[n] \mid n \in \mathbb{Z}, C \in \mathcal{C}\}$ as the Yoneda category of $\mathcal{C}$ in $\mathcal{T}$.

The Yoneda category is an additive category (with an autoequivalence). The extension-closure of $\mathcal{Y}(\mathcal{E})$ in $\mathrm{D}^b(\mathcal{E})$ is $\mathrm{D}^b(\mathcal{E})$.

Lemma 3.2. *Assume that $\mathcal{T}$ is triangulated category and that $n \geq 1$. For an admissible exact category $\mathcal{C}$ of a triangulated category $\mathcal{T}$ we consider:*

(1) $\text{Hom}(\mathcal{C}, \mathcal{C}[> n]) = 0$,
(2) $\mathcal{C}[n] * \mathcal{C} = \mathcal{C}[n] \oplus \mathcal{C}$,
(3) $\mathcal{C}[n] * \mathcal{C} \subseteq \mathcal{Y}_{\mathcal{T}}(\mathcal{C})$.

Then we have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3)*, and if $\mathcal{T}$ is also Krull-Schmidt then we also have* (3) $\Rightarrow$ (2)*.*

Proof. The equivalence of (1) and (2) is trivially true and also the implication from (2) to (3). We just show that (3) implies (2). As $\mathcal{T}$ is Krull-Schmidt and $\text{Hom}(\mathcal{C}[n], \mathcal{C}) = 0$, it follows by [17], Prop.2.1 (1) that $\mathcal{C}[n] * \mathcal{C}$ is closed under summands. Assume (3), let $C[a] \in \mathcal{C}[n] * \mathcal{C}$ for some $a \in \mathbb{Z}$ and $C \in \mathcal{C}$. To show (2), we need to conclude that $a \in \{n, 0\}$. There exists a triangle $C_1[n] \to C[a] \to C_2 \to C_1[n+1]$ with $C_1, C_2 \in \mathcal{C}$. For $a > 0$ we have $\text{Hom}(C[a], C_2) = 0$ and then $a = n$ or $C = 0$. If $a < n$ then $\text{Hom}(C_1[n], C[a]) = 0$ and therefore $a = 0$ or $C = 0$.

□

We may also recall here the following result:

Theorem 3.3. *([15], Cor.1.2) Let $\mathcal{C}$ be an admissible exact subcategory in a triangulated category. The following are equivalent*

(1) *$\mathcal{C}$ is h-admissible hereditary abelian*
(2) $\mathcal{C}[1] * \mathcal{C} = \mathcal{C}[1] \oplus \mathcal{C} = \mathcal{C} * \mathcal{C}[1]$
(3) $\mathcal{Y}_{\mathcal{T}}(\mathcal{C}) = \mathcal{T}$.

Remark 3.4. In the situation of the corollary, in loc. cit. a realization functor is constructed without the assumption that $\mathcal{T}$ is algebraic.

We can look at **Y** the category of Yoneda categories (of small exact categories), where morphisms are given by additive functors which preserve degree i-objects ($i \in \mathbb{Z}$) and commute with the shift functor.
Let **Ex** be the category of small exact categories with morphisms given by exact functors. We leave it to the reader to formulate this for 2-categories.

Remark 3.5. We consider the assignment $\mathcal{E} \mapsto \mathcal{Y}(\mathcal{E})$ and an exact functor f is mapped to the induced functor $\mathcal{Y}(f)$ on Yoneda categories. This defines a functor

$$\mathcal{Y} \colon \mathbf{Ex} \to \mathbf{Y}$$

Furthermore:
An exact functor f is homologically exact if and only if $\mathcal{Y}(f)$ is fully faithful. An exact functor f is an exact equivalence if and only if $\mathcal{Y}(f)$ is an equivalence (If $\mathcal{Y}(f)$ is an equivalence f is homologically exact and an equivalence as we get an induced equivalence on degree 0 elements).

4. Effaceable functors

Effaceable functors (also called *category of defects*) appeared prominently in several places in the literature. We collect here where it appears, examples and properties of these categories. Let us start with the definition.

Definition 4.1. Let $\mathcal{E}$ be an exact category, we define $\operatorname{eff}(\mathcal{E})$ to be the full subcategory of $\operatorname{mod}_1 \mathcal{E}$ given by functors $F\colon \mathcal{E}^{op} \to (Ab)$ such that there exists an exact sequence $X \to Y \to Z$ in $\mathcal{E}$ such that

$$\operatorname{Hom}_{\mathcal{E}}(-, Y) \to \operatorname{Hom}_{\mathcal{E}}(-, Z) \to F \to 0$$

is exact.
Dually, we define $\mathcal{E} - \operatorname{eff} \subseteq \mathcal{E}\operatorname{Mod}(= \operatorname{Mod}\mathcal{E}^{op})$ as the full subcategory of functors $F\colon \mathcal{E} \to (Ab)$ such that there exists an exact sequence $X \to Y \to Z$ in $\mathcal{E}$ such that

$$\operatorname{Hom}_{\mathcal{E}}(Y, -) \to \operatorname{Hom}_{\mathcal{E}}(X, -) \to F \to 0$$

Observe that by definition $\mathcal{E} - \operatorname{eff} = \operatorname{eff}(\mathcal{E}^{op})$ as we would expect.

Let us denote by $i\colon \mathcal{E} \to \mathcal{E}^{ic}, X \mapsto (X, 1)$ the idempotent completion of an exact category described in [8] - the functor i is fully faithful, exact and reflects exactness cf. loc. cit. The essential image of i is extension-closed, generating and cogenerating, so the induced derived functor $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}^{ic})$ is fully faithful (cp. [6, Cor 2.12]).
Furthermore, one can show using [7, Lemma 21] that $\mathcal{E}$ is weakly idempotent complete if and only if the essential image of i is deflation-closed if and only if it is inflation-closed.

Lemma 4.2. *Let $\mathcal{E}$ be an exact category.*

(i) $\operatorname{eff}(\mathcal{E})$ *is extension-closed in* $\operatorname{mod}_1 \mathcal{E}$*.*
(ii) *If $\mathcal{E}$ is idempotent complete then* $\operatorname{eff}(\mathcal{E})$ *is idempotent complete.*
(iii) *We have* $\operatorname{eff}(\mathcal{E}) = \operatorname{eff}(\mathcal{E}^{ic})$ *is idempotent complete.*

Proof. (i) The category $\operatorname{eff}(\mathcal{E})$ is extension-closed in the so called category of admissible presentable functors by [14, Proposition 3.6], and the latter is extension-closed subcategory of $\operatorname{Mod}(\mathcal{E})$ by [14, Proposition 3.5].
(ii) Assume $\mathcal{E}$ is idempotent complete. By [14, Corollary 3.18] $\operatorname{eff}(\mathcal{E})$ is an additively closed subcategory (e.g. it is part of a torsion pair) of an idempotent complete additive category, so it is idempotent complete itself.
(iii) We see $\operatorname{eff}(\mathcal{E})$ as a full subcategory of $\operatorname{eff}(\mathcal{E}^{ic})$ (using the universal property of the idempotent completion of an additive category).
Given a functor $F \in \operatorname{eff}(\mathcal{E}^{ic})$ we can choose an $\mathcal{E}^{ic}$-exact sequence $(Z, 1) \to (X, p) \xrightarrow{d} (Y, 1)$ such $F = \operatorname{coker}\operatorname{Hom}(-, d)$ (because given a short exact sequence $(A, a) \to (B, b) \xrightarrow{\varphi} (C, c)$ with $(C, c) \oplus (C, 1 - c) = (C, 1), (A, a) \oplus (A, 1 - a) = (A, 1)$ we can just add the following sequences $0 \to (C, 1 - c) \xrightarrow{id} (C, 1 - c)$ and $(A, 1 - a) \xrightarrow{id} (A, 1 - a) \to 0$, this does not change the cokernel of $\operatorname{Hom}(-, \varphi)$). As $\mathcal{E}$ is extension-closed in $\mathcal{E}^{ic}$ it follows that $F \in \operatorname{eff}(\mathcal{E})$. Then by (ii) it follows that $\operatorname{eff}(\mathcal{E})$ is idempotent complete.

$\square$

Theorem 4.3. *([20], Lemma 9) Let $\mathcal{E}$ be an exact category. Then* $\operatorname{eff}(\mathcal{E})$ *as fully exact subcategory of* $\operatorname{Mod}\mathcal{E}$*, is abelian.*

Remark 4.4. The previous result is proven only for idempotent complete exact categories but by Lemma 4.2, this implies it is true for all exact categories.

Lemma 4.5. *If $\phi\colon \mathcal{E} \to \mathcal{F}$ is an exact functor, there exists a well-defined exact functor*

$$\overline{\phi}\colon \operatorname{eff}(\mathcal{E}) \to \operatorname{eff}(\mathcal{F})$$

defined on objects via $\overline{\phi}(\operatorname{coker} \operatorname{Hom}_{\mathcal{E}}(-,d)) := \operatorname{coker} \operatorname{Hom}_{\mathcal{F}}(-,\phi(d))$ *for $\mathcal{E}$-deflations d.*

Proof. By [**14**], Thm 3.9 (2), there exists such an exact functor on the Auslander exact categories. As it restricts to the functor $\overline{\phi}\colon \operatorname{eff}(\mathcal{E}) \to \operatorname{eff}(\mathcal{F})$, it is automatically well-defined and exact. $\square$

Definition 4.6. Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then the **stable category** denoted by $\underline{\mathcal{E}}$ is defined as the ideal quotient category. For every two objects $X, Y \in \mathcal{E}$ let $\mathcal{P}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{E}}(X,Y)$ to be the subgroup of all morphisms factoring through a projective object. This defines an ideal in the category $\mathcal{E}$. Then $\underline{\mathcal{E}}$ has the same objects as $\mathcal{E}$ but morphisms are defined as

$$\operatorname{Hom}_{\underline{\mathcal{E}}}(X,Y) := \underline{\operatorname{Hom}}_{\mathcal{E}}(X,Y) = \operatorname{Hom}_{\mathcal{E}}(X,Y)/\mathcal{P}(X,Y)$$

This defines an additive category. Dually if $\mathcal{E}$ has enough injectives then we define $\overline{\mathcal{E}} = (\underline{\mathcal{E}^{op}})^{op}$.

Observe that the stable category of $\mathcal{E}$ is an additive category with an endofunctor, given by taking syzygies Ω. If in addition $\mathcal{E}$ is a Frobenius category then its stable category has the structure of a triangulated category with Ω^{-1} being the suspension functor and the distinguished triangles induced by short exact sequences (cp. [**13**]).
In general, one can either study this as a pretriangulated category or use the Heller stabilization to obtain a triangulated category from the stable category.

Let us observe the following easy

Lemma 4.7. *Let $\mathcal{E}$ be an exact category.*

(1) *If $\mathcal{E}$ has enough projectives then for every morphism $\underline{g}$ in the category $\underline{\mathcal{E}}$ there exists an $\mathcal{E}$-deflation d with $\underline{d} = \underline{g}$.*
(2) *If $\mathcal{E}$ has enough injectives then for every morphism $\overline{g}$ in the category $\overline{\mathcal{E}}$ there exists an inflation i such that $\overline{i} = \overline{g}$*

Proof. If $g\colon X \to Y$ and (1) if $\mathcal{E}$ has enough projectives, take a deflation $p\colon P \to Y$ with P projective and then using the pullback of p along g we have an induced deflation $d = [g,p]\colon X \oplus P \to Y$ with $\overline{d} = \overline{g}$. (2) If $\mathcal{E}$ has enough injectives, take an inflation $j\colon X \to I$ with I injective and form the pushout to obtain an inflation $i = \binom{g}{j}\colon X \to Y \oplus I$ with $\overline{i} = \overline{g}$. $\square$

For an additive category $\mathcal{P}$ we call $\operatorname{mod}_\infty \mathcal{P}$ to be the full subcategory of $\operatorname{Mod} \mathcal{P}$ of all additive functors $F\colon \mathcal{P}^{op} \to (Ab)$ such that there exists an exact sequence in

$$\operatorname{Hom}(-,P_n) \to \cdots \to \operatorname{Hom}(-,P_0) \to F \to 0$$

with $P_i \in \mathcal{P}$. This is a fully exact subcategory of $\operatorname{Mod} \mathcal{P}$ which has enough projectives and the Yoneda embedding $\mathcal{P}^{ic} \to \mathcal{P}(\operatorname{mod}_\infty \mathcal{P})$ induces an equivalence of additive categories.
Whenever we have an exact category $\mathcal{F}$ with enough projectives $\mathcal{P}$, then we have a functor

$$\mathbb{P}\colon \mathcal{F} \to \operatorname{mod}_\infty \mathcal{P}, \quad X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{P}^{op}}$$

which is homologically exact and induces an equivalence of $\mathcal{F}^{ic}$ to a resolving subcategory of $\operatorname{mod}_\infty \mathcal{P}$ but usually this is not essentially surjective.
Dually given an additive category $\mathcal{I}$ we define the category $\mathcal{I} \operatorname{mod}^\infty := (\operatorname{mod}_\infty \mathcal{I}^{op})^{op}$, this is an exact category with enough injectives and the Yoneda embedding

$\mathcal{I} \to (\mathrm{mod}_\infty \mathcal{I}^{op})^{op}, I \mapsto \mathrm{Hom}_{\mathcal{I}}(I,-)$ induces an equivalence $\mathcal{I}^{ic} \to \mathcal{I}((\mathrm{mod}_\infty \mathcal{I}^{op})^{op})$. Whenever an exact category $\mathcal{F}$ has enough injectives $\mathcal{I}$ then we consider

$$\mathbb{I}\colon \mathcal{F} \to (\mathrm{mod}_\infty \mathcal{I}^{op})^{op}, \quad X \mapsto \mathrm{Hom}(X,-)|_{\mathcal{I}}$$

this is homologically exact and induces an equivalence of $\mathcal{F}^{ic}$ to a coresolving subcategory of $(\mathrm{mod}_\infty \mathcal{I}^{op})^{op}$.

The following first part is [**10**], Lemma 2.13 (in the idempotent complete case)

Proposition 4.8. *Let $\mathcal{E}$ be an exact category.*

(1) *If $\mathcal{E}$ is an exact category with enough projectives, then* $\mathrm{eff}(\mathcal{E})$ *has enough projectives. The Yoneda functor $\underline{\mathcal{E}} \to \mathrm{mod}_1 \underline{\mathcal{E}}, X \mapsto \mathrm{Hom}_{\underline{\mathcal{E}}}(-,X)$ induces an equivalence of additive categories $(\underline{\mathcal{E}})^{ic} \to \mathcal{P}(\mathrm{eff}(\mathcal{E}))$. Furthermore, in this case, the functor $\mathbb{P}$ induces an equivalence*

$$\mathbb{P}\colon \mathrm{eff}(\mathcal{E}) \to \mathrm{mod}_\infty \underline{\mathcal{E}}$$

(2) *If $\mathcal{E}$ is an exact category with enough injectives, then* $\mathrm{eff}(\mathcal{E})$ *also has enough injectives. Furthermore, the functor $X \mapsto \mathrm{Ext}^1_{\mathcal{E}^{ic}}(-,X)$ gives an equivalence of additive categories $(\overline{\mathcal{E}})^{ic} \to \mathcal{I}(\mathrm{eff}(\mathcal{E}))$. Furthermore, we have an exact equivalence*

$$\mathbb{I}\colon \mathrm{eff}(\mathcal{E}) \to (\mathrm{mod}_\infty (\overline{\mathcal{E}})^{op})^{op}$$

PROOF. (1) The proof in [**10**, Lemma 2.13] works also if $\mathcal{E}$ is not idempotent complete. For $\mathbb{P}$ essentially surjective, the main argument is just Lemma 4.7,(1).
(2) Again the essentially surjectivity of $\mathbb{I}$ follows from Lemma 4.7, (2). □

Remark 4.9. In the light of the previous Proposition, it is sensible for arbitrary exact categories to define two exact categories as **stably equivalent** if there exists an equivalence ϕ between their effaceable functor categories (observe that additive equivalences between abelian categories are exact). In this case we would call ϕ the stable equivalence. It can be that a stable equivalence is induced by an exact functor as in Lemma 4.5 or it can also not be induced by a functor between the exact categories.
So continuing Auslander-Reiten's quest would mean: Try to classify/understand exact categories up to stable equivalence.

Corollary 4.10. *If $\mathcal{E}$ is an exact category with enough injectives and assume that* $\mathrm{gldim}\,\mathcal{E} \leq n$*, then we have*

$$\mathrm{gldim}\,\mathrm{eff}(\mathcal{E}) \leq 3n-1$$

This is an obvious generalization of [**1**, Prop. 10.2].

PROOF. Let $F \in \mathrm{eff}(\mathcal{E})$, then there exists an exact sequence $A \to B \xrightarrow{g} C$ such that $F = \mathrm{coker}\,\mathrm{Hom}_{\mathcal{E}}(-,g)$. So by definition, the long exact sequence when applying a functor $\mathrm{Hom}(X,-)$ induces an exact sequence of functors on $\underline{\mathcal{E}}$

$$\begin{aligned} 0 \to F \to \mathrm{Ext}^1_{\mathcal{E}}(-,A) \to \mathrm{Ext}^1_{\mathcal{E}}(-,B) \quad &\to \mathrm{Ext}^1_{\mathcal{E}}(-,C) \\ \to \mathrm{Ext}^2_{\mathcal{E}}(-,A) \to \cdots \qquad\qquad\quad &\to \mathrm{Ext}^n_{\mathcal{E}}(-,C) \to 0 \end{aligned}$$

as $\mathrm{Ext}^i_{\mathcal{E}}(-,X) \cong \mathrm{Ext}^1_{\mathcal{E}}(-,\Omega^{-(i-1)}X) \in \mathcal{I}(\mathrm{eff}(\mathcal{E}))$ by Prop. 4.8,(2), the claim follows. □

Remark 4.11. If $\mathcal{E}$ is an exact category with enough projectives, the category of effaceables are just the category $\mathrm{mod}_\infty \underline{\mathcal{E}}$, so its global dimension can be determined by *higher* weak kernels in the additive category $\underline{\mathcal{E}}$ (in the sense of Enomoto).
In particular, if $\mathcal{E}$ has enough projectives the following are equivalent

(1) $\mathrm{gldim}\,\mathrm{eff}(\mathcal{E}) = 0$
(2) $\underline{\mathcal{E}}$ is abelian semi-simple
(3) Every non-isomorphism in $\mathcal{E}$ factors through a projective

In particular, we can find examples of exact categories of all global dimension (including ∞) such that the category of effaceable functors is semi-simple abelian. Take $\mathcal{E}$ an abelian Krull-Schmidt category such that every indecomposable is either projective or simple and all simples Hom-orthogonal (e.g. take a finite dimension Nakayama algebra Λ and pass to the quotient $\Lambda/\operatorname{rad}^2$). We look at $\Lambda_n = k(1 \to 2 \to \cdots \to n)/\operatorname{rad}^2$ (of global dimension n) and $\Lambda_\infty = k[X]/(X^2)$ (of infinite global dimension), then $\mathcal{E}_n = \Lambda_n \operatorname{mod}$ has semi-simple effaceable functors for all $n \leq \infty$.

Lemma 4.12. *If $\mathcal{E}$ is a Frobenius category then* $\operatorname{eff}(\mathcal{E})$ *is also a Frobenius category and* $\operatorname{Hom}_{\underline{\mathcal{E}}}(-, \Omega^- X) \cong \operatorname{Ext}^1_{\mathcal{E}}(-, X)$ *for all $X \in \mathcal{E}$.*

Proof. By Happel [**13**], $\underline{\mathcal{E}}$ is triangulated, then there is a general result that $\operatorname{mod}_\infty \underline{\mathcal{E}}$ is a Frobenius category by a Theorem of Freyd (cf. [**11**], Thm 1.7). The last statement is more generally proven in Lemma 6.1. □

5. Yoneda-effaceable functors

Definition 5.1. We define the category of **Yoneda-effaceables** $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ to be the full subcategory of $\operatorname{mod}_\infty \mathcal{Y}(\mathcal{E})$ given by functors X such that there exists a triangle in $\mathrm{D}^b(\mathcal{E})$

$$A \to B \xrightarrow{f} C \to A[1]$$

with $A, B, C \in \mathcal{Y}(\mathcal{E})$ such that $X \cong \operatorname{Coker} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, f)$, that is, X admits a presentation as

$$\operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, B) \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, C) \to X \to 0$$

and we say that X is *presented* by f. In practice, we say that a Yoneda effaceable functor is presented by a map f and we implicitly assume that the domain, codomain and cone (in $\mathrm{D}^b(\mathcal{E})$) of f are in $\mathcal{Y}(\mathcal{E})$. In particular, any functor in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is finitely presented (a.k.a. coherent).

We also have the following harmless looking result - which has a lengthy proof which is only completed in Lemma 5.13.

Proposition 5.2. *Let $f\colon X \to Y$ a morphism in $\mathcal{Y}(\mathcal{E})$ and $F = \operatorname{coker} \operatorname{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, f)$. Then: $F|_{\mathcal{Y}(\mathcal{E})} \in \mathcal{Y}\mathrm{eff}(\mathcal{E})$ if and only if $\operatorname{cone}(f) \in \mathcal{Y}(\mathcal{E})$.*

Example 5.3. If $\mathcal{E}$ is hereditary abelian, we know by Thm 3.3 that $\mathcal{Y}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$, in particular for every morphism $f\colon X \to Y$ in $\mathcal{Y}(\mathcal{E})$ we have $\operatorname{cone}(f) \in \mathcal{Y}(\mathcal{E})$.
Therefore the category of Yoneda-effaceables $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ coincides with the (Frobenius exact) category $\operatorname{mod}_1 \mathrm{D}^b(\mathcal{E})$ (of finitely presented functors $\mathrm{D}^b(\mathcal{E})^{op} \to (Ab)$).

Definition 5.4. Let $\mathcal{E}$ be an exact category and $\mathcal{M}$ a full subcategory. We say $\mathcal{M}$ is **generating** if for every $E \in \mathcal{E}$ there exists a deflation $d\colon M \to E$ with $M \in \mathcal{M}$. It is called **deflation-closed** if for every short exact sequence $X \to Y \to Z$ with $Y, Z \in \mathcal{M}$ also $X \in \mathcal{M}$ holds.
A **resolving subcategory** in an exact category is fully exact subcategory which is deflation-closed and generating. It is a **coresolving subcategory** if it is resolving in the opposite category. A **biresolving subcategory** in an exact category is a fully exact subcategory which is resolving and coresolving.

We often use the following:

Remark 5.5. (cf. [**9**, 2.6]) If $\mathcal{E}$ is an exact category and $\mathcal{F}$ is a fully exact subcategory closed under direct summands.
If $\mathcal{E}$ has enough projectives $\mathcal{P}$, $\mathcal{P} \subseteq \mathcal{F}$ and $\mathcal{F}$ closed under syzygies then $\mathcal{F}$ is resolving.
If $\mathcal{E}$ has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ both contained in $\mathcal{F}$ and $\mathcal{F}$ is closed under syzygies and under cosyzygies, then $\mathcal{F}$ is biresolving.

Observe that biresolving subcategories in a Frobenius category are always again Frobenius categories (with the same projective-injectives).

Definition 5.6. Given an exact category $\mathcal{F}$ we denote by $\mathcal{P}(\mathcal{F})$ the full subcategory of projectives in $\mathcal{F}$. Let $\mathcal{C}$ be a fully exact subcategory. We call $\mathcal{C}$ **partially resolving** if $\mathcal{C}$ is deflation-closed, summand-closed and for every $C \in \mathcal{C}$ there exists an $\mathcal{F}$-deflation $d\colon P \to C$ with $P \in \mathcal{P}(\mathcal{F})$. Dually we define **partially coresolving** if $\mathcal{C}^{op}$ is partially resolving in $\mathcal{F}^{op}$.
We call $\mathcal{C}$ **partially biresolving** if it is partially resolving and partially coresolving.

Remark 5.7. We have the following (cf. Chapter 1)

(1) Let $\mathcal{C}$ be fully exact in an exact category $\mathcal{F}$ and closed under taking summands in $\mathcal{F}$. Then $\mathcal{C}$ is partially resolving if and only if for every $C \in \mathcal{C}$ there exists an $\mathcal{F}$-exact sequence $C' \rightarrowtail P \twoheadrightarrow C$ with $P \in \mathcal{P}(\mathcal{F}), C' \in \mathcal{C}$.
(2) If $\mathcal{C}$ is partially resolving then $\mathcal{C}$ has enough projectives with $\mathcal{P}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{F})$. If $\mathcal{C}$ is partially biresolving then it has enough projectives and enough injectives - therefore it is a Frobenius category if and only if $\mathcal{P}(\mathcal{C}) = \mathcal{I}(\mathcal{C})$ holds.
(3) If $\mathcal{C}$ is partially resolving in $\mathcal{F}$ then it is homologically exact in $\mathcal{F}$.

Lemma 5.8. *Let $f\colon \mathcal{E} \to \mathcal{E}'$ be an exact functor which is homologically exact. Then we have a fully faithful embedding $\mathcal{Y}(\mathcal{E}) \subseteq \mathcal{Y}(\mathcal{E}')$. The full subcategory*
$\mathcal{C} = \{F \in \mathcal{Y}\mathrm{eff}(\mathcal{E}') \mid \exists f\colon X \to Y \text{ in } \mathcal{Y}(\mathcal{E}), \mathrm{cone}(f) \in \mathcal{Y}(\mathcal{E})\}$ is partially resolving in $\mathcal{Y}\mathrm{eff}(\mathcal{E}')$ and the restriction functor $\mathcal{C} \to \mathcal{Y}\mathrm{eff}(\mathcal{E}), F \mapsto F|_{\mathcal{Y}(\mathcal{E})}$ is an exact equivalence.
In particular, the induced triangle functor

$$\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E}')}$$

is fully faithful

Proof. By the usual horseshoe argument $\mathcal{C}$ is extension-closed. By definition it has enough projectives and enough injectives equivalent to $\mathcal{Y}(\mathcal{E})$, therefore it partially biresolving. It is straight-forward to see that the restriction $\mathcal{C} \to \mathcal{Y}\mathrm{eff}(\mathcal{E}), F \mapsto F|_{\mathcal{Y}(\mathcal{E})}$ is an exact equivalence. As $\mathcal{C}$ is homologically exact in $\mathcal{Y}\mathrm{eff}(\mathcal{E}')$, we have induced isomorphisms on Ext-groups and by Lemma 6.1 these calculate the homomorphisms in the stable category, i.e. we have for all X, Y in $\mathcal{C}$ and $n \geq 1$

$$\mathrm{Hom}_{\underline{\mathcal{C}}}(X, \Omega^{-n}Y) \cong \mathrm{Ext}^n_{\mathcal{C}}(X,Y) = \mathrm{Ext}^n_{\mathcal{Y}\mathrm{eff}(\mathcal{E}')}(X,Y) \cong \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E}')}}(X, \Omega^{-n}Y)$$

as every object in $\mathcal{C}$ is a cosyzygy for some $n \geq 1$, the claim follows. □

Lemma 5.9. *We have $\mathcal{Y}\mathrm{eff}(\mathcal{E}) \subseteq \mathbf{Gp}(\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E}))$ is extension-closed. Furthermore, as fully exact category, $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is biresolving and therefore as a fully exact subcategory, it is a Frobenius exact category.*

Proof. Let $\mathcal{Y}$ denote $\mathcal{Y}(\mathcal{E})$. Let X be a Yoneda-effaceable functor presented by the presented by the map f fitting in the triangle $A \to B \xrightarrow{f} C \to A[1]$ in $\mathrm{D}^b(\mathcal{E})$ with A, B and C in $\mathcal{Y}$. Then the complex of finitely generated projective $\mathcal{Y}$–modules

$$\ldots \to \mathcal{Y}(-,A) \to \mathcal{Y}(-,B) \to \mathcal{Y}(-,C) \xrightarrow{d_0} \mathcal{Y}(-,A[-1]) \to \ldots$$

is totally acyclic and X is the image of d_0. Therefore X is in $\mathrm{GP}(\mathrm{mod}_\infty(\mathcal{Y}))$. Note that $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is extension closed. Indeed, consider and short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathbf{Gp}(\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E}))$ with X and Z in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ and chose a presenting maps $f\colon B \to C$ and $f'\colon B' \to C'$ of X and Z, respectively. Use the horseshoe lemma in $\mathrm{Mod}\,\mathcal{Y}(\mathcal{E})$ to produce a complex of the form

$$\ldots \to (-,\mathrm{A}'') \to (-, B \oplus B') \xrightarrow{(f'')^*} (-, C \oplus C') \to Y \to 0$$

where f'' is the matrix $\begin{pmatrix} f & 0 \\ g & f' \end{pmatrix}$ for some $g\colon B \to C'$. We claim that f'' is a presenting map for Y. Let D denote the cocone of f'' in $\mathrm{D}^b(\mathcal{E})$. Then comparing the long exact sequence obtained from the triangle induced by f'' and the previous complex obtained from the horseshoe lemma, we obtain that $(-,A'') \cong (-,D)$ as functors from $\mathcal{Y}(\mathcal{E})$ to (Ab). Then Lemma 5.10 will give us that $A'' \cong D$ in $\mathrm{D}^b(\mathcal{E})$. But A'' is in $\mathcal{Y}(\mathcal{E})$. This completes the claim.

It remains to show that $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is Frobenius exact as fully exact subcategory. In fact we need to see that it contains the projective-injectives and is closed under syzygies and cosyzygies but all three claims are clear when looking at the long exact sequence above. $\square$

Lemma 5.10. *Let $L, M \in \mathrm{D}^b(\mathcal{E})$ and $M \in \mathcal{Y}(\mathcal{E})$.*
If $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, L)|_{\mathcal{Y}(\mathcal{E})} \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, M)|_{\mathcal{Y}(\mathcal{E})}$ then $L \cong M$ as objects in $\mathrm{D}^b(\mathcal{E})$.

Proof. As $M \in \mathcal{Y}(\mathcal{E})$, there exists a morphism $f\colon M \to L$ (in $\mathcal{Y}(\mathcal{E})$) which corresponds to $\mathrm{id}_M\colon M \to M$ under the assumed natural isomorphism of functors. This induces a natural transformation $f^*\colon \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, M) \to \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, L)$ which is an equivalence when restricted to $\mathcal{Y}(\mathcal{E})$. The extension-closure of $\mathcal{Y}(\mathcal{E})$ in $\mathrm{D}^b(\mathcal{E})$ is $\mathrm{D}^b(\mathcal{E})$ and using the long exact sequences obtained when applying $\mathrm{Hom}(-, M)$ and $\mathrm{Hom}(-, L)$ we conclude that f^* is an isomorphism of functors. Then by the Yoneda-embedding, a quasi-inverse functor is given by an inverse morphism for f and f has to be an isomorphism in $\mathrm{D}^b(\mathcal{E})$. $\square$

The shift functor $[1]$ in $\mathcal{D} := \mathrm{D}^b(\mathcal{E})$ induces by precomposition an autoequivalence on $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ which maps representable (i.e. projectives) to projectives, therefore we have induced quasi-inverse autoequivalences

$$[1]_{\mathcal{D}}\colon \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \leftrightarrow \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}\colon [-1]_{\mathcal{D}}.$$

As $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is a Frobenius exact category we also have the quasi-inverse equivalences

$$\Sigma := \Omega^-\colon \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \leftrightarrow \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}\colon \Omega =: \Sigma^-$$

given by taking cosyzygies and syzygies (they are the suspension and cosuspension of the triangulated structure discussed before, therefore we will rename them as $\Sigma^{\pm}$).

Then the following corrollary is immediate from the previous lemma.

Corollary 5.11. *We have a natural isomorphism of functors $\Omega^3 = [1]_{\mathcal{D}}$ on $\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$. Furthermore, we have for all $F, G \in \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ there exists an $n = n_{F,G} << 0$ such that*

$$\mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(F, \Sigma^{<n} G) = \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(\Sigma^{>(-n)} F, G) = 0$$

Proof. The statement is obvious. $\square$

We make the following auxilliary definition.

Observe that $\mathcal{F} = \mathrm{mod}_1\, \mathrm{D}^b(\mathcal{E}) = \mathrm{mod}_\infty\, \mathrm{D}^b(\mathcal{E}) = \mathbf{Gp}(\mathrm{mod}_\infty\, \mathrm{D}^b(\mathcal{E})$ as every projective presentation of a functor can be extended to a complete projective resolution by taking the associated completion of a morphism to a distinguished triangle. This is a Frobenius category.

Definition 5.12. We define $\widetilde{\mathcal{Y}\mathrm{eff}}$ to be the full subcategory of $\mathrm{mod}_1\, \mathrm{D}^b(\mathcal{E})$ given by all F such that there exists an $f\colon X \to Y$ in $\mathcal{Y}(\mathcal{E})$ with $\mathrm{cone}(f) \in \mathcal{Y}(\mathcal{E})$ such that $F = \mathrm{coker}\,\mathrm{Hom}(-, f)$.

By the horseshoe lemma it is obvious that $\widetilde{\mathcal{Y}\mathrm{eff}}$ is extension-closed in $\mathcal{F}$.

Lemma 5.13. *Let $\mathcal{E}$ be idempotent complete.*

(1) *Then $\widetilde{\mathcal{Y}\mathrm{eff}}$ is partially biresolving in $\mathcal{F} = \mathrm{mod}_1\, \mathrm{D}^b(\mathcal{E})$. Furthermore, it is a Frobenius category.*
(2) *A morphism $f \in X \to Y$ in $\mathcal{Y}(\mathcal{E})$ the following are equivalent:*
   (a) $\mathrm{cone}_{\mathrm{D}^b(\mathcal{E})}(f) \in \mathcal{Y}(\mathcal{E})$
   (b) $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, f)$ *is $\widetilde{\mathcal{Y}\mathrm{eff}}$-admissible*
   (c) $\mathrm{coker}\,\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-, f) \in \widetilde{\mathcal{Y}\mathrm{eff}}$
(3) *The restriction functor $\widetilde{\mathcal{Y}\mathrm{eff}} \to \mathcal{Y}\mathrm{eff}(\mathcal{E})$, $F \mapsto F|_{\mathcal{Y}(\mathcal{E})}$ is an exact equivalence.*

We just state (2) in the previous Lemma, to combine it with the equivalence in (3) - then it implies Prop. 5.2.

Proof. To see (1) use the same argument as before, (2) follows by definition. The equivalence in (3) has been considered in bigger generality in Chapter 3. $\square$

Definition 5.14. Now we define $\widetilde{\mathrm{eff}} \subseteq \mathcal{Y}\mathrm{eff}(\mathcal{E})$ to be the full subcategory given by functors X such that there exists a triangle $A \to B \xrightarrow{g} C \to A[1]$ in $\mathrm{D}^b(\mathcal{E})$ with A, B, C in $\mathcal{E}$ such that $X \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, g)$.

The category $\widetilde{\mathrm{eff}}$ is extension-closed in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ (using the same horseshoe argument as in the Lemma above). But $\widetilde{\mathrm{eff}}$ does not contain any projectives.

Lemma 5.15. *The restriction functor $\widetilde{\mathrm{eff}} \to \mathrm{eff}(\mathcal{E}), F \mapsto F|_{\mathcal{E}^{op}}$ is an exact equivalence.*

Proof. As restriction functors on functor categories are exact, also their restriction to fully exact subcategories are exact functors.
By definition is this functor is essentially surjective and using the definition it is also straight-forward to see that for an additive functor $G\colon \mathcal{Y}(\mathcal{E})^{op} \to (Ab)$, and for X such that there exists an exact sequence $A \to B \xrightarrow{g} C$ in $\mathcal{E}$ such that $X = \operatorname{coker} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, g)$ we have isomorphisms

$$\operatorname{Hom}(X, G) = \ker_{(Ab)}(G(C) \xrightarrow{G(g)} G(B)) = \operatorname{Hom}(X|_{\mathcal{E}^{op}}, G|_{\mathcal{E}^{op}})$$

therefore the functor is an equivalence of categories. As every fully faithful exact functor it induces a monomorphism on Ext^1-groups. We need to see it is surjective.

Let $0 \to G \to H \to F \to 0$ be a complex in $\widetilde{\mathrm{eff}}$ such that when evaluated at objects of $\mathcal{E}(\subseteq \mathcal{Y}(\mathcal{E}))$ this yields an exact sequence of abelian groups. We need to see that $0 \to G \to H \to F \to 0$ evaluated at $E[i]$ with $E \in \mathcal{E}, i \in \mathbb{Z}, i \neq 0$ still gives an exact sequence of abelian groups. But this follows from the next Lemma 5.16. $\square$

Lemma 5.16. *Given a two composable morphisms of distinuished triangles $X_* \xrightarrow{f_*} Y_* \xrightarrow{g_*} Z_*$ which is degree-wise split exact in a triangulated category $\mathcal{T}$ with suspension $[1]$, i.e. we have commuting diagrams*

$$\begin{tikzcd}
X_1 \arrow[r, "a_1"] \arrow[d, "f_1"'] & X_2 \arrow[r, "a_2"] \arrow[d, "f_2"'] & X_3 \arrow[r, "a_3"] \arrow[d, "f_3"'] & X_1[1] \arrow[d, "f_1[1]"'] \\
Y_1 \arrow[r, "b_1"] \arrow[d, "g_1"'] & Y_2 \arrow[r, "b_2"] \arrow[d, "g_2"'] & Y_3 \arrow[r, "b_3"] \arrow[d, "g_3"'] & X_1[1] \arrow[d, "g_1[1]"'] \\
Z_1 \arrow[r, "c_1"] & Z_2 \arrow[r, "c_2"] & Z_3 \arrow[r, "c_3"] & X_1[1]
\end{tikzcd}$$

with (f_i, g_i) is a split exact sequence for all $i \in \{1, 2, 3\}$. Let A be an object in $\mathcal{T}$ and apply $(A, -) := \operatorname{Hom}_{\mathcal{T}}(A, -)$ to obtain two morphisms of long exact sequences. Assume that $0 \to \operatorname{coker}(A, a_2) \to \operatorname{coker}(A, b_2) \to \operatorname{coker}(A, c_2) \to 0$ is an exact sequence of abelian groups, then we have that also $0 \to \operatorname{coker}(A, a_i) \to \operatorname{coker}(A, b_i) \to \operatorname{coker}(A[i], c_i) \to 0$ is an exact sequence of abelian groups for $i \in \{1, 2, 3\}$. In particular, also $0 \to \operatorname{coker}(A[i], a_2) \to \operatorname{coker}(A[i], b_2) \to \operatorname{coker}(A[i], c_2) \to 0$ is an exact sequence of abelian groups for every $i \in \mathbb{Z}$.

Proof. Apply the snake lemma in the category of abelian groups. $\square$

Definition 5.17. Let $\mathcal{T}$ be a triangulated category (we will usually denote the suspension by [1]) and $\mathcal{C} \subseteq \mathcal{T}$ be a full additively closed subcategory. Then we say $\mathcal{C}$ is **admissible exact** in $\mathcal{T}$ if it is extension-closed and non-negative (i.e. $\operatorname{Hom}_{\mathcal{T}}(C, C'[-n]) = 0$ for all $n > 0$, $C, C' \in \mathcal{C}$).

Lemma 5.18. *The composition* $\mathrm{eff}(\mathcal{E}) \cong \widetilde{\mathrm{eff}} \to \mathcal{Y}\mathrm{eff}(\mathcal{E}) \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ *is fully faithful, furthermore its essential image is an admissible exact category.*

PROOF. First we proof that $\widetilde{\mathrm{eff}} \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ is fully faithful: Given a morphism $\phi\colon X \to Y$ in $\widetilde{\mathrm{eff}}$ which factors in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ as a composition $X \xrightarrow{a} \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-, E[t]) \xrightarrow{b} Y$ with $E \in \mathcal{E}, t \in \mathbb{Z}$. We claim that $\phi = 0$ holds in $\widetilde{\mathrm{eff}}$. Using the definition of objects in $\widetilde{\mathrm{eff}}$, it is easy to see that $\mathrm{Hom}(X, E[t]) = 0$ for $t < 0$ and $\mathrm{Hom}(E[t], Y) = 0$ for $t > 0$. For $t = 0$, we show that
$\mathrm{Hom}_{\mathcal{Y}\mathrm{eff}(\mathcal{E})}(\widetilde{\mathrm{eff}}, \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-, E)) = 0$ for all $E \in \mathcal{E}$, so take a projective resolution

$$\mathrm{Hom}_{\mathcal{Y}}(-, X) \to \mathrm{Hom}_{\mathcal{Y}}(-, Y) \to \mathrm{Hom}_{\mathcal{Y}}(-, Z) \to F \to 0$$

with $\sigma\colon X \to Y \to Z$ a short exact sequence in $\mathcal{E}$. When we apply $\mathrm{Hom}_{\mathcal{Y}}(-, E)$ with $E \in \mathcal{E}$, then the conclude that $\mathrm{Hom}_{\mathcal{Y}}(F, \mathrm{Hom}_{\mathcal{Y}}(-, E)) = 0$ as it has to be the zero to start the long exact sequence associated to $\mathrm{Hom}_{\mathcal{E}}(\sigma, E)$.
Next, we are going to see that the essential image of this functor is non-negative, i.e. we will show

$$\mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(F_1, \Sigma^{<0} F_2) = 0$$

for all $F_1, F_2 \in \widetilde{\mathrm{eff}}$. By definition of the ideal quotient (Hom-sets) it is enough to show that
$\mathrm{Hom}_{\mathcal{Y}\mathrm{eff}(\mathcal{E})}(F_1, \Omega^t F_2) = 0$ for all $t \geq 1$. For $t \geq 3$ this follows directly from lifting a morphism to projective resolutions and using that $\mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(E, E'[< 0]) = 0$. For $t = 1, 2$, let $X_i \xrightarrow{a_i} Y_i \xrightarrow{b_i} Z_i$ be the short exact sequences in $\mathcal{E}$ such that $F_i = \operatorname{coker} \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-, b_i)$. For $t \in \{1, 2\}$: By definition, we have a monomorphism $\Omega^t F_2 \to \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-, A)$ where $A = Z_2$ for $t = 2$ and $A = Y_2$ for $t = 2$, now by the previous discussion, we have that $\mathrm{Hom}(F_1, \mathrm{Hom}_{\mathcal{Y}}(-, A)) = 0$. This implies also
$\mathrm{Hom}(F_1, \Omega^t F_2) = 0$ for $t = 1, 2$.
Lastly, we still have to see that the essential image is extension closed. But this follows from the next Lemma (as $\widetilde{\mathrm{eff}}$ is extension-closed in the Frobenius exact category $\mathcal{Y}\mathrm{eff}(\mathcal{E})$).

□

Lemma 5.19. *Let* $\mathcal{F}$ *be a Frobenius exact category and let* $q\colon \mathcal{F} \to \underline{\mathcal{F}}$ *be the ideal quotient functor to its stable category. If* $\mathcal{C}$ *an extension closed full subcategory in* $\mathcal{F}$, *then the essential image of* $\mathcal{C}$ *is extension closed in* $\underline{\mathcal{F}}$.

PROOF. Given a standard triangle $X \to Y \to Z \to X[1]$ in $\underline{\mathcal{F}}$ with $X, Z \in q(\mathcal{C})$. We may assume that there exist injective-projective objects P and Q in $\mathcal{F}$ such that $X = C \oplus P, Z = C' \oplus Q$ with $C, C' \in \mathcal{C}$. By the construction of the triangulated structure on $\underline{\mathcal{F}}$, we have that $Y \to Z$ is the pushout of an inflation $X \to I$ into a projective-injective object in $\mathcal{F}$ along the morphism $X \to Y$. By [**8**, Proposition 3.1], this implies that there is a short exact sequence $C \oplus P \to Y \oplus I \to C' \oplus Q$. By [**8**, Proposition 2.12], the short exact sequence splits into a direct sum of of short exact sequences $P \to P \to 0$, $0 \to Q \to Q$ and $C \to \tilde{Y} \to C'$. Since $\mathcal{C}$ is extensions closed, it follows that $q(Y) \cong q(\tilde{Y})$ lies in the essential image of $\mathcal{C}$. □

Remark 5.20. Every extension-closed subcategory $\mathcal{C}$ in a triangulated category $\mathcal{T}$ can be equipped with the structure of an extriangulated structure by restricting the triangles to this category.
This extriangulated structure is an exact structure if and only if $\mathrm{Hom}_{\mathcal{T}}(C, C'[-1]) = 0$ for all C, C' in $\mathcal{C}$. In particular, every admissible exact subcategory has an exact structure given by all triangles $A \to B \to C \to A[1]$ in $\mathcal{T}$ such that A, B, C in $\mathcal{C}$. We will always equip an admissible exact subcategory with this exact structure.

We recall the following result:

THEOREM 5.21. *([**19**] or Chapter 9) For every admissible exact subcategory* $\mathcal{C}$ *in an algebraic triangulated category* $\mathcal{T}$ *there exists a triangle functor*

$$\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$$

which extends the inclusion $\mathcal{C} \subseteq \mathcal{T}$. *It is called a* realization functor *of* $\mathcal{C}$.

We will call a subcategory in a triangulated category **admissible abelian** if it is admissible exact and the induced exact structure from the triangles is abelian.

6. Fully faithfulness of the realization functor

We now want to see that $\mathrm{eff}(\mathcal{E})$ is h-admissible exact in $\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$. This means we need to see

$$\mathrm{Ext}^t_{\widetilde{\mathrm{eff}}}(F,G) \to \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(F, \Omega^{-t}_{\mathcal{Y}\mathrm{eff}(\mathcal{E})} G) \quad \forall t \geq 1$$

is an isomorphism for all $F, G \in \widetilde{\mathrm{eff}}$.

Lemma 6.1. *Let $\mathcal{F}$ be a Frobenius category and $X, Y \in \mathcal{F}$, then we have natural isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{F}}(X,Y) \to \underline{\mathrm{Hom}}(X, \Omega^{-n} Y) \quad \forall n \geq 1$$

where $\underline{\mathrm{Hom}} := \mathrm{Hom}_{\underline{\mathcal{F}}}$

Proof. We look at short exact sequences $\Omega^{-n}Y \to I^n \to \Omega^{-(n+1)}Y$, $n \geq 0$ with I^n projective-injective and apply $\mathrm{Hom}_{\mathcal{F}}(X,-)$. We have induced an exact sequence of abelian groups $\mathrm{Hom}(X, I^n) \to \mathrm{Hom}(X, \Omega^{-n}Y) \to \underline{\mathrm{Hom}}(X, \Omega^{-(n+1)}Y)$. Comparism with the long exact sequence gives an induced isomorphism $\mathrm{Ext}^1_{\mathcal{F}}(X, \Omega^{-n}Y) \to \underline{\mathrm{Hom}}(X, \Omega^{-(n+1)}Y)$. Now, the usual dimension shift argument using cosyzygies gives $\mathrm{Ext}^{n+1}_{\mathcal{F}}(X,Y) \cong \mathrm{Ext}^1_{\mathcal{F}}(X, \Omega^{-n}Y)$. $\square$

So, we are actually asking when $\widetilde{\mathrm{eff}}$ is an homologically exact subcategory of $\mathcal{Y}\mathrm{eff}(\mathcal{E})$.

Proposition 6.2. $\widetilde{\mathrm{eff}}$ *is an homologically exact subcategory of* $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ *(or equivalently:* $\widetilde{\mathrm{eff}}$ *is h-admissible exact in* $\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$*).*

Proof. We proceed by first making two general remarks in (1) and (2) before we proceed inductively in (3).
(1) We first remark that for every morphism $g\colon X[m] \to Y[m+1]$ in $\mathcal{Y}(\mathcal{E})$ with $X, Y \in \mathcal{E}$, $m \in \mathbb{Z}$ the following holds $\mathrm{Im}\,\mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-g) \in \Omega^{3m}_{\mathcal{Y}\mathrm{eff}(\mathcal{E})}\widetilde{\mathrm{eff}}$.
(2) We secondly remark that in $\mathcal{Y}(\mathcal{E})$ every morphism $f\colon X \to Y[n]$ with $X, Y \in \mathcal{E}, n \geq 1$ can be written as a composition $X = X_0 \xrightarrow{f_1} X_1[1] \xrightarrow{f_2} X_2[2] \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n[n] = Y[n]$ with $X_i \in \mathcal{E}$, $0 \leq i \leq n$. (3) Now, we claim the following: For every morphism $h\colon F \to \Omega^{-t}G$ in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ with $F, G \in \widetilde{\mathrm{eff}}$, $t \geq 2$ there exists an $s \in \mathbb{N}$, $h^s\colon \Omega^s F \to \Omega^{s-t}G$ with $\Omega^{-s}h_s = h$ in $\underline{\mathcal{Y}\mathrm{eff}}(\mathcal{E})$ such that h^s is a composition $\Omega^s F = \Omega^s F_0 \to \Omega^{s-1}F_1 \to \Omega^{s-2}F_2 \to \cdots \to \Omega^{s-t}F_t = \Omega^{s-t}G$.
We fix short exact sequences $X'' \xrightarrow{i} X \xrightarrow{p} X'$ and $Y'' \xrightarrow{j} Y \xrightarrow{q} Y'$ with $F = \mathrm{coker}\,\mathrm{Hom}(-,p)$, $G = \mathrm{coker}\,\mathrm{Hom}(-,q)$. The morphism $h\colon F \to \Omega^{-t}G$ induces morphisms between the long exact sequences of representable functors which induces morphisms $h_s\colon \Omega^s F \to \Omega^{s-t}G$ for all $s \in \mathbb{Z}$. Now we study the morphisms of long exact sequences to find the factorization, to shorten notation, we use $(-,?) := \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-,?)$.
If $t = 2$, we look at the commutative diagram

$$\begin{tikzcd}
(-,X'') \arrow[r, "{(-,i)}"] \arrow[d, "{(-,a'')}"'] & (-,X) \arrow[r, "{(-,p)}"] \arrow[d, "{(-,a)}"'] & (-X') \arrow[r] \arrow[d, "{(-,a')}"'] & F \arrow[d, "h"'] \\
(-,Y') \arrow[r, "\sigma"'] & (-,Y''[1]) \arrow[r, "{(-,j[1])}"'] & (-,Y[1]) \arrow[r] & \Omega^{-2}G
\end{tikzcd}$$

Then set $F_1 = \mathrm{Im}(-,j[1]) \circ (-,a) \in \widetilde{\mathrm{eff}}$ (by (1)) and using the exactness of the rows we conclude that h_1 factors as $\Omega F \to F_1 \to \Omega^{-1}G$.

If $t=3$, we look at the commutative diagram

$$\begin{tikzcd}
(-,X'') \arrow[r, "(-,i)"] \arrow[d, "(-,a'')"'] & (-,X) \arrow[r, "(-,p)"] \arrow[d, "(-,a)"'] & (-X') \arrow[r] \arrow[d, "(-,a')"'] & F \arrow[d, "h"'] \\
(-,Y''[1]) \arrow[r, "(-,j[1])"'] & (-,Y[1]) \arrow[r, "(-,q[1])"'] & (-,Y'[1]) \arrow[r] & \Omega^{-3}G
\end{tikzcd}$$

then with $F_1 = \mathrm{Im}(-,q[1]) \circ (-,a)$ we get a factorization of h_1 as $\Omega F \to F_1 \to \Omega^{-2}$.
For $t \geq 4$ we proceed inductively and find a factorization of the form $\Omega F \to F_1 \to \Omega^{-t+1}G$ with $F_1 \in \widetilde{\mathrm{eff}}$ as follows; Consider the commutative diagram

$$\begin{tikzcd}
(-,X'') \arrow[r, "(-,i)"] \arrow[d, "(-,a'')"'] & (-,X) \arrow[r, "(-,p)"] \arrow[d, "(-,a)"'] & (-X') \arrow[r] \arrow[d, "(-,a')"'] & F \arrow[d, "h"'] \\
(-,Y_1[n_1]) \arrow[r, "(-,\ell)"'] & (-,Y_2[n_2]) \arrow[r, "(-,m)"'] & (-,Y_3[n_3]) \arrow[r] & \Omega^{-t}G
\end{tikzcd}$$

with the second row is induced by the suitably-number rotated triangle, we have $\{Y_1, Y_2, Y_3\} = \{Y'', Y, Y'\}$ and certain $n_i \in \mathbb{N}_{\geq 1}$, $n_3 \geq 2$. By (2), the morphism $(-,a')$ factors as $(-,X') \xrightarrow{f_1'} (-,X_1[1]) \to (-,Y_3[n_3])$. We precompose with $(-,p)$ to obtain a factorization of $(-,m) \circ (-,a)\colon (-,X) \to (-,Y_3[n_3])$ as $(-,X) \xrightarrow{f_1} (-,X_1[1]) \to (-,Y_3[n_3])$. We define $F_1 := \mathrm{Im}\, f_1 \in \widetilde{\mathrm{eff}}$. As the second row is exact, we find an induced morphism $F_1 \to \Omega^{-t+1}$. By definition, we have $f_1 \circ (-,i) = 0$, this induces a morphism $\Omega F \to F_1$, this gives the factorization of h_1. The previous claim (3) implies that the maps $\mathrm{Ext}^t_{\widetilde{\mathrm{eff}}}(F,G) \to \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(F, \Omega^{-t}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}} G)$ are surjective for all $t \geq 1$ (as they are isomorphisms for $t=1$). Then it is a standard argument that this implies that they are isomorphisms for all $t \geq 2$. $\square$

7. The realization functor is essentially surjective

Is the inclusion $\mathrm{Thick}_\Delta(\widetilde{\mathrm{eff}}) \subseteq \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ an equality? We start with the following duality:

7.1. The Auslander-Bridger transpose.

$$\mathrm{Tr}\colon (\underline{\mathrm{mod}}_1 \mathcal{Y}(\mathcal{E}))^{op} \to \underline{\mathrm{mod}}_1 \mathcal{Y}(\mathcal{E}^{op})$$

maps $\mathrm{coker}\, \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-,f)$ to $\mathrm{coker}\, \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(f,-)$, compare [**14**, section 5.2]. It restricts to a duality, i.e. a functor

$$\mathrm{Tr}\colon \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}^{op} \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E}^{op})}$$

the quasi-inverse is given by the same transpose defined for $\mathcal{E}^{op}$ and by definition $\mathrm{Tr}(\Omega F) \cong \Omega^- \mathrm{Tr}\, F$. It restricts to a duality $\Omega^- \circ \mathrm{Tr}\colon \widetilde{\mathrm{eff}}(\mathcal{E}) \to \widetilde{\mathrm{eff}}(\mathcal{E}^{op})$.

Definition 7.1. Let $F\colon \mathcal{Y}(\mathcal{E}) \to (Ab)$ be a covariant additive functor, we define the graded support of F as

$$\mathrm{supp}(F) := \{i \in \mathbb{Z} \mid \exists E \in \mathcal{E} \mid F(E[i]) \neq 0\} \quad \subseteq \mathbb{Z}$$

If F is contravariant, we take the same definition but we write supp^{op} instead of supp.
Let $X \in \mathrm{D}^b(\mathcal{E})$ we have a covariant functor $F_X = \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(X,-)|_{\mathcal{Y}(\mathcal{E})}$ and a contravariant functor $F^X = \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,X)|_{\mathcal{Y}(\mathcal{E})}$. We call $\mathrm{supp}(F_X)$ resp. $\mathrm{supp}^{op}(F^X)$ the covariant resp. contravariant **graded support** of X.
We define the two **Yoneda degrees** of X via

(*) $\mathcal{Y}\mathrm{deg}(X) = n$ if $n \in \mathrm{supp}(F_X) \subseteq [n, \infty)$.
(*) $\mathcal{Y}\mathrm{deg}^{op}(X) = n$ if $n \in \mathrm{supp}^{op}(F^X) \subseteq (-\infty, n]$.

Remark 7.2. For $X \in \mathrm{D}^b(\mathcal{E})$.
If $\mathcal{Y}\deg(X) = n \in \mathbb{Z}$ then $\mathcal{Y}\deg(X[-n]) = 0$.
If $\mathcal{Y}\deg^{op}(X) = m$ then $\mathcal{Y}\deg^{op} X[-m] = 0$.
Given $X = A_i \oplus A_{i+1} \oplus \cdots \oplus A_j$, $i \leq j$ with $A_t \in \mathcal{E}[t]$ for all t and $A_i \neq 0$, $A_j \neq 0$, then we have $\mathcal{Y}\deg(X) = i$, $\mathcal{Y}\deg^{op}(X) = j$. Conversely, every $X \in \mathcal{Y}(\mathcal{E})$ with $\mathcal{Y}\deg(X) = i$, $\mathcal{Y}\deg^{op}(X) = j$ can be written in this way. In particular, for $X \in \mathcal{Y}(\mathcal{E})$, we have $X \in \mathcal{E}$ if and only if $\mathcal{Y}\deg(X) = 0 = \mathcal{Y}\deg^{op}(X)$.
Now, on $\mathrm{D}^b(\mathcal{E})$, for $n, m \in \mathbb{N}$, the conditions $\mathcal{Y}\deg \geq n$ and $\mathcal{Y}\deg^{op} \leq m$ are extension-closed. This implies that the Yoneda degrees are well-defined for all $X \in \mathrm{D}^b(\mathcal{E})$ because this is the extension-closure of $\mathcal{Y}(\mathcal{E})$.

Remark 7.3. We have a problem when we want to extend this definitions to the stable category of Yoneda-effaceables as then the support is no longer a well-defined invariant of the isomorphism class (e.g. the zero functor is isomorphic to every projective - but their supports vary).

To overcome this issue, we define these degree functions first for triangles.

Definition 7.4. Given a triangle without a split summand $\Delta \colon A \to B \to C \xrightarrow{+1}$, $A, B, C \in \mathcal{Y}(\mathcal{E})$. We number the objects as follows $A[n] =: D^{3n-2}, B[n] =: D^{3n-1}, C[n] =: D^{3n}$, $n \in \mathbb{Z}$. We define

$$\mathcal{Y}\deg(\Delta) := (\inf\{n \in \mathbb{Z} \mid \mathcal{Y}\deg(D^n) > 0\}) - 1$$

If $\mathcal{Y}\deg(\Delta) = t$ and Δ has no split triangles as summands, then we have for the suitably times rotated triangle $\Delta' \colon D^{t-2} \to D^{t-1} \to D^t \xrightarrow{+1}$ the following property

(*) $\mathcal{Y}\deg(D^t) = \mathcal{Y}\deg(D^{t-1}) = \mathcal{Y}\deg(D^{t-2}) = 0$.
To see this, by definition $\mathcal{Y}\deg(D^{t+1}) \geq 1$, so $\mathcal{Y}\deg(D^{t-2}) \geq 0$. But by definition $\mathcal{Y}\deg(D^{t-2})$ can not be > 0, so it has to be $= 0$. Also by definition $\mathcal{Y}\deg(D^t) \leq 0, \mathcal{Y}\deg(D^{t-1} \leq 0$. Assume that $\mathcal{Y}\deg(D^t) = (-n) < 0$, take $X[-n] \in \mathcal{E}[-n|$, then $(D^t, X[-n]) \cong (D^{t-1}, X[-n])$. Then one can get a contradiction to the assumption that Δ has no split summand. Therefore we have $\mathcal{Y}\deg(D^t) = 0$ and as $\mathcal{Y}\deg \geq 0$ is extension-closed, we conclude that $\mathcal{Y}\deg(D^{t-1}) = 0$

Furthermore, $\mathcal{Y}\deg(\Delta)$ only depends on the homotopy equivalence class of the complex $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(\Delta, -) \in K(\mathcal{P}(\mathrm{Mod}\,\mathcal{Y}(\mathcal{E}^{ic})^{op}))$. In particular, if $F \in \mathcal{Y}\mathrm{eff}(\mathcal{E})$ is represented by Δ, then $\mathcal{Y}\deg(\Delta)$ is well-defined for $\mathrm{Tr}\,\underline{F} \in \underline{\mathcal{Y}\mathrm{eff}}$ and Tr is a duality on $\underline{\mathcal{Y}\mathrm{eff}}$, so

$$\underline{\deg}(\underline{F}) := \mathcal{Y}\deg(\Delta) \quad \text{ for } \underline{F} \in \underline{\mathcal{Y}\mathrm{eff}}(\mathcal{E})$$

is a well-defined integer.
Dually, we may define for a triangle as before

$$\mathcal{Y}\deg^{op}(\Delta) = (\sup\{n \in \mathbb{Z} \mid \mathcal{Y}\deg^{op}(D^n) < 0\}) + 1$$

If $\mathcal{Y}\deg^{op}(\Delta) = s$, then for the suitable rotated triangle $\Delta'' \colon D^s \to D^{s+1} \to D^{s+2} \xrightarrow{+1}$ the following property holds

$(*)^{op}$ $\mathcal{Y}\deg^{op}(D^s) = \mathcal{Y}\deg^{op}(D^{s+1}) = \mathcal{Y}\deg^{op}(D^{s+2}) = 0$

In this case, also $\deg^{op}(\underline{F}) := \mathcal{Y}\deg^{op}(\Delta)$ is well-defined for F represented by Δ in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$.

Example 7.5. Let $\underline{F}$ be in $\Omega^t \widetilde{\mathrm{eff}}$, then $\underline{\deg}(\underline{F}) = t$, $\underline{\deg}^{op}(\underline{F}) = t - 2$.

Lemma 7.6. *Given a Yoneda-effaceable functor F and $t = \underline{\deg}(\underline{F})$. Then $\underline{\deg}(\underline{F}) \geq \underline{\deg}^{op}(\underline{F}) - 2$ and it is $=$ if and only if $F \in \Omega^t \widetilde{\mathrm{eff}}$.*

Proof. The inequality follows by definition. Equality means that the triangles Δ' and Δ'' coincide. But this means that $\mathcal{Y}\deg^{op} = \mathcal{Y}\deg = 0$ for D^{t-2}, D^{t-1}, D^t, i.e. we have a distinguished triangle with three consecutive terms in $\mathcal{E}$. Then the first two maps in such a triangle are given by a short exact sequence in $\mathcal{E}$ and the claim follows. $\square$

We think of the number $d_F = \underline{\deg}(\underline{F}) - \underline{\deg}^{op}(\underline{F}) + 2$ as the distance of F being a (co)syzygy of an effaceable. Now, the strategy is the following: Show that every $F \in \mathcal{Y}\mathrm{eff}(\mathcal{E})$ fits into a short exact sequence $G \rightarrowtail F \twoheadrightarrow \Omega^t E$ for some t such that $d_G < d_F$.

Remark 7.7. Observe that $\mathrm{mod}_{\infty}\mathcal{Y}(\mathcal{E})$ is deflation-closed in $\mathrm{Mod}\,\mathcal{Y}(\mathcal{E})$ and $\mathbf{Gp}(\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E}))$ is deflation-closed in $\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E})$ and $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is deflation-closed in $\mathbf{Gp}(\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E}))$. Therefore we have that arbitrary kernels of epimorphisms between Yoneda effaceable functors are again Yoneda effaceable.

Lemma 7.8. *Let $\mathcal{E}$ be weakly idempotent complete. Given a triangle $\Delta\colon Z \to Y \to X \xrightarrow{+1}, X, Y, Z \in \mathcal{Y}(\mathcal{E})$ without split summands of $\mathcal{Y}\deg(\Delta) = 0$. Then we have*

$$Z_{>0} \oplus Z_0 \xrightarrow{\begin{pmatrix} c & d \\ 0 & f \end{pmatrix}} Y = Y_{>0} \oplus Y_0 \xrightarrow{\begin{pmatrix} a & b \\ 0 & p \end{pmatrix}} X_{>0} \oplus X_0 \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} Z_{>0}[1] \oplus Z_0[1]$$

where $Z_{>0}, X_{>0}, Y_{>0} \in \bigvee_{i>0} \mathcal{E}[i]$, $X_0, Y_0, Z_0 \in \mathcal{E}$ and $p\colon Y_0 \to X_0$ an deflation.

Proof. That we can write it in this form follows from (*). Now, $\delta\colon X_0 \to Z_0[1]$ corresponds to a short exact sequence, say this is $Z_0 \overset{i}{\rightarrowtail} V_0 \overset{q}{\twoheadrightarrow} X_0$. Then, $\begin{pmatrix} 0 \\ q \end{pmatrix} : V_0 \to X_{>0} \oplus X_0$ satisfies $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore there exists a morphism $\begin{pmatrix} g \\ h \end{pmatrix} : V_0 \to Y_{>0} \oplus Y_0$ such that $\begin{pmatrix} a & b \\ 0 & p \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix}$, this implies $p \circ g = q$. Now, as $\mathcal{E}$ is weakly idempotent complete it follows from the obscure axiom [8, Prop. 7.6] that p is a deflation. $\square$

Remark 7.9. If we could show that p is a deflation, then the proof can be completed: let $u\colon Y \to Z$, $F = \mathrm{coker}(-, u)|_{\mathcal{Y}(\mathcal{E})}$ be as before, take $E = \mathrm{coker}\,\mathrm{Hom}(-, p)|_{\mathcal{Y}(\mathcal{E})}$ then the kernel can be described as $G = \mathrm{coker}(-, a)|_{\mathcal{Y}(\mathcal{E})}$. As we know that G is Yoneda eaffaceable, it follows from Prop. 5.2 that a has a cone in $\mathcal{Y}(\mathcal{E})$ and can be completed to a complete projective resolution of G. Then $\mathcal{Y}\deg(G) < 0 = \mathcal{Y}\deg(F)$ and $\mathcal{Y}\deg^{op}(G) = \mathcal{Y}\deg^{op}(F)$, so the induction would work.

Proposition 7.10. *Let $\mathcal{E}$ be weakly idempotent complete. Assume $F \in \mathcal{Y}\mathrm{eff}(\mathcal{E})$ is represented by a triangle Δ without split summands and $\mathcal{Y}\deg(\Delta) = 0$, then there exists a short exact sequence in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$*

$$\Omega G \rightarrowtail \Omega F \twoheadrightarrow \Omega E$$

with E in $\widetilde{\mathcal{E}}$ and if F is not in $\widetilde{\mathrm{eff}}$ then $d_G < d_F$.

Proof. We take for Δ the notation of the previous Lemma and set $u := \begin{pmatrix} a & b \\ 0 & p \end{pmatrix}, v = \begin{pmatrix} c & d \\ 0 & f \end{pmatrix}$, i.e. $F = \mathrm{coker}(-, u)|_{\mathcal{Y}(\mathcal{E})}$. By the previous Lemma, p is a deflation, say with $\mathcal{E}$-kernel $K_0 \xrightarrow{j} Y_0$, then $E = \mathrm{coker}(-, p)|_{\mathcal{Y}(\mathcal{E})}$ is in $\widetilde{\mathrm{eff}}$. Furthermore, we define $G = \mathrm{coker}(-, a)|_{\mathcal{Y}(\mathcal{E})}$. By the 3×3-Lemma for traingulated categories we find an object C and morphisms such that all rows and columns are distinguished triangles in the following diagram

$$\begin{tikzcd}
X_0[-1] \arrow[r] \arrow[d, "0"] & K_0[-1] \arrow[r] \arrow[d] & Y_0[-1] \arrow[r] \arrow[d, "0"] & X_0[-1] \arrow[d, "0"] \\
X_{>0}[-1] \arrow[r] \arrow[d, "\binom{1}{0}"] & C \arrow[r] \arrow[d, "z"] & Y_{>0} \arrow[r, "a"] \arrow[d, "\binom{1}{0}"] & X_{>0} \arrow[d, "\binom{1}{0}"] \\
X[-1] \arrow[r] \arrow[d, "(0,1)"] & Z \arrow[r, "v"] \arrow[d] & Y \arrow[r, "u"] \arrow[d, "(0,1)"] & X \arrow[d, "(0,1)"] \\
X_0[-1] \arrow[r] & K_0 \arrow[r, "j"] & Y_0 \arrow[r, "p"] & X_0
\end{tikzcd}$$

Observe that by definition $\Omega G = \mathrm{Im}(-,a)|_{\mathcal{Y}(\mathcal{E})}$, $\Omega F = \mathrm{Im}(-,u)|_{\mathcal{Y}(\mathcal{E})}$, $\Omega E = \mathrm{Im}(-,p)|_{\mathcal{Y}(\mathcal{E})}$. We now look at the induced diagramm of all representable functors $(-,A)|_{\mathcal{Y}}$ restricted to the Yoneda category $\mathcal{Y} = \mathcal{Y}(\mathcal{E})$.

$$\begin{tikzcd}
(-,X_{>0}[-1])|_{\mathcal{Y}} \arrow[r] \arrow[d, tail, "\binom{1}{0}"] & (-,C)|_{\mathcal{Y}} \arrow[r] \arrow[d, "(-,z)"] & (-,Y_{>0})|_{\mathcal{Y}} \arrow[r, "a"] \arrow[d, tail, "\binom{1}{0}"] & (-,X_{>0})|_{\mathcal{Y}} \arrow[d, tail, "\binom{1}{0}"] \\
(-,X[-1])|_{\mathcal{Y}} \arrow[r] \arrow[d, two heads, "(0,1)"] & (-,Z)|_{\mathcal{Y}} \arrow[r, "v"] \arrow[d, two heads] & (-,Y)|_{\mathcal{Y}} \arrow[r, "u"] \arrow[d, two heads, "(0,1)"] & (-,X)|_{\mathcal{Y}} \arrow[d, two heads, "(0,1)"] \\
(-,X_0[-1])|_{\mathcal{Y}} \arrow[r] & \mathrm{coker}(-,z)|_{\mathcal{Y}} \arrow[r, "j"] & (-,Y_0)|_{\mathcal{Y}} \arrow[r, "p"] & (-,X_0)|_{\mathcal{Y}}
\end{tikzcd}$$

Now, use the snake lemma (in $\mathrm{Mod}\,\mathcal{Y}(\mathcal{E})$) to obtain an exact sequence $\Omega G \rightarrowtail \Omega F \twoheadrightarrow \Omega E$. We conclude that G is also Yoneda effaceable and by Prop. 5.2 it follows C in $\mathcal{Y}(\mathcal{E})$. As $\mathcal{Y}\mathrm{deg}() > 0$ is extension-closed it follows $\mathcal{Y}\mathrm{deg}(C[1]) > 0$, so, let us denote Δ' the distinguished triangle $C \to Y_{>0} \to X_{>0} \xrightarrow{+1}$. By definition, if F is not in $\widetilde{\mathrm{eff}}$, then we have $\mathcal{Y}\mathrm{deg}(\Delta') \leq (-2)$ and we have $\mathcal{Y}\mathrm{deg}^{op}(\Delta') \geq \mathcal{Y}\mathrm{deg}^{op}(\Delta)$ (use the columns in the 3×3 diagram and the definition to see this). $\square$

Then just use the distinguished triangles induced by a the short exact sequence from the previous Proposition and an induction on d_F, to see the following corollary.

Corollary 7.11. *If $\mathcal{E}$ is weakly idempotent complete:* $\mathrm{Thick}(\widetilde{\mathrm{eff}}) = \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$

Remark 7.12. We conjecture $\mathcal{Y}(\mathcal{E}) = \mathcal{Y}(\mathcal{E}^{ic})$, this would imply that we can leave out the assumption $\mathcal{E}$ weakly idempotent complete in the corollary 7.11 is obsolete.

8. Main results

From the previous two sections we conclude the following theorem which is our main result

Theorem 8.1. *If $\mathcal{E}$ is a weakly idempotent complete exact category. Then the realization functor for the admissible exact category* $\mathrm{eff}(\mathcal{E}) \cong \widetilde{\mathrm{eff}} \subseteq \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ *is a triangle equivalence*

$$\mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}.$$

Theorem 8.2. *If $\mathcal{E} \to \mathcal{E}'$ is a homologically exact functor. Then the induced functor* $\mathrm{eff}(\mathcal{E}) \to \mathrm{eff}(\mathcal{E}')$ *is homologically exact.*

Proof. We get a commutative diagram

$$\begin{tikzcd}
\mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \arrow[r] \arrow[d] & \mathrm{D}^b(\mathrm{eff}(\mathcal{E}')) \arrow[d] \\
\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \arrow[r] & \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}
\end{tikzcd}$$

with the vertical arrows are fully faithful triangle equivalence and the lower one is fully faithful by Lemma 5.8. This implies the upper triangle functor is also fully faithful. $\square$

9. Some special situations

Definition 9.1. For an exact category we define a Frobenius pair (in the sense of Schlichting) by

$$\mathrm{eff}(\mathbf{Gp}(\mathcal{E})) \subseteq \mathbf{Gp}(\mathrm{eff}(\mathcal{E}))$$

The associated Verdier quotient

$$\underline{\mathbf{Gp}(\mathrm{eff}(\mathcal{E}))}/\underline{\mathrm{eff}(\mathbf{Gp}(\mathcal{E}))}$$

will be called the **Frobenius gap** of $\mathcal{E}$.

Open question: Is $\mathcal{E}$ a Frobenius category if and only if its Frobenius gap is zero? (This seems to be true for exact categories with enough projectives...)

Similary, if $\mathcal{E}$ is an exact category then $\mathcal{P}^{<\infty}(\mathcal{E}) = \{X \in \mathcal{E} \mid \operatorname{pd}_{\mathcal{E}} X < \infty\}$ is a thick subcategory We say $\mathcal{E}$ is **regular** if $\mathcal{E} = \mathcal{P}^{<\infty}(\mathcal{E})$. If we assume that $\mathcal{E}$ has enough projectives, then $\mathcal{P}^{<\infty}(\mathcal{E})$ is a resolving subcategory of $\mathcal{E}$, so in particular it is homologically exact. Then, we have a chain of homological exact functors $\operatorname{eff}(\mathcal{P}^{\infty}(\mathcal{E})) \subseteq \mathcal{P}^{<\infty}(\operatorname{eff}(\mathcal{E})) \subseteq \operatorname{eff}(\mathcal{E})$ this induces three short exact sequences of triangulated categories

$$\mathrm{D}^b(\operatorname{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))) \to \mathrm{D}^b(\mathcal{P}^{<\infty}(\operatorname{eff}(\mathcal{E}))) \to \mathrm{D}^b(\mathcal{P}^{<\infty}(\operatorname{eff}(\mathcal{E})))/\mathrm{D}^b(\operatorname{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))$$
$$\mathrm{D}^b(\operatorname{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))) \to \mathrm{D}^b(\operatorname{eff}(\mathcal{E})) \to \mathrm{D}^b(\operatorname{eff}(\mathcal{E}))/\mathrm{D}^b(\operatorname{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))$$
$$\mathrm{D}^b(\mathcal{P}^{<\infty}(\operatorname{eff}(\mathcal{E}))) \to \mathrm{D}^b(\operatorname{eff}(\mathcal{E})) \to D_{sg}(\operatorname{eff}(\mathcal{E}))$$

and we get an induced fourth exact sequence of exact categories

$$\mathrm{D}^b(\mathcal{P}^{<\infty}(\operatorname{eff}(\mathcal{E})))/\mathrm{D}^b(\operatorname{eff}(\mathcal{P}^{<\infty}(\mathcal{E})) \to \mathrm{D}^b(\operatorname{eff}(\mathcal{E}))/\mathrm{D}^b(\operatorname{eff}(\mathcal{P}^{<\infty}(\mathcal{E})) \to D_{sg}(\operatorname{eff}(\mathcal{E}))$$

If $\mathcal{E}$ is regular then $\operatorname{eff}(\mathcal{E})$ is regular and all three triangulated categories are zero.

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