The poset of exact structures

1. Synopsis

We survey the theory of exact structures on an essentially small idempotent complete additive category. We focus on explicit answers and examples. But we also collect/recall several lattice isomorphisms for the lattice of all exact structures. Several of these isomorphisms are induced by equivalences of 2-categories which we collect in an Appendix.
What is new here? The description of exact structures with enough projectives. The equivalence of 2-categories with tf-Auslander categories (i.e. the subcategory of torsionfree objects in the Auslander exact category) is new. Apart from the Auslander correspondence none of the equivalences of 2-categories are formulated as such in the literature that I know (we treat them for time reasons also somewhat sketchy). Furthermore, we look at all exact substructures in examples (e.g. finitely generated abelian groups) and establish some of their global dimensions.

2. Introduction

Now, we fix one essentially small, idempotent complete exact category $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ and introduce the following posets of exact substructures

$$\mathrm{ex}(\mathcal{E}) = \text{ exact substructures of } \mathcal{E}.$$

Here the poset structure is given by inclusion on the collection of short exact sequences, i.e. $\mathcal{E}_1 \leq \mathcal{E}_2$ means the identity functor $\mathcal{E}_1 \to \mathcal{E}_2$ is an exact functor. Rump showed [**35**] that for every essentially small additive category there always exists a maximal structure (independently this had been shown by Crivei [**16**] under the assumption that the underlying additive category is weakly idempotent complete). Therefore, we may as well define $\mathrm{ex}(\mathcal{A}) := \mathrm{ex}(\mathcal{E}_{max})$ where $\mathcal{E}_{max}$ is the maximal exact structure on the additive category $\mathcal{A}$.
As it is very easy to see that arbitrary intersections of exact structures give an exact structure, we have a complete meet semi-lattice. Using the existence of a maximal exact structure, this implies that $\mathrm{ex}(\mathcal{E})$ is a complete lattice, cf. [**8**]. We are interested in the following types of results:

(1) Explicit parametrizations and constructions of exact structures
(2) Lattice isomorphisms for $\mathrm{ex}(\mathcal{E})$

We survey three explicit answers in sections 2,3,4 respectively. Firstly, the easiest construction are exact substructures induced by subcategories, these include all exact structures with enough projectives (or resp. with enough injectives). They have been introduced by Auslander and Solberg in [**6**]. Secondly, in the representation-finite case, we obtain the very easy Boolean lattice of generators - first observed by Enomoto, cf. Enomoto's theorem 4.4.
Thirdly, for essentially small additive categories with weak cokernels, there exists a topological space called the Ziegler spectrum consisting of certain indecomposables in the ind-completion. The indecomposable injectives in the ind-completion of the maximal exact structure define a Ziegler-closed subset $\mathcal{U}_{max}$. Then there is a bijection between Ziegler-closed subsets containing $\mathcal{U}_{max}$ and exact structures. This connection has been observed by Schlegel (in [**36**]), cf. Theorem 5.6. We apply this result in some examples with known Ziegler spectrum to have an understanding all exact substructures.

Furthermore, we study lattice isomorphisms which are not leading (us) to explicit answers. First of all, all four equivalences of 2-categories from the Appendices A,B,C give such lattice isomorphisms. The most classical lattice isomorphism is the Butler-Horrock's theorem (Appendix A) which identifies $\mathrm{ex}(\mathcal{E})$ with the lattice of closed sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{E}_{max}}$. Auslander correspondence can be seen as an equivalence of 2-categories (cf. [22]) - as a companion we add the tf-Auslander correspondence (Appendix B). Ind-completion gives the forth equivalence of 2-categories (Appendix C).

(2a,2b,2c) We follow Auslander's idea to study associated functor categories. An exact structure is determined by three different classes of morphisms, the i) admissible morphisms, ii) the deflations, iii) the inflations. When looking at functors represented by either of these three classes we obtain three correspondences ,respectively, i) the Auslander correspondence (cf. also Appendix B), ii) Enomoto's correspondence and iii) a new one which we call tf-Auslander correspondence. The first and third are induced by the equivalences of 2-categories (cf. Appendix B). These lead to three further poset isomorphisms.

What remains open:

(2*)(Q1) Assuming the additive category has weak cokernels, [36] found several other lattice isomorphisms to $\mathrm{ex}(\mathcal{A})$ (with certain Ziegler-closed mentioned before, with fp-idempotent ideal, with torsion classes etc.). Can (some of it) be generalize to arbitrary small additive categories?

(Q2) Let $\mathcal{E}$ be an essentially small exact category and $\mathcal{I} = \mathcal{I}(\overrightarrow{\mathcal{E}})$ the injectives in the Ind-completion. Properties of the Ind-completion imply

$$\mathrm{gldim}\,\mathcal{E} \leq \mathrm{gldim}\,\overrightarrow{\mathcal{E}} \leq \mathrm{gldim}\,\mathcal{I}\,\mathrm{mod}_\infty$$

where $\mathcal{I}\,\mathrm{mod}_\infty$ is the category of all additive functors $F\colon \mathcal{I} \to (Ab)$ such that there exists an exact sequence

$$\cdots \to \mathrm{Hom}(I_2, -) \to \mathrm{Hom}(I_1, -) \to \mathrm{Hom}(I_0, -) \to F \to 0$$

with $I_n \in \mathcal{I}$. This is always an exact category with set-valued Ext-groups (even though $\mathcal{I}\,\mathrm{Mod}$ may not be).

Assume that there is a correspondence of exact structures with subsets of the Ziegler spectrum Zg (see Q1) by assigning $\mathcal{E} \to \mathcal{U}_{\mathcal{E}} = \mathcal{I} \cap \mathrm{Zg}$. Is there an upper bound for $\mathrm{gldim}\,\mathcal{E}$ using $\mathcal{U}_{\mathcal{E}}$?

3. Elementary constructions of exact substructures

Lemma 3.1. *([17], Section 1.2) Let $(\mathcal{A}, \mathcal{S})$ be an exact categeory. We have an obvious bijection between the following two sets*

(a) *(additive) subfunctors $F \subset \mathrm{Ext}^1_{\mathcal{S}}$*
(b) *subclasses $\mathcal{S}'$ of $\mathcal{S}$ closed under isomorphisms (and direct sums of short exact sequences), pullback and pushout of short exact sequences, i.e. (Ex2) holds for $\mathcal{S}'$.*

given by $F \mapsto \mathcal{S}_F$ where $\mathcal{S}_F$ consists of all exact pairs $Y \to E \to X$ in $\mathcal{S}$ such that its equivalence class is in $F(X,Y)$. Conversely, $\mathcal{S}' \mapsto F'$ with $F'(X,Y)$ consists of all equivalence classes of exact sequences in $\mathcal{S}'$.

As indicated by the brackets, the property of being an additive subfunctor translates into the property that the short exact sequences are closed under direct sums. To study the structures corresponding to additive sub(bi)functors the notion of **weakly exact structure** (i.e. those classes of kernel-cokernel pairs which fulfill (b) in the previous theorem) has been introduced and studied by [8].

Since exact structures are always closed under direct sums of short exact sequences, we will restrict to consider additive functors.

Definition 3.2. Given an exact category $(\mathcal{A}, \mathcal{S})$ and a sub(-bi)functors $F \subset \mathrm{Ext}^1_{\mathcal{S}}$. We call F **closed** if it is additive and $F(X, -)$ and $F(-, Y)$ are half exact for all objects X and Y in $\mathcal{A}$ (*here*: A functor is half exact if applied to a short exact sequence it gives a sequence which is exact in the middle).

Definition 3.3. We say an exact sequence $0 \to X \xrightarrow{i} E \xrightarrow{d} Y \to 0$ is **F-exact** if the equivalence class of (i, d) in $\mathrm{Ext}^1_{\mathcal{S}}(Y, X)$ lies in $F(Y, X)$. So $\mathcal{S}_F$ in Lemma 3.1 consists of F-exact sequences.

Then we have

THEOREM 3.4. *(**Butler-Horrock's Theorem**, [**17**, Prop.1.4]) Let $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignment $F \mapsto \mathcal{S}_F$ from Lemma 3.1 is a bijective map from*

(1) *closed sub(bi)functors of $\mathrm{Ext}^1_{\mathcal{S}}$ to*
(2) *exact structures $\mathcal{S}'$ on the additive category $\mathcal{A}$ with $\mathcal{S}' \subset \mathcal{S}$.*

Remark 3.5. Theorem 3.4 has been generalized to n-exangulated categories in [**23**], section 3.2. One can also assume that it was part of the inspiration to the definition of an extriangulated category.

Corollary 3.6. *If $\mathcal{A}$ is an additive category. Let $\mathcal{S}_{max}$ be its maximal exact structure. Then, the bijection of the Theorem 3.4 gives a $1-1$ correspondence between*

(1) *closed sub(bi)functors of $\mathrm{Ext}^1_{\mathcal{S}_{max}}$ and*
(2) *exact structures on $\mathcal{A}$.*

Continuing to ignore set-theoretic issues, we have the following:

Corollary 3.7. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category. The class of all closed sub(bi)functors of $\mathrm{Ext}^1_{\mathcal{S}}$ forms a* poset *with respect to inclusion of functors. It is even a lattice which is isomorphic via the bijection in Theorem 3.4 to the full sublattice of all exact structures which are contained in $\mathcal{S}$.*

Definition 3.8. Let F be an additive closed sub(bi)functor F of $\mathrm{Ext}^1_{\mathcal{S}}$. We write $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) for the category of projectives (resp. injectives) in $(\mathcal{A}, \mathcal{S}_F)$ We will say that a closed sub(bi)functor F of $\mathrm{Ext}^1_{\mathcal{S}}$ **has enough projectives** (resp. **has enough injectives**) whenever $\mathcal{S}_F$ has. Instead of the index $\mathcal{S}_F$ we write just F, e.g. $\mathrm{Ext}^1_F := \mathrm{Ext}^1_{\mathcal{S}_F}$ etc.

Lemma 3.9. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category.*

(a) *If $F \subset \mathrm{Ext}^1_{\mathcal{S}}$ has enough projectives, then an exact sequence (i, d) is F-exact if and only if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ applied to it gives a short exact sequence in abelian groups for every $P \in \mathcal{P}(F)$.*
(b) *If $F \subset \mathrm{Ext}^1_{\mathcal{S}}$ has enough injectives, then an exact sequence (i, d) is F-exact if and only if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ applied to it gives a short exact sequence in abelian groups for every $I \in \mathcal{I}(F)$.*

PROOF. The proof of [**6**], Prop. 1.5, also works for exact categories. □

Remark 3.10. One can prove a stronger statement than the previous lemma, see [**12**], Ex. 11.10: Let $(\mathcal{A}, \mathcal{S})$ with enough projectives. Given *any two composable* morphisms (i, d), then this is an exact sequence if and only if $\mathrm{Hom}(P, -)$ applied to it gives a short exact sequence of abelian groups for all $P \in \mathcal{P}(\mathcal{S})$.

3.1. Subfunctors from subcategories. We continue to look at an exact category $(\mathcal{A}, \mathcal{S})$. Let $\mathcal{X} \subseteq \mathcal{A}$ be a full subcategory of $\mathcal{A}$. We define two subfunctors $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ of $\mathrm{Ext}^1_{\mathcal{S}}$ for X, Z in $\mathcal{A}$

$$F_{\mathcal{X}}(Y, Z) := \{0 \to Z \to E \to Y \to 0 \text{ in } \mathrm{Ext}^1_{\mathcal{S}}(Y, Z) \mid \mathrm{Hom}_{\mathcal{A}}(X, -) \text{ exact on it for all } X \text{ in } \mathcal{X}\}$$

$$F^{\mathcal{X}}(Y, Z) := \{0 \to Z \to E \to Y \to 0 \text{ in } \mathrm{Ext}^1_{\mathcal{S}}(Y, Z) \mid \mathrm{Hom}_{\mathcal{A}}(-, X) \text{ exact on it for all } X \text{ in } \mathcal{X}\}$$

These are (the standard examples of) closed sub(bi)functors (closedness is proven in [**17**, Prop. 1.7]). The generalization of these functors to n-exangulated categories can be found in [**23**], Def. 3.16.

Definition 3.11. For two additive subcategories $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A}$ we write $\mathcal{C} \vee \mathcal{D}$ for the smallest additive subcategory containing $\mathcal{C}$ and $\mathcal{D}$. We call this the **join** of $\mathcal{C}$ and $\mathcal{D}$.

Remark 3.12. We remark that we have the obvious inclusions: $\mathcal{X} \vee \mathcal{P}(\mathcal{S}) \subset \mathcal{P}(F_{\mathcal{X}})$ (resp. dually $\mathcal{X} \vee \mathcal{I}(\mathcal{S}) \subset \mathcal{I}(F^{\mathcal{X}})$). Furthermore, it is clear that $F_{\mathcal{X}} = F_{\mathcal{X} \vee \mathcal{P}(\mathcal{S})}$ (resp. $F^{\mathcal{X}} = F^{\mathcal{X} \vee \mathcal{I}(\mathcal{S})}$). Also, one can see easily that any sub(bi)functor F of $\operatorname{Ext}^1_{\mathcal{S}}$ is also a sub(bi)functor of $F_{\mathcal{P}(F)}$ (resp. of $F^{\mathcal{I}(F)}$) since an F-exact sequence η fulfills that $\operatorname{Hom}_{\mathcal{A}}(P, \eta)$ is exact for any $P \in \mathcal{P}(F)$.

Remark 3.13. Let $(\mathcal{A}, \mathcal{S})$ be an exact category. It is obvious that the inclusion of two additive subcategories $\mathcal{X} \subset \mathcal{X}'$ of $\mathcal{A}$ implies $F_{\mathcal{X}} \supset F_{\mathcal{X}'}$ and $F^{\mathcal{X}} \supset F^{\mathcal{X}'}$.
There are two trivial examples

(1) $\mathcal{X} = \mathcal{P}(\mathcal{S})$, in this case $\mathcal{S}_{F_{\mathcal{X}}} = \mathcal{S}$ and $\operatorname{Ext}^1_{F_{\mathcal{X}}} = \operatorname{Ext}^1_{\mathcal{S}}$. This is the unique maximal element in the poset of exact structures induced by closed sub(bi)functors of $\operatorname{Ext}^1_{\mathcal{S}}$.
(2) $\mathcal{X} = \mathcal{A}$, in this case, the exact structure is the split exact structure and $\operatorname{Ext}^1_{F_{\mathcal{A}}} = 0$. This is the unique minimal element in the lattice of all exact structures.

One can ask now: When is an exact structure $\mathcal{S}' \subset \mathcal{S}$ on an exact category $(\mathcal{A}, \mathcal{S})$ is of the form $\mathcal{S}_{\mathcal{X}}$ for an additive subcategory $\mathcal{X} \subset \mathcal{A}$?

Definition 3.14. We call a subcategory $\mathcal{X}$ of $\mathcal{A}$ **projectively saturated** (resp. **injectively saturated**) if $\mathcal{P}(\mathcal{S}_{\mathcal{X}}) = \mathcal{X}$ (resp. if $\mathcal{I}(\mathcal{S}_{\mathcal{X}}) = \mathcal{X}$). We call an exact structure $\mathcal{S}' \subset \mathcal{S}$ **projectively determined** (resp. **injectively determined**) if it is of the form $\mathcal{S}_{\mathcal{X}}$ (resp. $\mathcal{S}^{\mathcal{X}}$) for some additive subcategory $\mathcal{X} \subset \mathcal{A}$.

Lemma 3.15. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category and $\mathcal{X} \subset \mathcal{A}$ an additive category. We have the following properties*

(1) *$\mathcal{P}(\mathcal{S}_{\mathcal{X}})$ is the smallest projectively saturated subcategory that contains $\mathcal{X}$.*
(2) *If $\mathcal{S}' \subset \mathcal{S}$ is an exact structure with enough projectives, then $\mathcal{S}'$ is projectively determined.*

Proof. (1) It is straight-forward to see that $F_{\mathcal{P}(\mathcal{S}_{\mathcal{X}})} = F_{\mathcal{X}}$ (since $\mathcal{X} \subset \mathcal{P}(\mathcal{S}_{\mathcal{X}})$ implies $F_{\mathcal{X}} \supset F_{\mathcal{P}(\mathcal{S}_{\mathcal{X}})}$ and conversely an $\mathcal{S}_{\mathcal{X}}$-exact sequence fulfills by definition of the projectives that it is $\mathcal{S}_{\mathcal{P}(\mathcal{S}_{\mathcal{X}})}$-exact). This implies that $\mathcal{P}(\mathcal{S}_{\mathcal{X}})$ is projectively saturated. If we have $\mathcal{X} \subset \mathcal{Y}$ with $\mathcal{Y}$ projectively saturated, then $\mathcal{S}_{\mathcal{X}} \supset \mathcal{S}_{\mathcal{Y}}$ and therefore $\mathcal{P}(\mathcal{S}_{\mathcal{X}}) \subset \mathcal{P}(\mathcal{S}_{\mathcal{Y}}) = \mathcal{Y}$.
(2) Follows from Lemma 3.9. □

Proposition 3.16. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignments $\mathcal{X} \mapsto \mathcal{S}_{\mathcal{X}}$ and $\mathcal{S}' \mapsto \mathcal{P}(\mathcal{S}')$ give inverse bijections between*

(1) *projectively saturated subcategories $\mathcal{X} \subset \mathcal{A}$*
(2) *projectively determined exact structures $\mathcal{S}' \subset \mathcal{S}$ on $\mathcal{A}$*

The proof is obvious. We leave the trivial dual statements to the imagination of the reader.

In [**11**], section 5 one can find an example of an exact structure on category of finite-dimensional modules over the Kronecker algebra which is not projectively determined.

3.2. Exact structures with enough projectives.

Definition 3.17. Let $\mathcal{A}$ be an additive category. We call a subcategory $\mathcal{M}$ **contravariantly** (resp. **covariantly**) **finite** in $\mathcal{A}$ if every object X in $\mathcal{A}$ admits a **right** (resp. **left**) **$\mathcal{M}$-approximation**, that is a morphism $\alpha\colon M \to X$ (resp. $\beta\colon X \to M$) with $M \in \mathcal{M}$ such that every $f\colon M' \to X$ with M' in $\mathcal{M}$ factors over α (resp. such that every $g\colon X \to M'$ factors over β). We say $\mathcal{M}$ is **functorially finite** if it is co- and contravariantly finite.

We remark that intersections of two contravariantly finite (resp. covariantly finite) subcategories do not necessarily have this property. We start with the following easy observation.

Lemma 3.18. *Let $\mathcal{A}$ be an additive category and $\mathcal{B}, \mathcal{C}$ two additive subcategories, we write $\mathcal{M} = \mathcal{B} \vee \mathcal{C}$ for their join. Then we have*

(1) *If $\mathcal{B}$ and $\mathcal{C}$ are contravariantly finite (resp. covariantly finite), then $\mathcal{M}$ too.*
(2) *If $\mathcal{M}$ is contravariantly finite (resp. covariantly finite) and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{C}) = 0$, then $\mathcal{B}$ contravariantly finite (resp. $\mathcal{C}$ covariantly finite).*

Proof. (1) Let X be an object and assume we have a right $\mathcal{B}$-approximation $b_X \colon B_X \to X$ and a right $\mathcal{C}$-approximation $c_X \colon C_X \to X$. Then we get an induced morphism $m_X := (b_X, c_X) \colon B_X \oplus C_X \to X$. One can check that (b_X, c_X) is a right approximation for $\mathcal{B} \vee \mathcal{C}$.
(2) Let $m_X = (b_X, c_X) \colon M_X = B_X \oplus C_X \to X$ be a right $\mathcal{M}$-approximation and $\operatorname{Hom}(\mathcal{B}, \mathcal{C}) = 0$. Then we have $b_X \colon B_X \to X$ is a right $\mathcal{B}$-approximation.

□

For the later part we need to understand what it means that right approximations of a contravariantly finite subcategory are deflations. So we look at this special situation.

Lemma 3.19. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category with enough projectives (resp. enough injectives). Let $\mathcal{X}$ be a contravariantly finite (resp. covariantly finite) additive subcategory. Then the following are equivalent:*

(a) *Any right (resp. left) $\mathcal{X}$-approximation is a deflation (resp. inflation).*
(b) *$\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$ (resp. $\mathcal{I}(\mathcal{S}) \subset \mathcal{X}$)*

In particular, if $(\mathcal{A}, \mathcal{S})$ has enough projectives and $\mathcal{X}$ is contravariantly finite with $\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$, then any right $\mathcal{X}$-approximation also admits a kernel in $\mathcal{A}$.

Proof. (a) implies (b) is clear. So assume (b). Let $d \colon X \to Z$ be a right $\mathcal{X}$-approximation of an object Z in $\mathcal{A}$. Let $\pi \colon P \to Z$ be a deflation with $P \in \mathcal{P}(\mathcal{S})$. Since, by assumption, $P \in \mathcal{X}$ the map $\operatorname{Hom}_{\mathcal{A}}(P, X) \to \operatorname{Hom}_{\mathcal{A}}(P, Z)$ is surjective because d is a right approximation. Therefore, there exists a $\widetilde{\pi} \colon P \to X$ such that $d \circ \widetilde{\pi} = \pi$. Since π is a deflation, it follows that d is a deflation by axiom $E2$ of an exact category. □

Remark 3.20. If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A}, \mathcal{S})$ an exact category. Then $\mathcal{P}(\mathcal{S})$ is closed under direct sums and summands (cf. [**12**], Rem 11.5, Cor 11.6).

Theorem 3.21. *Let $\mathcal{A}$ be weakly idempotent complete additive category and $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignments $\mathcal{X} \mapsto F_{\mathcal{X}} \mapsto S_{F_{\mathcal{X}}}$ gives a bijections from*

(1) *additively closed, contravariantly finite subcategories $\mathcal{X}$ of $\mathcal{A}$, closed under direct summands and whose right approximations are deflations to*
(2) *closed sub(bi)functors of $\operatorname{Ext}^1_{\mathcal{S}}$ with enough projectives and to*
(3) *exact structures $\mathcal{S}' \subset \mathcal{S}$ which have enough projectives.*

We consider the dual statement of the previous Proposition as obvious and leave it to the reader.

Proof. The bijection from (2) to (3) is clear from Theorem 3.4 and by definition of *having enough projectives*. The map from (1) to (2) is well-defined since $\mathcal{P}(F_{\mathcal{X}})$ contains by definition $\mathcal{X}$ and for any A in $\mathcal{A}$ we have a deflation $X \to A$ with $X \in \mathcal{X}$ given by the right $\mathcal{X}$-approximation. Now, the assignment $F \mapsto \mathcal{P}(F)$ goes from (2) to (1). We need to see that this is inverse to the previous map. By Lemma 3.9 we know $F = F_{\mathcal{P}(F)}$ since F has enough projectives. On the other hand, let $\mathcal{X}$ be as in (1). We clearly have $\mathcal{X} \subset \mathcal{P}(F_{\mathcal{X}})$. Let $P \in \mathcal{P}(F_{\mathcal{X}})$, we take the right $\mathcal{X}$-approximation $X \to P$ which is a deflation. By [12], Prop. 11.3, this map splits and P is a summand of X. Since $\mathcal{X}$ is closed under direct summands, we have $P \in \mathcal{X}$. □

Example 3.22. There exist projectively saturated categories which are not contravariantly finite. For example, take as $\mathcal{A}$ the category of finite-dimensional modules over the Kronecker algebra and consider the category $\mathcal{X}$ given by all the preprojective modules. This is projectively saturated but not contravariantly finite.

Given an additive category $\mathcal{A}$, we write $\mathrm{ex}^{ep}(\mathcal{A})$ (resp. $\mathrm{ex}_{ei}(\mathcal{A})$, resp. $\mathrm{ex}^{ep}_{ei}(\mathcal{A})$) for the subposet of all exact structures which have enough projectives (resp. enough injectives, resp. both).
For an interval J in the poset $\mathrm{ex}(\mathcal{A})$ we write $\mathrm{ex}^{ep}(\mathcal{A})_J$ (resp. $\mathrm{ex}_{ei}(\mathcal{A})_J$, resp. $\mathrm{ex}^{ep}_{ei}(\mathcal{A})_J$) for the intersection of these (respective) posets with the interval J.

Corollary 3.23. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category with enough projectives. Let $i \colon \mathcal{B} \to \mathcal{A}$ be an inclusion of a full additively closed, contravariantly finite subcategory which contains $\mathcal{P}(\mathcal{S})$. Then, $\mathcal{S}_{\mathcal{X}} \mapsto \mathcal{S}_{i(\mathcal{X})}$ is an isomorphism of posets*

$$\mathrm{ex}^{ep}(\mathcal{B}, \mathcal{S}^{\leq \mathcal{S} \cap \mathcal{B}}) \to \mathrm{ex}^{ep}(\mathcal{A})_{\mathcal{S}_{i(\mathcal{B})} \leq * \leq \mathcal{S}}$$

Proof. The inclusion functor $i \colon \mathcal{B} \to \mathcal{A}$ gives a natural bijection between

(1) contravariantly finite, additive, summand-closed subcategories $\mathcal{X}$ of $\mathcal{B}$ with $\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$.
(2) contravariantly finite, additive, summand-closed subcategories $\mathcal{X}$ of $\mathcal{A}$ with $\mathcal{P}(\mathcal{S}) \subset \mathcal{X} \subset i(\mathcal{B})$.

The rest of the claim follows from Prop.3.21 and Lem.3.19. □

Example 3.24. Let Λ be an artin algebra and $\mathcal{A} = \Lambda$-mod be the category of finitely generated left Λ-modules. Let C be a cotilting Λ-module (i.e. $\mathrm{id}\, C < \infty$, $\mathrm{Ext}^{>0}(C, C) = 0$ and there is an exact sequence $0 \to \mathrm{D}\,\Lambda \to C_0 \to \cdots \to C_r \to 0$ with $C_i \in \mathrm{add}(C)$). Then $\mathcal{B} = {}^{\perp}C := \bigcap_{i \geq 1} \ker \mathrm{Ext}^i(-, C)$ is full, extension-closed, summand-closed, contravariantly finite subcategory which contains Λ.

3.3. A classical situation. Let $\varphi \colon \mathcal{A} \to \mathcal{B}$ be an exact functor between exact categories $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$. Then we have maps natural in X and Y

$$\varphi_{X,Y} \colon \mathrm{Ext}^1_{\mathcal{S}}(X, Y) \to \mathrm{Ext}^1_{\mathcal{T}}(\varphi(X), \varphi(Y)).$$

This gives an additive sub(bi)functor $F := \ker \varphi_{*,*} \subset \mathrm{Ext}^1_{\mathcal{S}}$. It is closed by [17], Prop. 1.10. The F-exact sequences are the exact sequences in $(\mathcal{A}, \mathcal{S})$ which are split exact once we apply the functor φ.

Remark 3.25. If λ is a left adjoint functor to φ, then the counit $\lambda\varphi(X) \to X$ for an object X in $\mathcal{A}$ provides a right $\lambda(\mathcal{B})$-approximation of X. In particular, $\lambda(\mathcal{B})$ is contravariantly finite in $\mathcal{A}$.

Lemma 3.26. *If the functor φ has a left adjoint λ then*

(1) $F = F_{\lambda(\mathcal{B})} = F_{\lambda(\mathcal{B}) \vee \mathcal{P}(\mathcal{S})}$.
(2) *If all counits $\lambda\varphi(X) \to X$ are deflations in $(\mathcal{A}, \mathcal{S})$, then F has enough projectives and furthermore, $\mathcal{P}(F)$ consists of all direct summands of objects in $\lambda(\mathcal{B})$.*
(3) *If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A}, \mathcal{S})$ has enough projectives, then F has enough projectives and $\mathcal{P}(F)$ consists of direct summands of $\lambda(\mathcal{B}) \vee \mathcal{P}(\mathcal{S})$.*

Dually, if the functor φ has a right adjoint ρ then

(1') $F = F^{\rho(\mathcal{B})}$
(2') *If all units $X \to \rho\varphi(X)$ are inflations in $(\mathcal{A}, \mathcal{S})$, then F has enough injectives, and furthermore, $\mathcal{I}(F)$ consists of all direct summands of objects in $\rho(\mathcal{B})$.*

(3') *If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A}, \mathcal{S})$ has enough injectives, then F has enough injectives and $\mathcal{I}(F)$ consists of direct summands of $\rho(\mathcal{B}) \vee \mathcal{I}(\mathcal{S})$.*

Proof. (1) Let η be an exact sequence in $(\mathcal{A}, \mathcal{S})$. We have the adjunction property $\mathrm{Hom}_{\mathcal{A}}(\lambda(V), W) \cong \mathrm{Hom}_{\mathcal{B}}(V, \varphi(W))$ for all V in $\mathcal{B}$ and W in $\mathcal{A}$. Therefore, exactness of $\mathrm{Hom}_{\mathcal{A}}(\lambda(V), \eta) \cong \mathrm{Hom}_{\mathcal{B}}(V, \varphi(\eta))$ for all V in $\mathcal{B}$ means that $\varphi(\eta)$ is $F_{\mathcal{B}}$-exact is. But $F_{\mathcal{B}}$-exactness is the same as split exactness.

(2) By (1) we have $\lambda\mathcal{B} \subset \mathcal{P}(F)$. Let X be in $\mathcal{A}$. By assumption, the counit $\lambda\varphi(X) \to X$ is a deflation in $(\mathcal{A}, \mathcal{S})$ and as just observes $\lambda\varphi(X) \in \mathcal{P}(F)$. We want to see, that this map is split when we apply the functor φ. But by the second triangle identity of the adjunction, we have that $\varphi\lambda\varphi(X) \to \varphi(X)$ has a section and therefore it splits (true?). Now given any $P \in \mathcal{P}(F)$, we have just constructed an F-epimorphism $\lambda\varphi(P) \to P$ and so this map has to be split, i.e. P is a summand of an object in $\lambda\mathcal{B}$. Since $\lambda\mathcal{B} \subset \mathcal{P}(F)$ by (1) and $\mathcal{P}(F)$ is closed under summands, equality follows.

(3) Since $\lambda(\mathcal{B})$ is contravariantly finite (cf. Rem. 3.25) we have that $\mathrm{add}(\lambda(\mathcal{B}))$ is contravariantly finite. By Lemma 3.18 we also have $\mathrm{add}(\lambda(\mathcal{B})) \vee \mathcal{P}(\mathcal{S})$ is contravariantly finite. Therefore, the claim follows from Theorem 3.21 and Lemma 3.19.

The dual statement can be proven analogously. □

Example 3.27. Let $f\colon B \to A$ a ring homomorphism and $\varphi\colon$ A-Mod $\to$ B-Mod, $X \mapsto {}_BX$ the functor given by restriction of scalars along f.
Then, there is a left adjoint given by the following tensor functor $\lambda(X) := A \otimes_B X$ called the **induced module** and a right adjoint given by the following Hom-functor $\rho(X) := \mathrm{Hom}_B(A, X)$ called the **co-induced module**. The counits $\lambda\varphi(X) = A \otimes_B X \to X$ are epimorphisms since their restrictions of scalars are surjective maps, this follows from the triangle identity. The units $X \to \mathrm{Hom}_B(A, {}_BX)$ are monomorphisms since their restrictions of scalars are injective maps by the triangle identity. Therefore, by the previous lemma we have for $F = \ker \varphi_{*,*}$ the following

(1) $F = F_{A\otimes_B \text{B-Mod}} = F^{\mathrm{Hom}_B(A, \text{B-Mod})}$
(2) F has enough projectives and enough injectives. The F-projectives are the direct summands of $A \otimes_B$ B-Mod, the F-injectives are the direct summands of $\mathrm{Hom}_B(A, \text{B-Mod})$.

This exact structure on A-Mod has been introduced by Hochschild in [**24**] in 1956. In loc. cit. this has been used to define relative Hochschild homology, a Tor and Ext functor have been defined for this setup. A very nice application of the classical situation is the finite representation type classification for group algebras, cf. [**7**], chapter III, section 3. A recent application to Han's conjecture can be found in [**13**].

Example 3.28. Let Γ be a ring and $e \in \Gamma$ an idempotent, we define $\Lambda := e\Gamma e$. Then, the restriction functor $e\colon \Gamma\text{-Mod} \to \Lambda\text{-Mod}, X \mapsto eX$ has a left adjoint $\ell = \Gamma e \otimes_\Lambda (-)$ and right adjoint $r = \mathrm{Hom}_\Lambda(e\Gamma, -)$. Therefore, we have for $F = \ker e_{*,*}$ the following description (numbered by the parts of the lemma 3.26 that are used)

(1) $F = F_{\Gamma e \otimes_\Lambda \Lambda\text{-Mod}} = F^{\mathrm{Hom}_\Lambda(e\Gamma, \Lambda\text{-Mod})}$.
(3) Since Γ-Mod is abelian, it is weakly idempotent complete. It has enough projectives and enough injectives. So, it follows that F has enough projectives and enough injectives. We have $\mathcal{P}(F)$ consists of direct summands of $(\Gamma e \otimes_\Lambda \Lambda\text{-Mod}) \vee \mathrm{Add}(\Gamma)$ and $\mathcal{I}(F)$ consists of direct summands of $\mathrm{Hom}_\Lambda(e\Gamma, \Lambda\text{-Mod}) \vee \mathcal{I}(\Gamma\text{-Mod})$.

If we take a noetherian ring Γ and consider the abelian Γ-mod category given by finitely generated Γ-modules, then this category has not in general enough injectives but it has enough projectives given by $\mathrm{add}(\Gamma)$. Assume that $\Lambda = e\Gamma e$ is again noetherian, then the restriction functor $e\colon \Gamma\text{-mod} \to \Lambda\text{-mod}$ has a well-defined left adjoint functor $\ell = \Gamma e \otimes_\Lambda (-)$. We conclude that in this case F has enough projectives given by the direct summands of $(\Gamma e \otimes_\Lambda \Lambda\text{-Mod}) \vee \mathrm{add}(\Gamma)$.

4. The representation-finite case - Enomoto's result

Definition 4.1. For a ring Γ, we denote by $\mathrm{proj}(\Gamma)$ the category of finitely generated projective left Γ-modules. We say an idempotent complete additive category $\mathcal{A}$ is **representation-finite** if it is equivalent to $\mathrm{proj}(\Gamma)$ for some ring Γ.

By definition, $\mathcal{A} = \mathrm{proj}(\Gamma)$ is Krull-Schmidt if and only if Γ is semi-perfect.

We recall Enomoto's results from [**18**, section 3].

Lemma 4.2. *Let $\mathcal{A} = \mathrm{proj}(\Gamma)$ a Krull-Schmidt category and $\mathcal{E}$ an exact structure on $\mathcal{E}$. Then there exists an idempotent $e \in \Gamma$ such that $\mathcal{P}(\mathcal{E}) = \mathrm{add}(\Gamma e)$. Then $\mathcal{E}$ has enough projectives if and only if $\Gamma/\Gamma e\Gamma$ is a finite length left Γ-module.*

Assume additionally that we have a commutative artinian ring R and Γ is a finitely generated R-algebra with $R \subset Z(\Gamma)$. This is saying that $\mathcal{A} = \mathrm{proj}(\Gamma)$ is Hom-noetherian R-linear. Then every exact structure on $\mathcal{A} = \mathrm{proj}(\Gamma)$ has enough projectives and enough injectives.

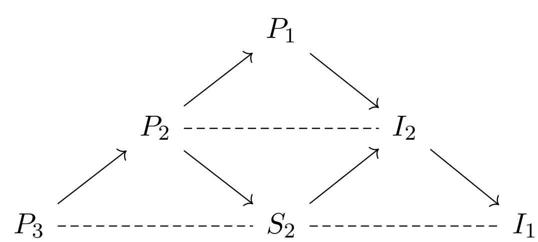
Definition 4.3. Let $\mathcal{E}$ be a an exact category and $\mathcal{M}$ a full subcategory. We say $\mathcal{M}$ is a **generator** if $\mathcal{M}$ is additively closed and for every X in $\mathcal{E}$ there exists a short exact sequence $Y \to M \to X$ with M in $\mathcal{M}$. A **cogenerator** is a generator in $\mathcal{E}^{op}$.
If $\mathcal{A}$ is an additive category and $\mathcal{M}$ a full subcategory, then we call $\mathcal{M}$ a generator (resp. cogenerator) if it is one in the maximal exact structure on $\mathcal{A}$.

Theorem 4.4. *(Enomoto's Theorem) Let R be a commutative artinian ring. Let $\mathcal{A}$ idempotent complete, representation-finite, Hom-noetherian R-linear. Let $\mathcal{P} = \mathrm{add}(\Gamma e)$ be the projectives in the maximal exact structure on $\mathcal{A}$. Then generators in $\mathcal{A}$ are given by the Boolean lattice of all additively closed subcategories containing $\mathcal{P}$, we denote it by* Generators($\mathcal{A}$). *Then*

$$\mathrm{ex}(\mathcal{A}) \to \mathrm{Generators}(\mathcal{A}),$$
$$\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$$

is an isomorphism of lattices.

Example 4.5. We look at the quiver $1 \to 2 \to 3$ and at its Auslander-Reiten quiver



To see the generators, fix the projectives P_1, P_2, P_3 and add any subset of $\{S_2, I_2, I_1\}$. So this is just the power set of this set with three elements. More interesting is to observe that we have seven hereditary exact substructures and one exact substructure of global dimension 2, corresponding to the generator $P_1 \oplus P_2 \oplus P_3 \oplus I_2$.

More generally for type $\mathbb{A}_n$-equioriented quivers the maximal global dimension is $n-1$.

Remark 4.6. We think that substructures of finite-dimensional Dynkin quiver representations are always of finite global dimension but we have not worked this out (except for type $\mathbb{A}$-equioriented). This should be mainly due to the Auslander-Reiten quiver has no oriented cycles. Then endomorphism rings of modules can be realized as upper triangular rings and should admit a quasi-hereditary structure which implies that they have finite global dimension.

If we are looking at representation-finite finite-dimensional algebras of global dimension 2, then we can already find examples of exact substructures of infinite global dimension.

Example 4.7. Let Γ be the Auslander algebra of $K[X]/(X^3)$. This has global dimension 2 and is representation-finite. Then look at the Frobenius exact substructure described in [3] on the Auslander algebra of a self-injective algebra.

5. Parametrization using the Ziegler spectrum - Schlegel's result

We refer to Appendix C for Ind-completion of additive and exact categories. For an essentially small additive category $\mathcal{C}$ we call $\overrightarrow{\mathcal{C}} =: \mathcal{A}$ its Ind-completion. Let $\mathcal{E}$ be an exact structure and $\overrightarrow{\mathcal{E}}$ be its ind-completion and $i\colon \mathcal{E} \to \overrightarrow{\mathcal{E}}$ the Yoneda embedding. We call X in $\overrightarrow{\mathcal{E}}$ is fp-$\overrightarrow{\mathcal{E}}$-injective if $\operatorname{Ext}^1_{\overrightarrow{\mathcal{E}}}(i(E), X) = 0$ for all E in $\mathcal{E}$.

Definition 5.1. An object in an additive category M is called **indecomposable** if $M = N \oplus L$ implies $N = 0$ or $L = 0$.
Let $\mathcal{A} = \overrightarrow{\mathcal{C}}$ be a locally finitely presented additive category. We define the **pure exact structure** to be $\mathcal{A}_p := \overrightarrow{\mathcal{C}_{\mathbf{split}}}$ to be the ind-completion of the split exact structure. An object in $\mathcal{A}$ is called **pure injective** if it is an injective in $\mathcal{A}_p$. Then we define the following class

$$\operatorname{ZSp}(\mathcal{A}) := \{M \in \mathcal{A} \mid M \text{ indecomposable and pure injective}\}$$

Remark 5.2. For general locally finitely presented additive categories we do not know of $\operatorname{ZSp}(\mathcal{A})$ is a set or if it is one if it is non-empty.

Definition 5.3. Let $\mathcal{C}$ be an essentially small additive category, $\mathcal{A} = \overrightarrow{\mathcal{C}}$ and S a class of morphisms in $\mathcal{C}$. Let $\mathcal{X}(S)$ be the full subcategory of $\mathcal{A}$ of all objects I with the following property: For any map $s\colon M \to M'$ in S and any map $f\colon M \to I$ there exists $f'\colon M' \to I$ such that $f's = f$. Alternatively, one can descibe this as

$$\mathcal{X}(\mathcal{S}) = \{I \in \mathcal{A} \mid \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(s, I) = 0 \quad \forall s \in S\}$$

A full subcategory $\mathcal{X}$ of $\overrightarrow{\mathcal{A}}$ is called **definable** if there exists a class of morphisms S in $\mathcal{A}$ such that $\mathcal{X} = \mathcal{X}(S)$.
Assume that $\operatorname{ZSp}(\mathcal{A})$ is a set, then a subset $\mathcal{U} \subseteq \operatorname{ZSp}(\mathcal{A})$ is called **Ziegler-closed** if there exists a definable subcategory $\mathcal{X}$ such that $\mathcal{U} = \operatorname{ZSp}(\mathcal{A}) \cap \mathcal{X}$.

From now on, we impose the condition that $\mathcal{C}$ has weak cokernels, this means that for every morphism $f\colon X \to Y$ in $\mathcal{C}$ there exists a morphism $g\colon Y \to Z$ such that the following sequence is exact (in the middle) in the abelian category $\operatorname{Mod} \mathcal{C}^{op}$ (all covariant, additive functors $\mathcal{C} \to (Ab)$)

$$\operatorname{Hom}_{\mathcal{C}}(Z, -) \xrightarrow{\operatorname{Hom}(g,-)} \operatorname{Hom}_{\mathcal{C}}(Y, -) \xrightarrow{\operatorname{Hom}(f,-)} \operatorname{Hom}_{\mathcal{C}}(X, -)$$

This condition is equivalent to $\operatorname{mod}_1 \mathcal{C}^{op}$ is abelian, in which case it also has enough projectives. This has been used in [14] and [36] to embed $\mathcal{A} = \overrightarrow{\mathcal{C}}$ in a locally coherent abelian category called the *purity category*. We are not going to explain this construction here, as we hope that these results can be generalized (without using this embedding).

Theorem 5.4. *(combine* [14, section (3.5), Lem 1] *with* [27, Lem. 12.1.12]*) If $\mathcal{C}$ has weak cokernels then* $\operatorname{ZSp}(\mathcal{A})$ *is a set and we have a topology on* $\operatorname{ZSp}(\mathcal{A})$ *with closed sets given by Ziegler-closed subsets. This topological space is called the* ***Ziegler spectrum*** *of $\mathcal{A}$.*

Lemma 5.5. *(*[36]*, proof of Lem. 2.9) Let $\mathcal{C}$ be essentially small, idempotent complete with weak cokernels and $\mathcal{E}$ an exact structure on it. Let $\overrightarrow{\mathcal{E}}$ be its Ind-completion.*

(1) *Let $\mathcal{X}_{\mathcal{E}}$ be the full subcategory fp-$\overrightarrow{\mathcal{E}}$-injectives, then this is a definable subcategory since it can be written as*

$$\mathcal{X}_{\mathcal{E}} = \mathcal{X}(\operatorname{Infl}_{\mathcal{E}})$$

where $\operatorname{Infl}_{\mathcal{E}}$ *denotes the $\mathcal{E}$-inflations.*

(2) *Let $\mathcal{U}_\mathcal{E}$ be the set of indecomposable injectives in $\overrightarrow{\mathcal{E}}$. Then $\mathcal{U}_\mathcal{E}$ is Ziegler-closed because*

$$\mathcal{U}_\mathcal{E} = \mathrm{ZSp}(\mathcal{A}) \cap \mathcal{X}_\mathcal{E}$$

Theorem 5.6. *([**36**], Thm B) Let $\mathcal{C}$ be an idempotent complete essentially small additive category with weak cokernels and $\mathcal{E}_{\max}$ its maximal exact structure. We write $\mathcal{X}_{max} := \mathcal{X}_{\mathcal{E}_{max}}, \mathcal{U}_{max} := \mathcal{U}_{\mathcal{E}_{max}}$. Then the assigment $\mathcal{E} \mapsto \mathcal{X}_\mathcal{E}$ and resp. $\mathcal{E} \mapsto \mathcal{U}_\mathcal{E}$ gives a an lattice isomorphism between $\mathrm{ex}(\mathcal{A})$ and*

(1) *the lattice of definable subcategories which contain $\mathcal{X}_{max}$ and resp.*
(2) *the lattice of Ziegler-closed subsets which contain $\mathcal{U}_{max}$.*

We want to understand the exact substructures in cases where the Ziegler spectrum is known. For this we need the following:

Proposition 5.7. *In the situation of the previous theorem, we have a map*

$$\{\mathcal{U} \text{ Ziegler-closed }, \mathcal{U}_{max} \subseteq \mathcal{U}\} \to \mathrm{ex}(\mathcal{A})$$
$$\mathcal{U} \mapsto (\mathcal{E}_{max})^{\mathcal{U}}$$

where $\mathcal{E}^{\mathcal{U}}_{max}$ consists of all $\mathcal{E}_{max}$-short exact sequences σ such that $\mathrm{Hom}_\mathcal{A}(\sigma, U)$ is exact for all $U \in \mathcal{U}$. This map is the inverse to the bijective map $\mathrm{ex}(\mathcal{A}) \to \{\mathcal{U} \text{ Ziegler closed }, \mathcal{U}_{max} \subseteq \mathcal{U}\}$ in Theorem 5.6.

We need the following easy lemma for the proof.

Lemma 5.8. *Let $\mathcal{C}$ be an idempotent complete essentially small category and $\mathcal{A} = \overrightarrow{\mathcal{C}}$ its Ind-completion. Assume we have an exact structure $\mathcal{E}$ on $\mathcal{C}$ and $\mathcal{U}$ some set of pure injective objects in $\mathcal{A}$. We denote $\mathcal{E}^{\mathcal{U}}$ the exact substructure of $\mathcal{E}$ consisting of $\mathcal{E}$-exact sequences σ such that $\mathrm{Hom}_\mathcal{A}(\sigma, U)$ exact for all $U \in \mathcal{U}$.*
Then all objects in $\mathcal{U}$ are $\mathrm{fp}(\overrightarrow{\mathcal{E}^{\mathcal{U}}})$-injectives.

Proof. (of Lemma 5.8) Let U be in $\mathcal{U}$ and X be an $\mathcal{C}$ and we take a $\overrightarrow{\mathcal{E}^{\mathcal{U}}}$-short exact sequence

$$\sigma \colon U \rightarrowtail Y \twoheadrightarrow X$$

We need to see it splits. We write $\sigma = \operatorname{colim} \sigma_i$ as a filtered colimit of $\mathcal{E}^{\mathcal{U}}$-short exact sequences $U_i \rightarrowtail Y_i \twoheadrightarrow X_i$. Now, we factorize the canonical morphisms $\sigma_i \to \sigma$, $i \in I$ of short exact sequences following [**12**]..

$$\begin{array}{ccccccc}
\sigma_i & \quad & U_i & \rightarrowtail & Y_i & \twoheadrightarrow & X_i \\
\downarrow & & \downarrow & & \downarrow & & \| \\
\eta_i & & U & \rightarrowtail & Z_i & \twoheadrightarrow & X_i \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\sigma & & U & \rightarrowtail & Y & \twoheadrightarrow & X
\end{array}$$

This means η_i is the push-out of σ_i along the canonical morphism $U_i \to U$. As $\mathrm{Hom}_\mathcal{A}(\sigma_i, U)$ is exact, it follows that η_i is split exact. Now, it is a straight forward observation to see that we have $\operatorname{colim}_I \eta_i = \sigma$. As η_i are split exact they are also pure exact sequences. Now, filtered colimits of pure exact sequences are again pure exact as the pure exact structure is a locally coherent exact structure (cf. Appendix..). In particular σ is pure exact and U is pure injective, it splits. ☐

Let us come back to:

Proof. (of Prop. 5.7) Let $\mathcal{E}$ be an exact structure on $\mathcal{C}$ and we set $\mathcal{U} := \mathcal{U}_\mathcal{E}$. As $\mathcal{E}$ is fully exact in $\overrightarrow{\mathcal{E}}$ we have that $\mathcal{E} \leq \mathcal{F} := \mathcal{E}^{\mathcal{U}}_{max}$ is an exact substructure. To see that they are equal, it is enough to see that $\mathcal{U} = \mathcal{U}_\mathcal{F}$. As $\mathcal{E} \leq \mathcal{F}$ we have that $\overrightarrow{\mathcal{E}} \leq \overrightarrow{\mathcal{F}}$ this implies that the subcategory of injectives fulfill $\mathcal{I}(\overrightarrow{\mathcal{E}}) \supseteq \mathcal{I}(\overrightarrow{\mathcal{F}})$ and therefore $\mathcal{U} \supseteq \mathcal{U}_\mathcal{F}$. Now, for the other inclusion we conclude from Lemma 5.8 that $\mathcal{U} \subseteq \mathcal{X}_\mathcal{F}$. This implies $\mathcal{U} \subseteq \mathcal{X}_\mathcal{F} \cap \mathrm{ZSp}(\mathcal{A}) = \mathcal{U}_\mathcal{F}$ by Lemma 5.5. ☐

Let us also note the following corollary.

Corollary 5.9. *Let $\mathcal{C}$ be an idempotent complete essentially small category and $\mathcal{A} = \overrightarrow{\mathcal{C}}$ its Ind-completion. Assume we have an exact structure $\mathcal{E}$ on $\mathcal{C}$ and $\mathcal{U}$ some set of pure injective objects in $\mathcal{A}$. Then*

$$\mathcal{E}^{\mathcal{U}} = \mathcal{E}^{\overline{\mathcal{U}}}$$

where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ in the Ziegler spectrum.

Proof. By definition $\mathcal{E}^{\overline{\mathcal{U}}}$ is an exact substructure of $\mathcal{E}^{\mathcal{U}}$. This implies that $\mathcal{U} \subseteq \mathcal{X}_{\mathcal{E}^{\mathcal{U}}} \subseteq \mathcal{X}_{\mathcal{E}^{\overline{\mathcal{U}}}}$. This implies $\mathcal{U} \subseteq \mathcal{U}_{\mathcal{E}^{\mathcal{U}}} \subseteq \mathcal{U}_{\mathcal{E}^{\overline{\mathcal{U}}}} = \overline{\mathcal{U}}$ but as $\mathcal{U}_{\mathcal{E}^{\mathcal{U}}}$ is Ziegler-closed, it has to be equal to $\overline{\mathcal{U}}$. Since we have a bijection it follows that $\mathcal{E}^{\overline{\mathcal{U}}} = \mathcal{E}^{\mathcal{U}}$. □

Let Λ be a ring, then we define the (left) Ziegler spectrum of Λ as $\mathrm{Zg}_\Lambda := \mathrm{ZSp}(\Lambda\,\mathrm{Mod})$.

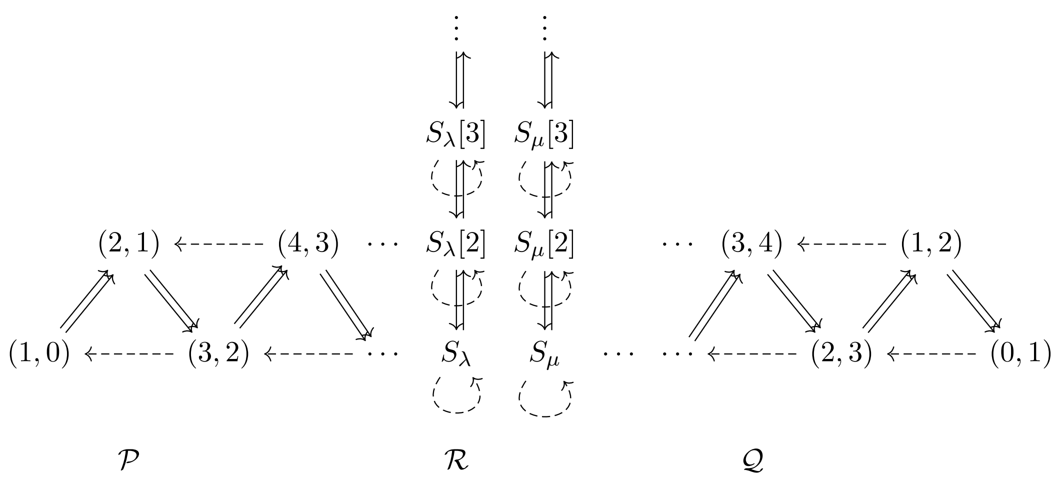
5.1. Examples.

Example 5.10. As a consequence of [**31**, Cor. 5.3.36, Cor. 5.3.37, Thm 5.1.12] one obtains: For a finite-dimensional algebra Λ the following are equivalent

(1) Λ is of finite representation-type
(2) Zg_Λ is a finite set
(3) Zg_Λ does not contain any infinite-dimensional modules.

In this case, Zg_Λ is a discrete topological space, $\mathcal{U}_{max}$ consists of the indecomposable injectives in $\Lambda\,\mathrm{mod}$. So, Ziegler-closed subsets containing $\mathcal{U}_{max}$ are in bijection with basic cogenerators in $\Lambda\,\mathrm{mod}$. This is easily seen to be an equivalent description to Enomoto's theorem in this case.

Example 5.11. Ziegler spectrum in tame hereditary case has been described by Ringel in [**33**], we just look here at the easiest case:
We define Q to be the Kronecker quiver $1 \leftleftarrows 2$ and $\Lambda = KQ$ for some infinite field K. Its Auslander-Reiten quiver (see picture below) has as vertices the indecomposables in $\Lambda\,\mathrm{mod}$, they are divided in three types 1) $\mathcal{P}$ preprojectives (in the τ^--orbit of the projectives), they are denoted by their dimension vector $(n+1, n)$, 2) $\mathcal{R}$ regulars, they are determined by the regular simple which they contain and their dimension vector, the regular simples are denoted by $S_\lambda, \lambda \in K \cup \{\infty\} =: \Omega$, 3) $\mathcal{Q}$ preinjectives (in the τ-orbit of the injectives), they are denoted by their dimension vector $(n, n+1)$.



The arrows between the vertices indicate irreducible maps between the indecomposables and the dotted arrow the Auslander-Reiten translate, for every dotted arrow there is an almost split sequence. For more details look into [**32**].

Then Zg_Λ consists of the following points

(1) indecomposables in $\Lambda\,\mathrm{mod}$
(2) For every $\lambda \in \Omega$ a Prüfer module $S_\lambda[\infty]$, which is the filtered colimit (union) over $S_\lambda \rightarrowtail S_\lambda[2] \rightarrowtail S_\lambda[3] \rightarrowtail \cdots$
(3) For every $\lambda \in \Omega$ an adic module $\hat{S_\lambda}$, which is the limit over $\cdots \twoheadrightarrow S_\lambda[3] \twoheadrightarrow S_\lambda[2] \twoheadrightarrow S_\lambda$
(4) The generic module G, it is characterized by being an indecomposable module with $\mathrm{Hom}_\Lambda(G, S_\lambda) = 0 = \mathrm{Hom}_\Lambda(S_\lambda, G)$ for all $\lambda \in \Omega$

Now, $\mathcal{U}_{max} = \{(1, 2), (0, 1)\}$ only consists of the two indecomposable injectives in $\Lambda\,\mathrm{mod}$. Given a subset $\mathcal{U} \subseteq \mathrm{Zg}_\Lambda$ containing $\mathcal{U}_{max}$ we find $T, M \subseteq \Omega$

$$\mathcal{U}^{fin} := \{U \in \mathcal{U} \mid U \in \Lambda\,\mathrm{mod}, U \notin \mathcal{U}_{max}\}$$
$$\mathcal{U}_{T,M} := \mathcal{U}_{max} \cup \{S_t[\infty] \mid t \in T\} \cup \{\hat{S_m} \mid m \in M\} \cup \{G\}$$

such that $\mathcal{U} = \mathcal{U}_{max} \cup \mathcal{U}^{fin} \cup \mathcal{U}_{T,M}$ or $\mathcal{U} = \mathcal{U}_{max} \cup \mathcal{U}^{fin} \cup \mathcal{U}_{T,M} \setminus \{G\}$. Following Ringel's characterization in [**33**] we find that $\mathcal{U}$ is Ziegler-closed iff

(a) $\mathcal{U}^{fin}$ finite, then it and T, M can be arbitily chosen (also empty is allowed) and $\mathcal{U} = \mathcal{U}_{\max} \cup \mathcal{U}^{fin} \cup \mathcal{U}_{T,M}$ or if $T = M = \emptyset$ we can also have $\mathcal{U} = \mathcal{U}_{\max} \cup \mathcal{U}^{fin}$.
(b) $\mathcal{U}^{fin}$ infinite, then always $G \in \mathcal{U}$ but T, M must satisfy the following.
  (c1) If $\mathcal{U}^{fin} \cap \mathcal{P}$ is infinite, then $M = \Omega$ (all adics in)
  (c2) If $\mathcal{U}^{fin} \cap \mathcal{Q}$ is infinite, then $T = \Omega$ (all Prüfer in)
  (c3) For every $\lambda \in \Omega$, if $\mathcal{U}^{fin} \cap \{S_\lambda[n] \mid n \in \mathbb{N}\}$ is infinite, then $\lambda \in T \cap M$

Before we start we need to understand some properties of the functors $\mathrm{Hom}_\Lambda(-, U)$ and $\mathrm{Ext}^1_\Lambda(-, U)$ for points of type (2,),(3),(4). In [**34**, p. 46] and [**15**, section 3] we found the following vanishing where we set $S = S_\lambda$ and denote by $\mathcal{R}_\lambda$ be (the tube of) all regular modules with S as a submodule.

$$\mathrm{Hom}_\Lambda(\mathcal{R}, G) = \mathrm{Hom}_\Lambda(\mathcal{Q}, G) = 0 = \mathrm{Ext}^1_\Lambda(\mathcal{R}, G) = \mathrm{Ext}^1_\Lambda(\mathcal{P}, G)$$
$$\mathrm{Hom}_\Lambda(\mathcal{R}, \hat{S}) = \mathrm{Hom}_\Lambda(\mathcal{Q}, \hat{S}) = 0 = \mathrm{Ext}^1_\Lambda(\mathcal{P}, \hat{S}) = \mathrm{Ext}^1_\Lambda(\mathcal{R}_\mu, \hat{S}) \quad \mu \neq \lambda$$
$$\mathrm{Hom}_\Lambda(\mathcal{R}_\mu, S[\infty]) = \mathrm{Hom}_\Lambda(\mathcal{Q}, S[\infty]) = 0 = \mathrm{Ext}^1_\Lambda(\mathcal{R}, S[\infty]) = \mathrm{Ext}^1_\Lambda(\mathcal{P}, S[\infty]) \quad \mu \neq \lambda$$

As a consequence we see: If $U \in \{G, \hat{S}, S[\infty]\}$ and σ a short exact sequence in $\Lambda\,\mathrm{mod}$ with all three terms in either $\mathrm{add}(\mathcal{P})$, $\mathrm{add}(\mathcal{R})$ or $\mathrm{add}(\mathcal{Q})$, then $\mathrm{Hom}(\sigma, U)$ is exact.
As there are very many Ziegler-closed sets in this case, we focus on two types:

(I) Either $\mathcal{U}^{fin} = \emptyset$, these give exact structures containing all almost split sequences.
(II) $\mathcal{U} = \overline{\mathcal{U}^{fin}}$, these are so-called Auslander-Solberg exact structures. Here, this is still an Auslander-Reiten category, the almost split sequences are precisely the ones of $\Lambda\,\mathrm{mod}$ not starting in $\mathcal{U}^{fin}$.

Now, we look at the exact structures in these cases:

(I) For $\mathcal{U} = \overline{\{U\}}$, we set $\mathrm{Ext}^1_U := \mathrm{Ext}^1_{\mathcal{E}\mathcal{U}}$.
We start with the unique maximal not abelian exact structure in this case $\mathcal{U} = \{G\}$, then $\mathrm{Ext}^1_G(\mathcal{X}, \mathcal{Y}) = \mathrm{Ext}^1_\Lambda(\mathcal{X}, \mathcal{Y})$ for all $(\mathcal{X}, \mathcal{Y}) \in \{\mathcal{P}, \mathcal{R}, \mathcal{Q}\}^2 \setminus \{(\mathcal{Q}, \mathcal{P})\}$ and $\mathrm{Ext}^1_G(\mathcal{Q}, \mathcal{P}) = 0$ (for this we leave the proof out). The interesting thing is that this is an exact substructure of global dimension ≥ 2 (probably $= 2$), since the following exact sequence σ is not zero in $\mathrm{Ext}^2_G(\mathcal{Q}, \mathcal{P})$: Let R be a regular module, take a projective Λ-module resolution and an injective Λ-module resolution of R and concatenate to an exact sequence σ

$$P_1 \rightarrowtail P_0 \to I_0 \twoheadrightarrow I_1$$

Observe, this implies for all not abelian exact structures of type (I) that $\mathrm{Ext}^1(\mathcal{Q}, \mathcal{P}) = 0$.

Next, we consider $\mathcal{U} = \overline{\{\hat{S}\}}$, we have $\mathrm{Ext}^1_{\hat{S}}(\mathcal{X}, \mathcal{Y}) = \mathrm{Ext}^1_G(\mathcal{X}, \mathcal{Y})$ for all $(\mathcal{X}, \mathcal{Y}) \neq (\mathcal{R}_\lambda, \mathcal{P})$ and $\mathrm{Ext}^1_{\hat{S}}(\mathcal{R}_\lambda, \mathcal{P}) = 0$.

Now, we consider $\mathcal{U} = \overline{\{S[\infty]\}}$. Then $\mathrm{Ext}^1_{S[\infty]}(\mathcal{Q}, \mathcal{R}_\lambda) = 0$ and $\mathrm{Ext}^1_{S[\infty]}(\mathcal{X}, \mathcal{Y}) = \mathrm{Ext}^1_G(\mathcal{X}, \mathcal{Y})$ for all $(\mathcal{X}, \mathcal{Y}) \in \{\mathcal{P}, \mathcal{Q}, \mathcal{R}_\mu, \mu \in \Omega\}^2 \setminus \{(\mathcal{P}, \mathcal{R}_\lambda)\}$.

In both cases $\mathcal{U} = \overline{\{\hat{S}\}}$ or $\mathcal{U} = \overline{\{S[\infty]\}}$ is the global dimension of $\mathcal{E}^{\mathcal{U}}$ still ≥ 2. Just look at the exact sequence σ as above. Choose in its definition R to be a regular module R with no summand in the tube λ, then for both exact substructures it gives an exact sequence which is not 2-split.

Now, we look at intersections of these exact structures and respectively unions of the Ziegler-closed sets.

When $M = \Omega$ and $T = \emptyset$ then the exact structure consists of ses $\sigma = \sigma_p \oplus \sigma_{rq}$ such that σ_p is an exact sequence in $\mathrm{add}(\mathcal{P})$ and σ_{rq} is an exact sequence in $\mathrm{add}(\mathcal{R} \cup \mathcal{Q})$. It is very easily seen to be hereditary exact.

When $M = \emptyset$ and $T = \Omega$ then the exact structure consists of ses $\sigma = \sigma_{pr} \oplus \sigma_q$ such that σ_{pr} is an exact sequence in $\mathrm{add}(\mathcal{P} \cup \mathcal{R})$ and σ_q is an exact sequence in $\mathrm{add}(\mathcal{Q})$. It is very easily seen to be hereditary exact.

The case $T = M = \Omega$, then this is the minimal exact structure containing all almost split sequences. The short exact sequences in this structure are $\sigma_p \oplus \sigma_r \oplus \sigma_q$ with σ_p is an exact sequence in $\mathrm{add}(\mathcal{P})$, σ_r is an exact sequence in $\mathrm{add}(\mathcal{R})$ and σ_q is an exact sequence in $\mathrm{add}(\mathcal{Q})$. Again, we easily see that this is hereditary exact.

(II) $\mathcal{U}^{fin} =: \mathcal{H}$. The exact structure corresponding to $\mathcal{U}$ is just given by all short exact sequences such that $\mathrm{Hom}(-, H)$ is exact on it for all H in $\mathcal{H}$, we write $\mathcal{E} = (\Lambda \operatorname{mod}, F^{\mathcal{H}})$. This case is well-studied in [**6**], [**4**], [**5**]. If $\mathcal{H}$ is finite, then the exact structure always has enough projectives and enough injectives given by $\mathrm{add}(\mathcal{H})$. Its global dimension can be characterized as follows $\operatorname{gldim} \mathcal{E} \leq k$ is equivalent to the following two conditions (i) $\operatorname{gldim} \mathrm{End}_\Lambda(\bigoplus_{H \in \mathcal{H}} H) \leq k + 2$ and (ii) $\mathrm{id}_{\mathcal{E}} \Lambda \leq k$.

For $\Lambda = KQ$ with Q the Kronecker quiver every global dimension can occur. This can be seen directly, just take $\mathcal{H} = \{(0, 1), (1, 2), (3, 4), (5, 6), (7, 8), \cdots, (2n - 1, 2n)\}$. Then injective coresolutions are calculated via left $\mathrm{add}(\mathcal{H})$-approximations and it can be easily seen that minimal injective coresolutions have at most $(n + 1)$-injective modules, e.g.

$$(2n, 2n + 1) \rightarrowtail (2n - 1, 2n)^{\oplus 2} \to (2n - 3, 2n - 2)^{\oplus 2} \to \cdots \to (1, 2)^{\oplus 2} \twoheadrightarrow (0, 1)$$

If you take $\mathcal{H} = \{S_\lambda, S_\lambda[3]\}$, then you find $\mathrm{id}\, S_\lambda[2] = \infty$ and therefore we have infinite global dimension.

Another class of examples always gives hereditary exact substructures, take $\mathcal{H} = \{(n, n + 1) \mid 0 \leq n \leq N\}$ for some $N \in \mathbb{N}$, then the cogenerator $\mathrm{add}(\mathcal{H})$ is closed under quotients, this is easily seen to imply that the corresponding exact structure is hereditary exact.

Once you take $\mathcal{H}$ infinite, it is also easy to find infinite global dimensions:

$$\mathcal{H} = \{(2n - 1, 2n) \mid n \in \mathbb{N}\}$$

Using minimal injective coresolutions for $(2n, 2n + 1)$ for all $n \in \mathbb{N}$, we find objects of injective dimension n for every $n \in \mathbb{N}$, this implies $\mathrm{gldim} = \infty$.

Example 5.12. We describe all exact substructures on finitely generated modules over a commutative discrete valuation ring R with maximal ideal P. We recall the description of the Ziegler spectrum from [**31**, Section 5.2]:
The points in Zg_R are:

(a) indecomposable modules of finite length R/P^n, $n \geq 1$
(b) the P-adic completion $\overline{R} = \lim R/P^n$ (this is the limit over $\cdots \to R/P^2 \to R/P = k$)
(c) The Prüfer module $R_{P^\infty} = \operatorname{colim} R/P^n$ (this is the colimit over $k = R/P \to R/P^2 \to \cdots$)
(d) the quotient division ring $Q = Q(R)$ of R

Now, $\mathcal{U}_{max} = \{Q, R_{P^\infty}\}$ is the Ziegler-closed set given by the indecomposable injective R-modules. We also observe that $\mathrm{Zg}'_R := \{\overline{R}\} \cup \mathcal{U}_{max}$ is Ziegler-closed.
Next, all Ziegler-closed subsets containing $\mathcal{U}_{max}$ are given by:

(1) for $\emptyset \neq L \subseteq \mathbb{N}$ *finite*, we have

$$\mathcal{U}_L := \{R/P^n \mid n \in L\} \cup \mathcal{U}_{max}$$

(2) for $\emptyset \subseteq L \subseteq \mathbb{N}$ *arbitrary subset* we have

$$\mathcal{V}_L := \{R/P^n \mid n \in L\} \cup \mathrm{Zg}'_R$$

So let us describe the exact structures on $\mathcal{C} = R\,\mathrm{mod}$ the category of finitely generated left R-modules corresponding to these closed sets:

(max) Trivially $\mathcal{U}_{max}$ corresponds to the abelian structure on $\mathcal{C}$, this is hereditary and with enough projectives (but not with enough injectives)
(min) and Zg_R corresponds to the split exact structure.
(Zg') The Ziegler-closed set Zg'_R corresponds to the exact structure $\mathcal{E}'$ making the torsion functor exact. This is a hereditary exact structure , cp. example...
($\mathcal{U}_L$) This corresponds to the exact substructure $\mathcal{E}_L$ such that $\mathrm{Hom}_R(-, R/P^n)$, $n \in L$ are exact functors to abelian groups.
($\mathcal{V}_L$) This corresponds to the exact substructure $\mathcal{E}'_L$ such that the torsion functor and $\mathrm{Hom}_R(-, R/P^n)$, $n \in L$ are exact.

First, of all in general how can one see that for $\emptyset \neq L \subseteq \mathbb{N}$ finite: $\mathcal{E}_L$ and $\mathcal{E}'_L$ are different exact structures?
Take a short exact sequence $R \rightarrowtail R \twoheadrightarrow R/P^n$ and the pushout along $R \twoheadrightarrow R/P^m$, this gives an exact sequence $R/P^m \rightarrowtail R/P^{m+n} \twoheadrightarrow R/P^n$. But the cartesian square induces another exact sequence

$$R \rightarrowtail R/P^m \oplus R \twoheadrightarrow R/P^n$$

It is easily seen to be not exact in $\mathcal{E}'$ if $n \neq m$ and we conclude that $\mathrm{Ext}^1_{\mathcal{E}'}(R/P^n, R) = 0$. But if you apply $\mathrm{Hom}_R(-, R/P^\ell)$ using $\mathrm{Hom}_R(R/P^s, R/P^\ell) = R/P^{\min(s,\ell)}$ we see that this is exact for n, m both larger or equal than ℓ.
We say that L has *gaps* if there is an interval $[a, b]$ such that $[a, b] \cap L = \emptyset$ and $b + 1 \in L$ and $a - 1 \in L$ if $a > 1$. If L has gaps then $\mathrm{gldim}\,\mathcal{E}_L = \infty = \mathrm{gldim}\,\mathcal{E}'_L$ (for $\mathcal{E}'_L$ we also allow L to be an infinite subset).
In this case one can always find an infinite injective $\mathcal{E}_L^{(')}$-coresolution for an R/P^s some $s \in [a, b]$. We give them as sequence of short exact sequences.

($a = 1$) Take $s = b$ and $R/P^b \rightarrowtail R/P^{b+1} \twoheadrightarrow R/P$, then continue with $R/P \rightarrowtail R/P^{b+1} \twoheadrightarrow R/P^b$ and repeat with the first short exact sequence etc.
(2) If $a > 1$ and $b + a$ even then we take $s = \frac{1}{2}(a + b)$ and the short exact sequence

$$R/P^s \to R/P^{a-1} \oplus R/P^{b+1} \twoheadrightarrow R/P^s$$

and then continue with the same sequence.
(3) If $a > 1$ and $b + a$ uneven then we take $s = \frac{1}{2}(b + a - 1)$ and first $R/P^s \rightarrowtail R/P^{a-1} \oplus R/P^{b+1} \twoheadrightarrow R/P^{s+1}$ then $R/P^{s+1} \rightarrowtail R/P^{a-1} \oplus R/P^{b+1} \twoheadrightarrow R/P^s$ and then repeat with the first sequence.

If L has no gaps and $L \neq \mathbb{N}$ then $L = [1, n]$ for some $n \in \mathbb{N}$. The set $\{R/P^\ell \mid \ell \in L\}$ is closed under quotients, so injective coresolutions in this class of modules will always end after one short exact sequence. We show in the next Lemma that these exact structures are hereditary exact.

Lemma 5.13. *Let R denote a commutative discrete valuation ring with maximal ideal P and let $n \in \mathbb{N}$. We have a functor* $\mathrm{rad}^n \colon R\,\mathrm{mod} \to R\,\mathrm{mod}$ *defined by* $\mathrm{rad}^n M = P^n M$. *Let $L = [1, n] \subseteq \mathbb{N}$.*

(1) *Then we have* $\operatorname{Ext}^1_{\mathcal{E}_L}(-,-) = \operatorname{rad}^n \operatorname{Ext}^1_R(-,-)$ *and* $\operatorname{Ext}^1_{\mathcal{E}'_L}(-,-) = \operatorname{rad}^n_R \operatorname{Ext}^1_{\mathcal{E}'}(-,-)$

(2) *For every* $\mathcal{E}_L$*-exact sequence* σ *and every object* X*, the sequences* $\operatorname{Ext}^1_{\mathcal{E}_L}(X,\sigma)$ *is right exact. For every* $\mathcal{E}'_L$*-exact sequence* σ *and every object* X*, the sequences* $\operatorname{Ext}^1_{\mathcal{E}'_L}(X,\sigma)$ *is right exact.*

In particular, $\mathcal{E}_L$ *and* $\mathcal{E}'_L$ *are hereditary exact.*

Proof. (1) We first describe the R-module structure on

(a) $\operatorname{Ext}^1_R(R/P^m, R/P^\ell) \cong R/P^s$ where $s = \min(m,\ell)$. For $a,b \in \mathbb{N}$ we write $\sigma_{a,b} \in \operatorname{Ext}^1_R(R/P^m, R/P^\ell)$ for an exact sequence with middle term $R/P^a \oplus R/P^b$ (whenever this exist). In R/P^s we pick $P = (p)$ and we have the following mult. by p $1 \mapsto p \mapsto p^2 \mapsto \cdots \mapsto p^{s-1} \mapsto 0$ this corresponds to the following on the Ext-group

(a1) If $s = \ell \leq m$ we have
$\sigma_{0,m+\ell} \mapsto \sigma_{1,m+\ell-1} \mapsto \cdots \mapsto \sigma_{\ell-1,m+1} \mapsto 0$
Then we have $\operatorname{rad}^n \operatorname{Ext}^1_R(R/P^m, R/P^\ell) \cong R/P^{s-n}$ whenever $n < s$ and zero otherwise. For $n < s = \ell$ it is the image of p^n, i.e.
$\sigma_{n,m+\ell-n} \mapsto \sigma_{n+1,m+\ell-n-1} \mapsto \cdots \mapsto \sigma_{\ell-1,m+1}$

(a2) If $\ell > m = s$ we have
$\sigma_{\ell+m,0} \mapsto \sigma_{\ell+m-1,1} \mapsto \cdots \mapsto \sigma_{\ell+1,m-1} \mapsto 0$
Then we have $\operatorname{rad}^n \operatorname{Ext}^1_R(R/P^m, R/P^\ell) \cong R/P^{s-n}$ whenever $n < s$ and zero otherwise. For $n < s = \ell$ it contains the following elements
$\sigma_{\ell+m-n,n} \mapsto \sigma_{\ell+m-n-1,n+1} \mapsto \cdots \mapsto \sigma_{\ell+1,m-1}$

(b) $\operatorname{Ext}^1_R(R/P^m, R) \cong R/P^m$ we write σ_a for the extension with $R \oplus R/P^a$ as middle term. Then the multiplication by p is given by
$\sigma_0 \mapsto \sigma_1 \mapsto \cdots \mapsto \sigma_{m-1} \mapsto 0$
The submodule $\operatorname{rad}^n \operatorname{Ext}^1_R(R/P^m, R)$ is of course zero is $n \geq m$ and if $n < m$ is given by the following
$\sigma_n \mapsto \sigma_{n+1} \mapsto \cdots \mapsto \sigma_{m-1}$

We claim $\operatorname{Ext}^1_{\mathcal{E}_{[0,n]}} = \operatorname{rad}^n \operatorname{Ext}^1_R$. First observe that in $\mathcal{E}_L$: R/P^a $1 \leq a \leq n$ are injectives and they are also projectives. So for $s = \min(\ell, m) \leq n$ we have $\operatorname{Ext}^1_{\mathcal{E}_{[0,n]}}(R/P^m, R/P^\ell) = 0$ and for $m \leq n$ we have $\operatorname{Ext}^1_{\mathcal{E}_{[0,n]}}(R/P^m, R) = 0$.

So we may always assume wlog that $n < \min(\ell, m)$, then proceed by induction over n. For $n = 1$, a short exact sequence of in $R\,\mathrm{mod}$ is in $\mathcal{E}_{[0,1]}$ iff the indcomposable summands of number of the indec. summands of the outer terms add up to the indec summands of the middle term. This means in case (a) all exact sequence are in this exact structure except $\sigma_{0,m+n}$ and $\sigma_{m+n,0}$, in case (b) all except σ_0.

For $n > 1$, it is enough to observe that $\operatorname{Hom}(-, R/P^n)$ is

(ad a1) exact on $\sigma_{n,m+\ell-n}$ and not exact on $\sigma_{n-1,m+\ell-n+1}$

(ad a2) exact on $\sigma_{\ell+m-n,n}$ and not exact on $\sigma_{\ell+m-n+1,n-1}$

(ad b) exact on σ_n and not exact on σ_{n-1}

Then the rest follows by induction hypothesis.

Now, for $\mathcal{E}'_L$ one observes that $\operatorname{Ext}^1_{\mathcal{E}'_L}(X,Y) = \operatorname{Ext}^1_{\mathcal{E}_L}(X,Y)$ for all X, Y torsion, and for Y free it is zero.

As $\operatorname{Ext}^1_{\mathcal{E}'}(X,Y) = \operatorname{Ext}^1_R(X,Y)$ for all X,Y torsion and for Y free it is zero. Then the claim $\operatorname{Ext}^1_{\mathcal{E}'_L}$ follows from the proof for $\operatorname{Ext}^1_{\mathcal{E}_L}$.

(2) Taking n-th radical of an epimorphism in $R\,\mathrm{mod}$ is again an epimorphism - as the n-th radical can be described as the image of multiplication by p^n (making it also a quotient and not ony a submodule). As $\operatorname{Ext}^1_R(X,\sigma)$ (resp. $\operatorname{Ext}^1_R(\sigma, X)$) is right exact for all exact sequences σ, we conclude that $\operatorname{rad}^n \operatorname{Ext}^1_R(X, f)$ (resp. $\operatorname{rad}^n \operatorname{Ext}^1_R(g, X)$) are epimorphisms for f an epimorphism and g a monomorphism.

Of course, on general exact sequence rad^n is not a middle-exact functor (but this follows from (1) since for $\sigma \in \mathcal{E}_L$ the sequences $\operatorname{Ext}^1_{\mathcal{E}_L}(X,\sigma)$ and $\operatorname{Ext}^1_{\mathcal{E}_L}(\sigma, X)$ are middle exact). In particular, the right exactness claim follows.

For $\mathrm{Ext}^1_{\mathcal{E}'_L}(X,\sigma)$ we can restrict to X indecomposable torsion and as σ in $\mathcal{E}'$, it fits into an exact sequence of ses $\sigma_{tor} \rightarrowtail \sigma \twoheadrightarrow \sigma_{free}$ with σ_{tor} the torsion part and σ_{free} the free part, and we conclude that $\mathrm{Ext}^1_{\mathcal{E}'_L}(X,\sigma) = \mathrm{Ext}^1_{\mathcal{E}'_L}(X,\sigma_{tor}) = \mathrm{Ext}^1_{\mathcal{E}_L}(X,\sigma_{tor})$ is right exact.

□

After understanding exact substructures of $R\,\mathrm{mod}$ for R a commutative discrete valuation ring, we are ready to generalize this to commutative Dedekind domains:

Example 5.14. We describe all exact structures on finitely generated left modules over a commutative Dedekind domain.
The Ziegler spectrum is studied as a more general case of the discrete valuation ring, again we follow [31, Section 5.2] for its description: Let $\mathrm{mSpec}(R) := \{P \mid \text{ max. ideal in } R\}$. The points in Zg_R are:

(a) indecomposable modules of finite length R/P^n, $n \geq 1$, and $P \in \mathrm{mSpec}(R)$.
(b) the P-adic completion $\overline{R}_P = \lim R/P^n$ (this is the limit over $\cdots \to R/P^2 \to R/P$) for $P \in \mathrm{mSpec}(R)$.
(c) The Prüfer module $R_{P^\infty} = \mathrm{colim}\, R/P^n$ (this is the colimit over $R/P \to R/P^2 \to \cdots$) for $P \in \mathrm{mSpec}(R)$.
(d) the quotient division ring $Q = Q(R)$ of R

Now, $\mathcal{U}_{max} = \{Q\} \cup \{R_{P^\infty} \mid P \in \mathrm{mSpec}(R)\}$ is the Ziegler-closed set given by the indecomposable injective R-modules. We describe all Ziegler-closed subsets containing $\mathcal{U}_{max}$ (following loc. cit.).
First we fix some notation, let $L \subseteq \mathrm{mSpec}(R) \times \mathbb{N}$ always denote such a subset and for $P \in \mathrm{mSpec}(R)$ let $L_P := \{\ell \in \mathbb{N} \mid (\ell, P) \in L\}$. Subsets of indecomposable finite length modules are of the form $\mathcal{F}_L = \{R/P^\ell \mid (P,\ell) \in L\}$. We fix a closed subset $\mathcal{U}$ and denote by $\mathcal{F}_L$ its points of finite length.
(type 1) $L = \emptyset$. For every $M \subseteq \mathrm{mSpec}(R)$ we have a closed subset $\mathrm{Zg}'_M = \mathcal{U}_{max} \cup \{\overline{R}_P \mid P \in M\}$. We define $\mathrm{Zg}' := \mathrm{Zg}'_{\mathrm{mSpec}(R)}$.
(type 2) $0 < |L| < \infty$. Then $\mathcal{U} = \mathcal{F}_L \cup \mathrm{Zg}'_M$ for an arbitrary subset $M \subseteq \mathrm{mSpec}(R)$ (the sets L and M are independent from each other).
(type 3) $|L| = \infty$. We define $M_L := \{P \in \mathrm{mSpec}(R) \mid \exists\, n \in \mathbb{N} \colon (P,n) \in L\}$ and in this case $\mathcal{U} = \mathcal{F}_L \cup \mathrm{Zg}'_{M_L}$.
Let us look at the corresponding exact substructures:

(type 1) Make all P-torsion functors exact for all $P \in M$. As we are dealing with hereditary torsion pairs, the torsion functors are left exact and .. applies to show that these are hereditary exact structures.
(gaps) Let us assume we are in type 2 or type 3.
If for some $P \in \mathrm{mSpec}(R)$ we have L_P has a gap (see previous example), then we find an infinite injective coresolution as in the previous example and it follows $\mathrm{gldim} = \infty$.
(no gaps) Let us assume we are in type 2 or type 3. If all non-empty L_P have no gaps for every maximal ideal P, then we find $L_P = [1, n_P]$ for an $n_P \in \mathbb{N}$. We claim that we only have hereditary exact structures in this case. We give the proof in the next Lemma.

Now, let R be a commutative Dedekind domain. Observe that Ext^1_R only takes values in torsion modules. We write $()_P$ for its P-torsion submodule and $()_{tor}$ for the remaining torsion summands. So, for two finitely generated R-modules $X = R^t \oplus X_P \oplus X_{tor}, Y = R^s \oplus Y_P \oplus Y_{tor}$ we have

$$\mathrm{Ext}^1_R(X,Y) = \mathrm{Ext}^1_R(X,Y)_P \oplus \mathrm{Ext}^1_R(X,Y)_{tor}$$
$$\mathrm{Ext}^1_R(X,Y)_P = \mathrm{Ext}^1_R(X_P, Y_P \oplus R^s)$$
$$\mathrm{Ext}^1_R(X,Y)_{tor} = \mathrm{Ext}^1_R(X_{tor}, Y_{tor} \oplus R^s)$$

Then $\mathrm{Ext}^1_R = (\mathrm{Ext}^1_R)_P \oplus (\mathrm{Ext}^1_R)_{tor}$ is a direct sum decomposition of bifunctors (but this does not imply that these subfunctors are middle exact for the abelian structure on $R\,\mathrm{mod}$). Let $F_P \colon R\,\mathrm{mod} \to R\,\mathrm{mod}$ be the functor $X \mapsto R^t \oplus X_P$ (resp. $F_{tor}(X) = R^t \oplus X_P$), induced by the projection into the torsionfree part (in the split hereditary torsion pairs considered in (type 1)). They preserve epimorphisms. In particular, if $f\colon A \twoheadrightarrow B$ is an epimorphism, then the following are also epimorphism as $\operatorname{Ext}^1_R(M,-)$ preserves epimorphisms for all objects M

$$\operatorname{Ext}^1_R(X,f)_P = \operatorname{Ext}^1_R(F_P(X),F_P(f))$$
$$\operatorname{Ext}^1_R(X,f)_{tor} = \operatorname{Ext}^1_R(F_{tor}(X),F_{tor}(f))$$

As it looks simpler let us look at the case of only one prime:

Lemma 5.15. *Let R be a commutative Dedekind domain. Let P be a fixed maximal prime ideal. We define for $M \in R\operatorname{mod}, P \in \operatorname{mSpec}(R), n \in \mathbb{N}$ the following $\operatorname{rad}^n_P M := P^n M$. If $L = \{P\} \times [1,n]$ we denote by $\mathcal{E}_L$ the exact structure corresponding to $\mathcal{F}_L \cup \mathcal{U}_{max}$.*

1. *Then $\operatorname{Ext}^1_{\mathcal{E}_L} = (\operatorname{rad}^n_P \operatorname{Ext}^1_R) \oplus (\operatorname{Ext}^1_R)_{tor}$*
2. *The exact structure $\mathcal{E}_L$ is hereditary exact.*

Proof. (1) As for different primes the torsion submodules are Hom_R- and Ext_R-orthogonal, the exactness of $\operatorname{Hom}(-,R/P^a)$ for some $a \in \mathbb{N}$ only depends on the P-torsion and the free module summand. The same proof as in Lemma 5.13 applies.

(2) By the discussion before the Lemma and knowing that rad^n preserves epimorphisms, it follows from (1) that $\operatorname{Ext}^1_{\mathcal{E}_L}$ preserves epimorphisms. Therefore $\mathcal{E}_L$ is hereditary exact.

□

But it actually is the same for an arbitrary subset of primes:

Lemma 5.16. *Let R be a commutative Dedekind domain. Let $L = \bigcup_{P\in M}\{P\} \times [1,n_P] \subseteq \operatorname{mSpec}(R)\times\mathbb{N}$ and $\mathcal{E} = \mathcal{E}^{\mathcal{U}}$ for a Ziegler-closed subset $\mathcal{U}$ with modules of finite length given by $\mathcal{F}_L$, then $\mathcal{E}$ is hereditary exact.*

Proof. (of Cor. 5.16) $\mathcal{U} = \mathcal{F}_L \cup \operatorname{Zg}'_M$ for some subset $M \subseteq \operatorname{mSpec}(R)$ (type 2 or type 3). Let us denote by an index $tor(M^c)$ the torsion summand corresponding to the complement of M. The above Lemma generalizes to

$$\operatorname{Ext}^1_{\mathcal{E}_L} = \bigoplus_{P\in M} (\operatorname{rad}^{n_P}_P \operatorname{Ext}^1_R) \oplus (\operatorname{Ext}^1_R)_{tor(M^c)}$$

This is clear as intersection of exact substructures correspond to intersecting the corresponding Ext^1-subfunctors.
This functor is still preserving epimorphisms as before. □

6. The functorial point of view

Let $\mathcal{E}$ be an essentially small exact category. We consider three classical assignments for $\mathcal{F} \in \operatorname{ex}(\mathcal{E})$

(i) the Auslander category
(ii) its category of inflation represented functors
(iii) the category of deflation represented functors (called effaceable functors)

all will be considered fully exact subcategories in $\operatorname{mod}_1 \mathcal{A}$ (where $\mathcal{A}$ is the underlying additive category). By a results of [**22**] and [**18**], we have characterizations of the subcategories when we look only at exact substructures of $\mathcal{E}$.

Recall a Serre subcategory is a full additive subcategory $\mathcal{F}$ in an exact category $\mathcal{E}$ with the following property: For every $\mathcal{E}$-short exact sequence $X \to Y \to Z$ we have $Y \in \mathcal{F}$ if and only if $X, Z \in \mathcal{F}$.

Definition 6.1. We denote by $\mathcal{P}^2(\mathcal{A})$ the full subcategory of $\operatorname{Mod}\mathcal{A}$ given by all functors F such that there exists an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(-,X) \to \operatorname{Hom}_{\mathcal{A}}(-,Y) \to \operatorname{Hom}_{\mathcal{A}}(-,Z) \to F \to 0$$

for some X, Y, Z in $\mathcal{A}$.

$$\mathcal{G}^2(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d),(j,p) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{j}\circ \mathrm{d})\}$$
$$\mathcal{C}^2(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{d})\}$$
$$\mathcal{J}^1(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (j,p) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{j})\}$$

Apriory these are additive categories. The **grade** of $F \in \operatorname{Mod}\mathcal{A}$ is defined as the supremum of all natural numbers $i \geq 0$ such that $\operatorname{Ext}^j_{\operatorname{Mod}\mathcal{A}}(F, \operatorname{Hom}_{\mathcal{A}}(-,A)) = 0 \forall A \in \mathcal{A}$ for all $j < i$ (of course, only if this exists, else we define it to be ∞). Categories defined in terms of grade equalities are by definition extension-closed in $\operatorname{Mod}-\mathcal{A}$. We have

$$\mathcal{G}^2(\mathcal{A}) \subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} \in \{0,2\}\}$$
$$\mathcal{C}^2(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} = 2\}$$
$$\mathcal{J}^1(\mathcal{A}) \subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} = 0\}$$

The two inclusions follow from the definition. The equality in the middle is proven in [18], Lemma 2.3, so $\mathcal{C}^2(\mathcal{A})$ is extension-closed in $\mathcal{P}^2(\mathcal{A})$. (These subcategories $\mathcal{G}^2(\mathcal{A}), \mathcal{J}^1(\mathcal{A})$ can be fully characterized using the Auslander-Bridger transpose, if one is keen to avoid $\mathrm{KC}(\mathcal{A})$ - look at [22], chapter 5 and the HKR-bijection for more details). We do not mind working with $\mathrm{KC}(\mathcal{A})$ and use these not always extension-closed subcategories.

Now we define our categories of interest.

Definition 6.2. Let $\mathcal{E}$ be essentially small exact. The **Auslander exact category** of $\mathcal{E}$ is defined as the full subcategory of $\mathcal{P}^2(\mathcal{A})$ given by

$$\mathrm{AE}(\mathcal{E}) = \{\mathrm{F} \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{F} = \operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{f})), \text{ with f } \mathcal{E}\text{-admissible}\}$$

It has as a subcategory

$$\mathrm{eff}(\mathcal{E}) = \mathrm{AE}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}) = \{\operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{d}) \mid \mathrm{d}\ \mathcal{E}\text{-deflation}\}$$

called the subcategory of **effaceable functors** and another subcategory

$$\mathrm{H}(\mathcal{E}) := \{\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, i) \mid i\ \mathcal{E}\text{-inflation}\}$$

which we refer to as **tf-Auslander category** (cf. Appendix B).

Obviously: $\mathrm{AE}(\mathcal{E}) \subseteq \mathcal{G}^2(\mathcal{A}), \mathrm{eff}(\mathcal{E}) \subseteq \mathcal{C}^2(\mathcal{A}), \mathrm{H}(\mathcal{E}) \subseteq \mathcal{J}^1(\mathcal{A})$.
In [22, Prop. 3.5, Prop.3.6, Prop.5.4], it is shown that $\mathrm{AE}(\mathcal{E})$ is extension-closed in $\operatorname{mod}_1\mathcal{A}$, even resolving in $\mathcal{P}^2(\mathcal{A})$, and $(\mathrm{H}(\mathcal{E}), \mathrm{eff}(\mathcal{E}))$ is a torsion pair in $\mathrm{AE}(\mathcal{E})$. In particular, $\mathrm{H}(\mathcal{E})$ is also a resolving subcategory of $\mathrm{AE}(\mathcal{E})$.
To state the results that we look at two (different) dualities. Enomoto found the following duality between $\mathcal{C}^2(\mathcal{A})$ and $\mathcal{C}^2(\mathcal{A}^{op})$

THEOREM 6.3. *(Enomoto) Let $\mathcal{A}$ be an idempotent complete small additive category. There exists a duality $E\colon \mathcal{C}^2(\mathcal{A})^{op} \to \mathcal{C}^2(\mathcal{A}^{op})$ such that*
$E(\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,d)) \cong \operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(i,-))$ for every kernel-cokernel pair (i,d) in $\mathcal{A}$.

Then he can characterize exact structures as follows

THEOREM 6.4. *(Enomoto's bijection) Given a small idempotent complete additive category $\mathcal{A}$. Then the assignments $\mathcal{E} \mapsto \mathrm{eff}(\mathcal{E})$ and $\mathcal{C} \mapsto \mathcal{S} := \{(i,d) \in \mathrm{KC}(\mathcal{A}) \mid \operatorname{coker}\operatorname{Hom}(-, \mathrm{d}) \in \mathcal{C}\}$ give inverse bijections between*

(1) *exact structures on $\mathcal{A}$*

(2) *full subcategories* $\mathcal{C} \subseteq \mathcal{C}^2(\mathcal{A})$ *with* $\mathcal{C}$ *is a Serre subcategory in* $\operatorname{mod}_1 \mathcal{A}$ *and* $E(\mathcal{C})$ *is a Serre subcategory in* $\operatorname{mod}_1 \mathcal{A}^{op}$

If we denote by $\mathcal{E}_{max}$ *the maximal exact structure on* $\mathcal{A}$ *with corresponding Serre subcategory* $\mathcal{C}_{max}$ *then (2) coincides with the following.*

(2') *Serre subcategories* $\mathcal{C}$ *of* $\mathcal{C}_{max}$

Where (2') is an observation of Kevin Schlegel (cf. [**36**, Cor. 2.3]).

The second duality is Auslander-Bridger transpose. We need the ideal quotient with respect to the projectives (called the stable category) - this exists even if the category has not enough projectives. As there are no grade 0 objects in $\mathcal{C}^2(\mathcal{A})$ the composition

$$\mathcal{C}^2(\mathcal{A}) \to \operatorname{mod}_1 \mathcal{A} \to \underline{\operatorname{mod}_1}\mathcal{A}$$

is still fully faithful.

THEOREM 6.5. *Let* $\mathcal{A}$ *be an idempotent complete small additive category. Then we have a duality* $\operatorname{Tr}\colon (\underline{\operatorname{mod}_1}\mathcal{A})^{\mathrm{op}} \to \underline{\operatorname{mod}_1}(\mathcal{A}^{\mathrm{op}})$ *which maps* $\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,f)$ *to* $\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(f,-)$.

Remark 6.6. On objects $(\mathrm{Tr}\circ\Omega)(\mathrm{C}) \cong \mathrm{E}(\mathrm{C})$ for C in $\mathcal{C}^2(\mathcal{A})$ but in general Ω is not an endofunctor on the stable category. (But on stable categories of exact categories *with* enough projectives, Ω defines an endofunctor.)

For a subcategory $\mathcal{X} \subseteq \operatorname{mod}_1 \mathcal{A}$, we denote by $\mathrm{Tr}(\mathcal{X})$ the full subcategory of $\operatorname{mod}_1 \mathcal{A}^{op}$ consisting of objects X such that $X \cong \mathrm{Tr}(\mathrm{X}')$ in $\underline{\operatorname{mod}_1}\mathcal{A}^{op}$ for some X' in $\mathcal{X}$.

Remark 6.7. By definition

$$\mathrm{Tr}(\mathcal{G}^2(\mathcal{A})) = \mathcal{G}^2(\mathcal{A}^{\mathrm{op}}), \quad \Omega\mathrm{Tr}(\mathcal{J}^1(\mathcal{A})) = \mathcal{J}^1(\mathcal{A}^{\mathrm{op}})$$

For every $\mathcal{X} \subseteq \mathcal{G}^2(\mathcal{A})$ containing all representables: $\mathrm{Tr}\mathrm{Tr}(\mathcal{X}) = \mathcal{X}$.
For every $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A})$ containing all representables: $\Omega\mathrm{Tr}\Omega\mathrm{Tr}(\mathcal{J}) = \mathcal{J}$.

THEOREM 6.8. *(HKR-bijection) Given a small idempotent complete additive category* $\mathcal{A}$*. Then the assignments* $\mathcal{E} \mapsto \mathrm{AE}(\mathcal{E})$ *gives a bijection between*

(1) *exact structures on* $\mathcal{A}$
(2) *resolving subcategories* $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A})$ *with all objects have either grade* 0 *or* 2 *such that* $\mathrm{Tr}(\mathcal{X}) \subseteq \mathcal{P}^2(\mathcal{A}^{\mathrm{op}})$ *is resolving and all objects have either grade* 0 *or* 2.

Furthermore, in this case, the full subcategory of grade 2 *objects is* $\mathrm{AE}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}) = \mathrm{eff}(\mathcal{E})$ *and the one of grade* 0*-objects is* $\mathrm{AE}(\mathcal{E}) \cap \mathcal{J}^1(\mathcal{A}) = \mathrm{H}(\mathcal{E})$.

Open question 6.9. In (2) we could use $\mathcal{G}^2(\mathcal{A})$ as well. Furthermore, we can also use the maximal exact structure, let $\mathcal{X}_{max}$ be the resolving subcategory of $\mathcal{P}^2(\mathcal{A})$ corresponding to the maximal exact structure. Is the following (2') equivalent to (2)?
(2') Resolving subcategories $\mathcal{X}$ in $\mathcal{X}_{max}$.

We add the following third bijection.

THEOREM 6.10. *Given a small idempotent complete additive category* $\mathcal{A}$*. Then the assignments* $\mathcal{E} \mapsto \mathrm{H}(\mathcal{E})$ *gives a bijection between*

(1) *exact structures on* $\mathcal{A}$
(2) *full subcategories* $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A})$ *such that* $\mathcal{J} \subseteq \mathcal{P}^1(\mathcal{A})$ *and* $\Omega\mathrm{Tr}(\mathcal{J}) \subseteq \mathcal{P}^1(\mathcal{A}^{\mathrm{op}})$ *are both resolving.*

Open question 6.11. Again, we look at $\mathcal{J}_{max}$ to be the resolving subcategory of $\mathcal{P}^1(\mathcal{A})$ corresponding to the maximal exact structure, then can we describe the subcategories in (2) also as the following?
(2') Resolving subcategories $\mathcal{J}$ in $\mathcal{J}_{max}$.

Remark 6.12. As $\mathcal{P}^1(\mathcal{A})$ is an hereditary exact (i.e. gldim ≤ 1) with enough projectives we have that a full subcategory is resolving if and only if it is fully exact, closed under summands and contains the projectives.

We need the following lemma for the proof.

Lemma 6.13. *If i is a monomorphism in $\mathcal{A}$ such that $F = \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, i) \in \mathcal{J}^1(\mathcal{A})$ then there is $(i, p) \in \mathrm{KC}(\mathcal{A})$. Also, if $\mathcal{A}$ is idempotent complete then $\mathcal{J}^1(\mathcal{A})$ is closed under taking direct sums and summands (in $\mathcal{P}^1(\mathcal{A})$).*

Proof. By assumption we can find a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \xrightarrow{q} & Z \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \\ U & \xrightarrow{i} & V & & \end{array}$$

with $(j, q) \in \mathrm{KC}(\mathcal{A})$ and $X \xrightarrow{(a,j)^t} U \oplus Y \xrightarrow{(i,-b)} V$ (call this (*)) a split exact sequence (this implies in particular that the commuting square is a pullback-pushout diagramm in $\mathcal{A}$). Then there exists $p \colon V \to Z$ with $q = pb$ and $p = \operatorname{coker}(i)$ - see e.g. [16], Lemma 2.3 (or prove this directly). We claim that (using the split exact sequence) we can also show $i = \ker(p)$ (i.e. $(i, p) \in \mathrm{KC}(\mathcal{A})$.
Given $r \colon R \to V$ with $pr = 0$. We claim that r factors over i (as i is a monomorphism, such a factorization is unique). We form the pullback (see below) of (*) along r in the split exact category and consider the commutative diagramm:

$$\begin{array}{ccccc} X & \longrightarrow & E & \xrightarrow{d} & R \\ = \downarrow & & \downarrow{\scriptstyle (u,y)^t} & & \downarrow{\scriptstyle r} \\ X & \longrightarrow & U \oplus Y & \xrightarrow{(i,-b)} & V \\ & & & {\scriptstyle (0,-q)}\searrow & \downarrow{\scriptstyle p} \\ & & & & Z \end{array}$$

As $(0, -q) \circ (u, y)^t = 0$ we find that $(u, y)^t$ factors uniquely through $(1, j)^t = \ker(0, -q) \colon U \oplus X \to U \oplus Y$, i.e. $y = jx$ for an $x \colon E \to X$. Therefore, we have (using the two commuting squares from before)

$$rd = iu - bjx = iu - iax = i(u - ax)$$

As d is a split epimorphism there exists an $s \colon R \to E$ with $ds = 1_R$ and therefore $r = i(u - ax)s$ as claimed.
Closed under direct sums: Straight forward using the horseshoe lemma and the fact that direct sums of kernel-cokernel pairs are again kernel-cokernel pairs.
Now assume $F \oplus G \in \mathcal{J}^1(\mathcal{A})$. As $\mathcal{P}^1(\mathcal{A})$ is closed under taking summands, choose monomorphisms i, j such that $F = \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, i), G = \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, j)$ and by the horseshoe lemma we conclude $F \oplus G = \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, i \oplus j)$. By the previous part it follows that there exists $(i \oplus j, g) \in \mathrm{KC}(\mathcal{A})$. We look at projection onto and then inclusion of the summand i of the two-term complex given by $i \oplus j$. This induces an idempotent endomorphism e on the cokernel $Z \to Z$. By assumption this idempotent is split admissible, i.e. factors as $Z \xrightarrow{\pi} Z_1 \xrightarrow{\iota} Z$ with π split epimorphism and ι split monomorphism. Then, it is straight forward to see that $g = p \oplus q$ with $(i, p), (j, q) \in \mathrm{KC}(\mathcal{A})$ and therefore $F, G \in \mathcal{J}^1(\mathcal{A})$.

$\square$

Proof. (of Thm 6.10). Given an exact structure $\mathcal{E}$, the category $\mathcal{J} = \mathrm{H}(\mathcal{E}) \subseteq \mathcal{J}^1(\mathcal{A})$ and $\Omega\mathrm{Tr}(\mathcal{J}) = \mathrm{H}(\mathcal{E}^{\mathrm{op}}) \subseteq \mathcal{J}^1(\mathcal{A}^{\mathrm{op}})$ by definition. As observed before $\mathcal{J}$ is resolving in $\mathrm{AE}(\mathcal{E})$ which is resolving in $\mathcal{P}^2(\mathcal{A})$ by Thm 6.8, this implies that it is also resolving in $\mathcal{P}^1(\mathcal{A})$. Dually, we also have $\Omega\mathrm{Tr}(\mathcal{J})$ is resolving in $\mathcal{P}^1(\mathcal{A}^{op})$, so the map is well-defined.
Conversely given $\mathcal{J}$ as in (2), we consider $\mathcal{S} = \{(i,d) \in \mathrm{KC}(\mathcal{A}) \mid \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-,\mathrm{i}) \in \mathcal{J}\}$ and claim that $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ is an exact structure. As $\mathcal{J}$ contains the projectives, all split exact sequences are contained in $\mathcal{S}$. For $(i,d) \in \mathcal{S}$ we call i an $\mathcal{E}$-inflation and d an $\mathcal{E}$-deflation. We claim the following

(e1) composition ji of two (composable) $\mathcal{E}$-inflations j,i are $\mathcal{E}$-inflations
(e2) for every morphism f and every $\mathcal{E}$-inflation i (starting at the same object), the morphism $\left(\begin{smallmatrix} i \\ -f \end{smallmatrix}\right)$ is an $\mathcal{E}$-inflation with cokernel (f', i') where i' is again an $\mathcal{E}$-inflation.

Together with the dual statements for the deflations implied by $\Omega\mathrm{Tr}(\mathcal{J})$ having the same properties it follows that $\mathcal{E}$ is an exact category.
We use the following notation $P_X = \operatorname{Hom}_{\mathcal{A}}(-, X)$ and for a morphism $f\colon X \to Y$ we have $P_f = \operatorname{Hom}_{\mathcal{A}}(-, f)\colon P_X \to P_Y$.

(e1) Let $i\colon X \to Y$ and $j\colon Y \to V$ be two $\mathcal{E}$-inflations in $\mathcal{A}$, we look at the commutative diagram where F, G, H are defined as $\operatorname{coker} P_i, \operatorname{coker} P_{ji}, \operatorname{coker} P_j$ respectively.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_X & \xrightarrow{P_i} & P_Y & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow{P_j} & & \downarrow & & \\
0 & \longrightarrow & P_X & \xrightarrow{P_{ji}} & P_V & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow{P_i} & & \downarrow{=} & & \downarrow & & \\
0 & \longrightarrow & P_Y & \xrightarrow{P_j} & P_V & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

In particular i, j, ji are monomorphisms and the three rows are exact (in $\mathcal{P}^1(\mathcal{A})$). Now, using the *ker-coker sequence* (e.g. [**12**], Prop. 8.11), in $\mathrm{Mod}-\mathcal{A}$, we can deduce that $0 \to F \to G \to H \to 0$ is a short exact sequence on $\mathcal{P}^1(\mathcal{A})$. As $\mathcal{J}$ is extension-closed, it follows that G is an object in $\mathcal{J}$. By Lemma 6.13 it follows that ji is an $\mathcal{E}$-inflation.

(e2) We have $\begin{pmatrix} i \\ -f \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and as composition and direct sums of $\mathcal{E}$-inflations are $\mathcal{E}$-inflations by (e1) and Lemma 6.13 we conclude that $\begin{pmatrix} i \\ -f \end{pmatrix}$ is again an $\mathcal{E}$-inflation. So there exists $(\begin{pmatrix} i \\ -f \end{pmatrix}, (g \quad j)) \in \mathrm{KC}(\mathcal{A})$, i.e. we have a pullback-pushout diagramm in $\mathcal{A}$

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{f} & & \downarrow{g} \\
U & \xrightarrow{j} & V
\end{array}$$

We need to see that j is an $\mathcal{E}$-inflation. It is again a monomorphism in $\mathcal{A}$, therefore we have $F = \operatorname{coker} P_j \in \mathcal{P}^1(\mathcal{A})$. We will show $F \in \mathcal{J}$:

As we have an $(i, p\colon Y \to Z) \in \mathrm{KC}(\mathcal{A})$ we can find a cokernel $q = \operatorname{coker}(j)$ with $p = qg$. But $j = \ker q$ is not directly clear.

We look at the covariant functors $P^A = \operatorname{Hom}_{\mathcal{A}}(A, -)$ and define $G := \operatorname{coker} P_i$. We find a commutative diagram with rows exact in $\mathrm{Mod}-\mathcal{A}^{op}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P^Z & \longrightarrow & P^Y & \longrightarrow & P^X & \longrightarrow & \mathrm{Tr(G)} & \longrightarrow & 0 \\
 & & \uparrow{=} & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^Z & \longrightarrow & P^V & \longrightarrow & P^U & \longrightarrow & \mathrm{Tr(F)} & \longrightarrow & 0
\end{array}$$

We get an induced commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^Z & \longrightarrow & P^Y & \longrightarrow & \Omega\mathrm{Tr}(\mathrm{G}) & \longrightarrow & 0 \\ & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & P^Z & \longrightarrow & P^V & \longrightarrow & \Omega\mathrm{Tr}(\mathrm{F}) & \longrightarrow & 0 \end{array}$$

Therefore, the right hand square is a pullback-pushout square and we get an induced exact sequence

$$0 \to P^V \to P^Y \oplus \Omega\mathrm{Tr}(\mathrm{F}) \to \Omega\mathrm{Tr}(\mathrm{G}) \to 0$$

As $\Omega\mathrm{Tr}(\mathcal{J})$ is resolving, it follows that $\Omega\mathrm{Tr}(\mathrm{F}) \in \Omega\mathrm{Tr}(\mathcal{J})$. This implies $F \in \mathcal{J}$.

If we denote this exact category by $\mathcal{E} = \mathcal{E}_{\mathcal{J}}$, then we easily see by definition $\mathcal{J} = \mathrm{H}(\mathcal{E}_{\mathcal{J}})$.
Also, by definition we have for an exact structure $\mathcal{E}$ that $\mathcal{E} \leq \mathcal{E}_{\mathrm{H}(\mathcal{E})}$ is an exact substructure. For equality, we need to see: If $(i\colon X \to Y, p) \in \mathrm{KC}(\mathcal{A})$ such that $F := \operatorname{coker} P_i \in \mathrm{H}(\mathcal{E})$ then i is an $\mathcal{E}$-inflation. By definition there exist an $\mathcal{E}$-inflation $j\colon U \to V$ such that $\operatorname{coker} P_j = F$. Looking at the projective presentations we get a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_U & \xrightarrow{P_j} & P_V & \longrightarrow & F & \longrightarrow & 0 \\ & & {\scriptstyle P_u}\downarrow & & {\scriptstyle P_v}\updownarrow & & {\scriptstyle =}\downarrow & & \\ 0 & \longrightarrow & P_X & \xrightarrow{P_i} & P_Y & \longrightarrow & F & \longrightarrow & 0 \end{array}$$

Since the right hand square has to be bicartesian it follows that we have a split exact sequence $P_U \rightarrowtail P_X \twoheadrightarrow P_V \to P_Y$ which has to be split as P_Y is projective. This means we have split exact sequence $U \rightarrowtail X \oplus V \twoheadrightarrow Y$, in particular we have a pullbak-pushout diagram in $\mathcal{A}$

$$\begin{array}{ccc} U & \xrightarrow{j} & V \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

As j is an $\mathcal{E}$-inflation it follows that i is also one. □

7. Appendix on equivalences of categories

We shortly review three equivalences of 2-categories for small exact categories.
We will only consider strict 2-categories, i.e. they are enhanced in the category of small categories. This means 1-morphisms will be certain functors between categories and 2-morphisms will be natural transformations between them. We only consider strict 2-functors, i.e. these are which preserve compositions of 1-morphisms and compositions of 2-morphisms.

(A) The Butler-Horrock theorem (seeing exact categories as extriangulated categories)
(B) The Auslander correspondence (going to functor categories)
(C) Ind-completion (passes from small exact to locally coherent exact structures)

8. Appendix A: Going into extriangulated - The Butler Horrocks Theorem

For a small additive category $\mathcal{A}$ we denote by $\mathcal{E}_{max}$ its maximal exact structure and we set $\mathrm{Ext}^1_{\mathcal{A}} := \mathrm{Ext}^1_{\mathcal{E}_{max}}$.
The Butler Horrock's theorem (Thm. 3.4) gives for a small additive category a one-to-one correspondence between closed sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{A}}$ and exact structures on $\mathcal{A}$. By [**28**], the pair $(\mathcal{A}, \mathrm{Ext}^1_{\mathcal{E}})$ gives an extriangulated category.
We have $\mathbb{E}\mathbf{x}$ is the 2-category of small exact categories, 1-morphisms are exact functors and 2-morphisms are natural transformations.

In [**9**] the 2-category $\mathbb{E}$**xtri** is defined and exact categories are embedded via the Butler-Horrocks theorem as objects in it. Let $\mathbb{BH}$ be the full 2-subcategory of the 2-category of extriangulated categories with objects in exact categories. It can be described as follows:

(1) **Objects** are pairs $(\mathcal{A}, \mathbb{E})$ of a small additive category $\mathcal{A}$ together with an additive bifunctor $\mathbb{E}\colon \mathcal{A} \times \mathcal{A}^{op} \to (Ab)$ which is a closed sub-bifunctor of $\mathrm{Ext}^1_{\mathcal{A}}$.
(2) **1-Morphisms** $(\mathcal{A}, \mathbb{E}) \to (\mathcal{A}', \mathbb{E}')$ are pairs of an additive functor $f\colon \mathcal{A} \to \mathcal{A}'$ and a natural transformation $\varphi\colon \mathbb{E} \to \mathbb{E}'(f(-), f(-))$ of functors on $\mathcal{A}$ which satisfies the following *connecting-property*: Let $\mathcal{E}, \mathcal{E}'$ be the exact categories corresponding to $(\mathcal{A}, \mathbb{E}), (\mathcal{A}', \mathbb{E}')$. For every $\mathcal{E}$-exact sequence (i, p) we have an associated distinguished triangle in $D^b(\mathcal{E})$, with $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\sigma} X[1]$, then we have a distinguished triangle in $D^b(\mathcal{E}')$
$$f(X) \xrightarrow{f(i)} f(Y) \xrightarrow{f(p)} f(Z) \xrightarrow{\varphi(\sigma)} f(X)[1]$$
A morphism (f, φ) induces a functor $\mathrm{Ses}(f)\colon \mathrm{Ses}(\mathcal{E}) \to \mathrm{Ses}(\mathcal{E}')$ on the categories of short exact sequences.
(3) **2-Morphisms** between two 1-morphisms $(f, \varphi), (g, \psi)\colon (\mathcal{A}, \mathbb{E}) \to (\mathcal{A}', \mathbb{E}')$ are given by natural transformations $\Phi\colon f \to g$. These always induce natural transformations $\mathrm{Ses}(f) \to \mathrm{Ses}(g)$, i.e. for every $\mathcal{E}$-short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ we have a commutative diagram with $\mathcal{E}'$-exact rows
$$\begin{array}{ccccc} f(X) & \rightarrowtail & f(Y) & \twoheadrightarrow & f(Z) \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_Y} & & \downarrow{\scriptstyle \Phi_Z} \\ g(X) & \rightarrowtail & g(Y) & \twoheadrightarrow & g(Z) \end{array}$$
such that for every morphism of short exact sequences are mapped into the corresponding 3-dimensional diagram.

The connecting property is a reformulation of: The composition $\mathcal{E} \to \mathcal{E}' \to D^b(\mathcal{E}')$ is a δ-functor in the sense of Keller ([**26**]) - or in a modern language: We only want extriangulated functors as morphisms in $\mathbb{BH}$ in the sense of [**10**, Def. 2.32].

THEOREM 8.1. *(cf. Thm 3.4 together with* [**10**, Thm 2.34]*) The assignment $\mathcal{E} = (\mathcal{A}, \mathcal{S}) \mapsto (\mathcal{A}, \mathrm{Ext}^1_{\mathcal{E}})$, and mapping an exact functor to the underlying additive functor gives an equivalence of strict 2-categories $\mathbb{E}\boldsymbol{x} \to \mathbb{BH}$.*

As morphisms in $\mathbb{BH}$ are defined, we can obviously write down the inverse 2-functor.

9. Appendix B: Auslander correspondences as equivalence(s) of 2-categories

We recall the equivalence of 2-categories to Auslander exact categories from [**22**] and on the way the explain the similar equivalence of 2-categories to torsionfree subcategories in Auslander exact categories.
Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an essentially small exact category. We have the Auslander exact category and the tosionfree subcategory assigned to $\mathcal{E}$
$$\mathrm{AE}(\mathcal{E}) := \mathrm{mod}_{\mathrm{adm}} \mathcal{A} \quad \supseteq \quad \mathrm{mod}_{\mathrm{infl}} \mathcal{A} =: \mathrm{H}(\mathcal{E})$$
Here H stands for **hereditary** (i.e. gldim ≤ 1)[1]. We endow both with the fully exact substructure restricted from $\mathrm{Mod}\,\mathcal{A}$. As we have no suitable name for it, we call $\mathrm{H}(\mathcal{E})$ the **tf-Auslander category** (tf stands for torsionfree).

Definition 9.1. We call an additive functor between to exact categories $f\colon \mathcal{E} \to \mathcal{F}$ **inflation-preserving** if it maps $\mathcal{E}$-inflation to $\mathcal{F}$-inflation. We call it **left exact** if every $\mathcal{E}$-short

[1]please do not confuse this with the Hall algebra of the exact category

exact sequence (i, p) is mapped to a pair $(f(i), f(p) = j \circ q)$ with $f(i), j$ $\mathcal{F}$-inflations and $q = \operatorname{coker} f(i)$ an $\mathcal{F}$-deflation.

Obviously, every left exact functor is inflation-preserving.

We define the following 2-categories

$$\mathbb{E}\mathbf{x} \subseteq \mathbb{E}\mathbf{x}_L \subseteq \mathbb{E}\mathbf{x}_{\text{inf}}$$

with

(a) $\mathbb{E}\mathbf{x}$ is the 2-category of small exact categories, 1-morphisms are exact functors and 2-morphisms are natural transformations.
(b) $\mathbb{E}\mathbf{x}_L$ is the 2-category of small exact categories, 1-morphisms are left exact functors and 2-morphisms are natural transformations.
(c) $\mathbb{E}\mathbf{x}_{\text{inf}}$ is the 2-category of small exact categories, 1-morphisms are inflation-preserving additive functors and 2-morphisms are natural transformations.

Lemma 9.2. (1) *The assignment* $\text{mod}_{\text{adm}} \colon \mathcal{E} \mapsto \text{AE}(\mathcal{E})$ *defines a* 2*-functor which is left adjoint to* $\mathbb{E}\boldsymbol{x} \subseteq \mathbb{E}\boldsymbol{x}_L$
(2) *The assignment* $\text{mod}_{\text{inf}} \colon \mathcal{E} \mapsto \text{H}(\mathcal{E})$ *defines a* 2*-functor which is left adjoint to the inclusion* $\mathbb{E}\boldsymbol{x} \subseteq \mathbb{E}\boldsymbol{x}_{\text{inf}}$

For (1), look at [**22**, Cor 3.15] and (2) is completely analogue - just observe that the Yoneda embedding into the Auslander exact category is left exact and the Yoneda embedding into the tf-Auslander category is only inflation-preserving (and usually not left exact).
Recall the intrinsic definition of an Auslander exact category

Definition 9.3. An exact category $\mathcal{E}$ is called an **Auslander exact** category if it is an exact category with enough projectives $\mathcal{P}$ such that

(1) $({}^{\perp}\mathcal{P} =: \text{eff}, \text{cogen}(\mathcal{P}) =: \text{H})$ is a torsion pair (here H is the *torsionfree* subcategory)
(2) Every morphism to an object in eff is admissible with image also in this category
(3) $\text{Ext}^1_{\mathcal{E}}(\text{eff}, \mathcal{P}) = 0$
(4) $\operatorname{gldim} \mathcal{E} \leq 2$

Now we define the following two 2-categories

(d) $\mathbb{AE}$ is the 2-category of Auslander exact categories with 1-morphsims are exact functors mapping projectives to projectives and 2-morphisms are natural transformations.
(e) $\mathbb{H}$ is the 2-category of tf-Auslander categories with 1-morphisms are a exact functors mapping projectives to projectives and 2-morphisms are natural transformations.

Observe that we have a 2-functor

$$\text{Res} \colon \mathbb{AE} \to \mathbb{H}$$

assigning to an Auslander exact category its torsionfree subcategory. Exact functors preserving projectives restrict to the torsionfree subcategory (as it can be presented as objects which admit an inflation to a projective).

Theorem 9.4. *The* 2*-functors* $\text{mod}_{\text{adm}}, \text{mod}_{\text{inf}}$ *induce equivalences of* 2*-categories*

$$\text{mod}_{\text{adm}} \colon \mathbb{E}\boldsymbol{x}_L \to \mathbb{AE} \qquad \text{mod}_{\text{inf}} \colon \mathbb{E}\boldsymbol{x}_{\text{inf}} \to \mathbb{H}$$

These fit into a diagram which commutes up to

$$\begin{array}{ccc} \mathbb{E}\boldsymbol{x}_L & \xrightarrow{\text{mod}_{\text{adm}}} & \mathbb{AE} \\ \subseteq \downarrow & & \downarrow \text{Res} \\ \mathbb{E}\boldsymbol{x}_{\text{inf}} & \xrightarrow{\text{mod}_{\text{inf}}} & \mathbb{H} \end{array}$$

This is all adapted from [22, Thm 4.8]. For the second equivalence of 2-categories, we just define the inverse 2-functor $\mathbb{H} \to \mathbb{E}\mathbf{x}_{\text{inf}}$. On objects assign to a tf-Auslander category H its category of projectives $\mathcal{P}(H)$ and the exact structure such that the inflations are the H-inflations in $\mathcal{P}(H)$.
On morphisms, an exact functor $f\colon H \to H'$ which restricts to projectives, is restricted to projectives $f|_{\mathcal{P}}\colon \mathcal{P}(H) \to \mathcal{P}(H')$ an inflation-preserving functor.
A natural transformation $\Phi\colon f \to g$ between two exact functors which preserve the projectives, restricts to a natural transformation $\Phi|_{\mathcal{P}}\colon f|_{\mathcal{P}} \to g|_{\mathcal{P}}$ between the restricted functors.

9.0.1. *Properties of tf-Auslander algebras.*

Remark 9.5. The category $\mathcal{H} = \mathrm{H}(\mathcal{E})$ is always hereditary exact with enough projectives $\mathcal{P}$. Every object admits a monomorphism (in $\mathcal{H}$) $m\colon X \to P_X$ with P_X in $\mathcal{P}$ such that $\mathrm{Hom}_{\mathcal{H}}(m, P)$ is an isomorphism for all P in $\mathcal{P}$.
If $X = \operatorname{coker} i$ for a $\mathcal{P}$-monomorphism $i\colon P_1 \to P_0$, then a $\mathcal{P}$-cokernel for i is obtained by the composition $p\colon P_0 \to X \to P_X$. This way we find the short exact sequences (i, p) for the exact structure on $\mathcal{P}$.

Observe in a hereditary exact category with enough projectives: If a monomorphism $X \to P$ is an inflation then X has to be projective as well (this follows since the category is hereditary exact). As a consequence we see.

Remark 9.6. If $\mathrm{H}(\mathcal{E})$ is abelian then it is semi-simple and this implies $\mathcal{E}$ is split exact.

We link this with the following concept.

Definition 9.7. ([20], Def.1.1) An exact category is called a **0-Auslander category** if it is a hereditary exact category with enough projectives and for every projective P there exists a short exact sequence

$$P \to I \to X$$

with I projective-injective.
We say an exact category is **torsionfree 0-Auslander category** if it is a 0-Auslander category which is also hereditary torsionfree.

Remark 9.8. We recall [37], Thm B: Let $\mathcal{Q}$ be a quasi-abelian category and $(\mathcal{T}, \mathcal{F})$ a torsion-pair in $\mathcal{Q}$, then $\mathcal{T}$ and $\mathcal{F}$ are also quasi-abelian.

We also easily deduce the following special cases.

Lemma 9.9. *If $\mathcal{E}$ is abelian then $\mathrm{H}(\mathcal{E})$ is quasi-abelian.*
If $\mathcal{E}$ has enough injectives then $\mathrm{H}(\mathcal{E})$ is a 0-Auslander exact category.

This is particularly interesting as 0-Auslander exact categories have a very strong mutation theory for tilting subcategories, cf. [20].

Open question 9.10. We are missing an intrinsic characterization of tf Auslander categories.

10. Appendix C: Ind-Completion of (small) exact categories

These notes are based on the recent preprint of Positselski [29] - but we prefer the construction using the Gabriel-Quillen embedding (this way, we extend Crawley-Boeveys classical dictionary to exact structures [14]).

Locally finitely presented additive categories. Here, we give a quick summary of the *correspondence* from [14].
Let $\mathcal{C}$ an essentially small additive category. We define $\hat{\mathcal{C}} := \operatorname{Mod}\mathcal{C}$ to be the category of all additive functors $\mathcal{C}^{op} \to (Ab)$, we call this the category of (left) $\mathcal{C}$-**modules**. It is easily seen to be an abelian category.We have the (covariant) Yoneda embedding

$$\mathbb{Y}\colon \mathcal{C} \to \hat{\mathcal{C}}, \quad X \mapsto (-,X) := \operatorname{Hom}_{\mathcal{C}}(-,X)$$

this is fully faithful, the essential image consists of (some) projective objects which we call **representable functors**.
Every object in $\hat{\mathcal{C}}$ is as a small colimit of representables - for $F \in \hat{\mathcal{C}}$ define the *slice* category $\mathcal{C}/F$ for $F \in \hat{\mathcal{C}}$ (objects: (X,x), $X \in \mathcal{C}, x \in F(X)$, morphisms $f\colon X \to X'$ in $\mathcal{C}$ such that $F(f)(x') = x$), then we have a small category and a functor $\Phi\colon \mathcal{C}/F \to \hat{\mathcal{C}}, (X,x) \mapsto (-,X)$. Its colimit is $F = \operatorname{colim}_{\mathcal{C}/F} \Phi$.

Definition 10.1. ([2], Expose I) We define the **ind-completion** $\overrightarrow{\mathcal{C}}$ (in the literature denoted as $\operatorname{Ind}(\mathcal{C})$) as the closure of $\mathcal{C}$ under arbitrary directed colimits: Objects are functors $D\colon I \to \mathcal{C}$ from small filtered categories I. Morphisms are defined as

$$\begin{aligned}\operatorname{Hom}(D\colon I \to \mathcal{C}, E\colon J \to \mathcal{C}) &:= \operatorname{Hom}_{\operatorname{Mod}\mathcal{C}}(\operatorname*{colim}_I \mathbb{Y}D, \operatorname*{colim}_J \mathbb{Y}E)\\ &= \lim_{i\in I} \operatorname*{colim}_{j \in J} \operatorname{Hom}_{\mathcal{C}}(D(i), E(j))\end{aligned}$$

Observe that the Yoneda embedding factors over $\overrightarrow{\mathcal{C}}$. Via the Yoneda embedding, we can identify this with the following full subcategory of $\hat{\mathcal{C}}$

$$\overrightarrow{\mathcal{C}} := \{\operatorname*{colim}_{i\in I}(-,X_i) \mid \quad (X_i)_{i\in I}\ I\text{−shaped diagram in } \mathcal{C} \text{ with } I \text{ directed set }\}$$

Remark 10.2. The second description uses that closure under small filtered colimits it the same as closure under small directed colimits, cf. [1], Thm 1.5.

Proposition 10.3. *([2], Expose I, Prop. 8.6.4) Ind-completion is a 2-functorial.*
An additive functor f is faithful (resp. fully faithful) if and only if $\overrightarrow{f}$ is faithful (resp. fully faithful). Furthermore the ind-completion $\overrightarrow{f}$ of an additive functor $f\colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only it is fully faithful and the essential image inclusion $\operatorname{Im} f \to \mathcal{D}$ induces an equivalence on idempotent completions.

Let $\mathcal{D}$ be an additive category, we denote by $\operatorname{Add}(\mathcal{C},\mathcal{D})$ the category of additive functors from $\mathcal{C}$ to $\mathcal{D}$ and $\operatorname{Add}_{fc}(\mathcal{C},\mathcal{D})$ the subcategory of functors which preserve directed colimits (whenever these exist).

Ind-completion can be defined for arbitrary additive and even arbitary categories categories and can be characterized by a universal property such as:

Lemma 10.4. *(**Universal property of ind-completion**, cf. [2], Expose I, Prop. 8.7.3) Let $\mathcal{C}$ be a small additive category, then $\overrightarrow{\mathcal{C}}$ has all directed colimits.*
Assume that $\mathcal{D}$ is an additive category which is closed under arbitrary directed colimits.
Precomposition with $\mathcal{C} \to \overrightarrow{\mathcal{C}}$ is an equivalence of categories

$$\operatorname{Add}_{fc}(\overrightarrow{\mathcal{C}},\mathcal{D}) \to \operatorname{Add}(\mathcal{C},\mathcal{D})$$

Furthermore, it also has the following property

Lemma 10.5. *([14], Lem. 1) $\overrightarrow{\mathcal{C}}$ is idempotent complete.*

For small additive categories, we have an alternative description of the ind-completion found in [14].

Definition 10.6. Let $\mathcal{C}$ be a small additive category. We say that an object F in $\hat{\mathcal{C}}$ is **flat** if the tensor functor $F \otimes_{\mathcal{C}} -\colon \hat{\mathcal{C}^{op}} \to (Ab)$ is exact. We denote by $\operatorname{Flat}(\mathcal{C}^{op}, Ab)$ the full subcategory of flat functors.

THEOREM 10.7. *([**14**], Thm p. 1646)*
$\overrightarrow{\mathcal{C}} = \mathrm{Flat}(\mathcal{C}^{op}, Ab)$ *and* $F \in \overrightarrow{\mathcal{C}}$ *is equivalent to*

(1) $\mathcal{C}/F$ *is filtered*
(2) *Every natural transformation* $\mathrm{coker}(-,f) \to F$ *factors over a representable.*

Definition 10.8. Let $\mathcal{A}$ be an additive category. We say an object X in $\mathcal{A}$ is **finitely presented** if $\mathrm{Hom}_{\mathcal{A}}(X,-)$ commutes with arbitrary filtered colimits. We denote by $\mathrm{fp}(\mathcal{A})$ the full subcategory of finitely presented objects in $\mathcal{A}$.
The additive category $\mathcal{A}$ is called **locally finited presented** if $\mathrm{fp}(\mathcal{A})$ is essentially small and $\mathcal{A}$ is equivalent to $\overrightarrow{\mathrm{fp}(\mathcal{A})}$.

Remark 10.9. If $\mathcal{A}$ is locally finitely presented then $\mathrm{fp}(\mathcal{A})$ is essentially small, closed under direct sums and summands. In particular by Lemma 10.5, it is idempotent complete.

Lemma 10.10. *(cf. [**14**], part of Thm on p.1647)*
For an essentially small category $\mathcal{C}$ *we have* $\mathrm{fp}(\overrightarrow{\mathcal{C}}) \cong \mathcal{C}^{ic}$ *is equivalent to the idempotent completion of* $\mathcal{C}$.

THEOREM 10.11. *([**14**], Thm. in (1.2), p.1645) If* $\mathcal{C}$ *is essentially small, then*

$$\mathrm{fp}(\hat{\mathcal{C}}) = \{F \in \hat{\mathcal{C}} \mid F \cong \mathrm{coker}(-,f),\ f \in \mathrm{Mor}(\mathcal{C})\} \quad =: \mathrm{mod}_1 \mathcal{C}$$

Furthermore, $\hat{\mathcal{C}}$ *is locally finitely presented.*

Example 10.12. Locally finitely presented abelian categories are **Grothendieck categories** (i.e. (1) abelian, (2) with arbitrary small coproducts, (3) directed colimits are exact, (4) has a generating object G). Here, the generator can be chosen as

$$G = \bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} (-,C) \quad \in \overrightarrow{\mathcal{C}}$$

For the converse: If a Grothendieck category admits a set of finitely presented objects whose coproduct is a generator, then it is locally finitely presented.
Grothendieck categories always have enough injectives (often they are hard to find), have arbitrary small limits and colimits.

Remark 10.13. For a not necessarily small category $\mathcal{C}$ we can still define its ind-completion. If $\mathcal{C}$ is abelian, this is an abelian category - but it may not have enough injectives (cf. [**25**], Prop. 15.1.2).

The following is a consequence of Lem. 10.4 together with [**14**], Thm in (1.4), p. 1647.

THEOREM 10.14. *(equivalence of (2-)categories)*
The assignments $\mathcal{C} \mapsto \overrightarrow{\mathcal{C}}$ *and* $\mathcal{A} \mapsto \mathrm{fp}(\mathcal{A})$ *are* $2-$*functorial and give rise to an equivalence of (strict)* $2-$*categories between*

(1) *essentially small, idempotent complete additive categories* $\mathcal{C}$ *with additive functors*
(2) *Locally finitely presented additive categories* $\mathcal{A}$ *with additive functors that preserve arbitrary filtered colimits and restrict to the subcategories of finitely presented functors.*

Let $\mathcal{C}$ be idempotent complete, essentially small additive category and $\mathcal{A}$ a locally finitely presented (additive) category. We assume $\mathcal{C} = \mathrm{fp}(\mathcal{A})$ and $\mathcal{A} = \overrightarrow{\mathcal{C}}$. Then the following holds (by restricting further and further):

(i) $\mathcal{C}$ left abelian $\Leftrightarrow$ $\mathcal{A}$ abelian
 the definition of **left abelian** (cf. [**14**], (2.4)): Every morphisms has a cokernel, every epi is a cokernel and whenever $A \xrightarrow{f} B \xrightarrow{c} C$ with $c = \mathrm{coker}(f)$ and $g\colon D \to B$, $cg = 0$ then there exists an epi $d\colon E \to D$ such that gd factors over f.
(ii) $\mathcal{C}$ abelian $\Leftrightarrow$ $\mathcal{A}$ locally coherent

(iii) $\mathcal{C}$ abelian and all objects noetherian $\Leftrightarrow$ $\mathcal{A}$ locally noetherian abelian
(iv) $\mathcal{C}$ is length abelian $\Leftrightarrow$ $\mathcal{A}$ is locally finite abelian

Example 10.15. R a ring:
(i) $R\,\mathrm{Mod}$ abelian and $R\,\mathrm{Mod}_1$ is left abelian
(ii) for R left coherent
(iii) for R left noetherian
(iv) e.g. for R left artinian (with Loewy length)

10.1. Gabriel Quillen embedding. We review this well-known embedding of an essentially small exact category as a fully exact subcategory in an abelian category.
Let $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ be an essentially small exact category.

Definition 10.16. We define the category $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ to be the category of all additive functors $F\colon \mathcal{C}^{op} \to (Ab)$ which map short exact sequences $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{E}$ to left exact sequences $0 \to F(Z) \xrightarrow{F(d)} F(Y) \xrightarrow{F(i)} F(X)$ in abelian groups. We will call this the category of **left exact functors on** $\mathcal{E}$. We define the category of **locally effaceble functors** $\mathrm{Eff}_{\mathcal{E}}$ to be the full subcategory of $\hat{\mathcal{C}}$ of objects F such that for every pair (X, x) of an object X in $\mathcal{C}$ and $x \in F(X)$ there exists an $\mathcal{E}$-deflation $d\colon Z \to X$ with $F(d)(x) = 0$.

Lemma 10.17. [27]*, Prop. 2.3.7 (1),(2) (with intermediate steps)*

(i) *(Prop 2.2.16)* $\mathcal{D} = \mathrm{Eff}_{\mathcal{E}}$ *is a Serre subcategory of* $\hat{\mathcal{C}}$ *closed under coproducts. Therefore, the Serre quotient functor* $Q\colon \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}/\mathcal{D}$ *admits a right adjoint.*
(ii) *(Lem. 2.2.10) Let* Q_ρ *be the right adjoint. It factors as* $\hat{\mathcal{C}}/\mathcal{D} \xrightarrow{\Phi} \mathcal{D}^{\perp} \xrightarrow{I} \hat{\mathcal{C}}$ *with* Φ *an equivalence of categories and* I *the inclusion functor. The quasi-inverse of* Φ *is given by* $Q \circ I$.
(iii) $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}) = \mathcal{D}^{\perp} := \{\mathrm{Y} \in \hat{\mathcal{C}} \mid \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}(\mathrm{E}, \mathrm{Y}) = 0 = \mathrm{Ext}^1_{\mathrm{Mod}\,\mathcal{C}}(\mathrm{E}, \mathrm{Y}) \forall \mathrm{E} \in \mathcal{D}\}$.

Remark 10.18. $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ is a Grothendieck category (as it is the localization of a Grothendieck category by a Serre subcategory?). In the abelian structure it has a generator

$$G = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} (-, X)$$

As $(-, X)$ are (some) finitely presented objects in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, it follows that $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$ is locally finitely presented abelian.
It also has an exact substructure as fully exact category in $\hat{\mathcal{C}}$ but these two exact structures usually do not coincide.

Remark 10.19. The inclusion $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}) \to \hat{\mathcal{C}}$ is not an exact functor (if we consider $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ with its abelian structure). Yet it reflects exactness in the following sense: If $0 \to F \to G \to H \to 0$ in $\hat{\mathcal{C}}$ with F, G, H in $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$, then this is a short exact sequence in $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.

Corollary 10.20. $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}) \subseteq \hat{\mathcal{C}}$ *is a deflation-closed subcategory: Given a short exact sequence* $0 \to F \to G \to H \to 0$ *in* $\mathrm{Mod}\,\mathcal{A}$.
If $G, H \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$*, then also* $F \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.
If $F, G \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ *and* $\mathrm{Ext}^2(E, F) = 0$ *for all* E *effaceable, then* $H \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.

In general we can characterize short exact sequences in $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ (in the abelian structure) as follows:

Lemma 10.21. *Given two composable maps* $0 \to F \xrightarrow{i} G \xrightarrow{p} H \to 0$ *in* $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$. *TFAE*

(1) (i, p) *are an exact sequence in* $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$
(2) *In* $\hat{\mathcal{C}}$ *we have a exact sequence* $0 \to F \xrightarrow{i} G \xrightarrow{p} H$ *and* $\mathrm{coker}(p)$ *is locally effaceable.*

THEOREM 10.22. *Let $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ be an exact category. The Yoneda functor gives a functor*

$$i\colon \mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}), \quad \mathrm{X} \mapsto (-, \mathrm{X}) = \mathrm{Hom}_{\mathcal{C}}(-, \mathrm{X})$$

with the following properties

(1) *i is exact, reflects exactness and the essential image of i is extension-closed, its idempotent completion is deflation-closed.*
(2) *i induces isomorphisms on all extension-groups.*

PROOF. (1) [**12**], Appendix A and [**27**], Prop. 2.3.7.(3). The last statement follows from [**29**], Prop. 6.1, (e).
(2) [**27**] in Lem 4.2.17 it is shown that $\mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ is right cofinal (Keller's definition) that implies the statement. □

The following result is also relevant for us:

Proposition 10.23. *([**29**], Prop. 6.2) For every X in the category $\mathcal{E}$, the functor $\mathrm{Ext}^n_{\mathrm{Lex}(\mathcal{E}^{op}, Ab)}((-, X), -)$ preserves filtered colimits.*

10.2. Locally coherent exact. The ind-completion of a small exact category $\mathcal{E}$ has a natural exact structure (namely as fully exact in the Gabriel-Quillen embedding), this exact structure can also be described as directed colimits of short exact sequences in $\mathcal{E}$ and is called **locally coherent exact structure**.
The CB-correspondence (section 1) is extended to i.c. small exact categories.

We begin with the following observation.

Lemma 10.24. *Let $\mathcal{C}$ be a small additive category, F a flat functor and $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g = \mathrm{coker}(f)$ in $\mathcal{C}$ then $0 \to F(Z) \to F(Y) \to F(X)$ is exact in abelian groups.*

PROOF. Define $(X, -) := \mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to (Ab)$ and the contravariant Yoneda embedding $\mathcal{C}^{op} \to \hat{\mathcal{C}^{op}}, X \mapsto (X, -)$. By assumption we have an exact sequence $0 \to (Z, -) \to (Y, -) \to (X, -)$. Since F is flat, the functor $F \otimes_{\mathcal{A}}$ is exact and we have $F \otimes_{\mathcal{C}} (E, -) \cong F(E)$. Therefore, we obtain the exact sequence $0 \to F(Z) \to F(Y) \to F(X)$. □

Now for an exact category $\mathcal{E} = (\mathcal{C}, \mathcal{S})$, by the previous lemma $\mathrm{Flat}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \subseteq \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.

Lemma 10.25. *(and definition.) $\mathrm{Flat}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is closed under extensions in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$. We define $\overrightarrow{\mathcal{E}}$ to be the fully exact structure on $\overrightarrow{\mathcal{C}}$ and call this the **ind-completion of the exact category** $\mathcal{E}$.*

PROOF. (sketch) (of Lemma 10.25) Given a short exact sequence $\sigma\colon F \to G \to H$ in (the abelian structure on) $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$ with F, H flat.
First assume $H = (-, X)$, write F as filtered colimit and use Prop 10.23 and the fact that the essential image of $\mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{op}, Ab)$ is extension-closed to conclude the claim.
In general, we use Thm 10.7, (2). Given a morphism $\theta\colon \mathrm{coker}(-, f) \to G$. Postcompose to $\mathrm{coker}(-, f) \to H$. As H is flat, this factors over a morphism $g\colon (-, X) \to H$ for some X in $\mathrm{Ob}(\mathcal{E})$. Now form the pull-back of σ along g in the abelian category $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, say this is a short exact sequence $F \to E \to (-, X)$. The universal property of the pull-back gives a morphism $\theta'\colon \mathrm{coker}(-, f) \to E$ and a morphism $u\colon E \to G$ with $\theta = u\theta'$. By the first case θ' factors over a representable, therefore θ does so too. □

Remark 10.26. (and definition) Let $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ be an essentially small exact category, then $\mathcal{E}$ is fully exact in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$. As the latter is abelian, it is idempotent complete, therefore $\mathcal{E}^{ic}$ is also a fully exact subcategory in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$.

But this means $\mathrm{fp}(\overrightarrow{\mathcal{C}})$ is an extension-closed subcategory in $\overrightarrow{\mathcal{E}}$. We define $\mathrm{fp}(\overrightarrow{\mathcal{E}})$ to be the fully exact structure on $\mathrm{fp}(\overrightarrow{\mathcal{C}})$.

Definition 10.27. Let $\mathcal{F} = (\mathcal{A}, \mathcal{T})$ be an exact category. We say it is **locally coherent exact** if $\mathrm{fp}(\mathcal{A})$ is essentially small and extension-closed in $\mathcal{F}$ -in this case we denote by $\mathrm{fp}(\mathcal{F})$ the fully exact subcategory- and $\mathcal{F} = \overrightarrow{\mathrm{fp}(\mathcal{F})}$.

In an ess. small exact category $\mathcal{E}$, the category of short exact sequences $\mathrm{Ses}(\mathcal{E})$ is again an essentially small exact category (with degree-wise short exact sequences).

THEOREM 10.28. *([29], proof of Lemma 1.2) A filtered colimit of short exact sequences in $\overrightarrow{\mathcal{E}}$ is a short exact sequence in $\overrightarrow{\mathcal{E}}$.*
The universal property of the ind-completion yields an equivalence of categories

$$\overrightarrow{\mathrm{Ses}(\mathcal{E})} \to \mathrm{Ses}(\overrightarrow{\mathcal{E}})$$

We also observe the following:

Lemma 10.29. *For an essentially small exact category $\mathcal{E}$, we have that $\overrightarrow{\mathcal{E}}$ is a resolving subcategory in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, in particular it is homologically exact.*

PROOF. $\overrightarrow{\mathcal{E}}$ is extension-closed, idempotent complete and contains a generator of $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, so it is enough to see that it is also deflation-closed. Now given a short exact sequence $F \to G \to H$ in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, it gives a 4-term exact sequence in $\hat{\mathcal{C}}$: $0 \to F \to G \to H \to E \to 0$ withe E locally effaceble. Assume that G, H are flat, we want to see that E is so too. As every finietly presented effaceable functor has projecive dimension 2, every locally effaceable has flat dimension 2 and this implies F is flat. □

Furthermore, since $(-, X) \in \mathrm{Flat}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ for all X in $\mathcal{C}$, we get a fully faithful exact functor with extension-closed essential image

$$i \colon \mathcal{E} \to \overrightarrow{\mathcal{E}}, \quad X \mapsto (-, X)$$

Lemma 10.30. *The functor i is homologically exact.*

PROOF. As $\mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{op}, Ab)$ is homologically exact and also $\overrightarrow{\mathcal{E}} \to \mathrm{Lex}(\mathcal{E}^{op}, Ab)$, this is immediate. □

We directly get the following from the previous corollary and Prop. 10.23.

Corollary 10.31. *For X in $\mathrm{Ob}(\mathcal{E})$, the functor $\mathrm{Ext}^n_{\overrightarrow{\mathcal{E}}}((-, X), -)$ commutes with filtered colimits.*

Let $\mathrm{Ex}(\mathcal{E}, \mathcal{F})$ be the category of exact functors between two exact categories and $\mathrm{Ex}_{fc}(\mathcal{E}, \mathcal{F})$ be the full subcategory of exact functors which preserve filtered colimits.

Lemma 10.32. *(**Universal property of ind-completion for exact categories**) Let $\mathcal{E}$ be an essentially small exact category. Then $\overrightarrow{\mathcal{E}}$ is closed under directed colimits and directed colimits are exact functors.*
Let $\mathcal{F}$ be an exact category closed under all directed colimits and they are exact functors. Then precomposition $\mathcal{E} \to \overrightarrow{\mathcal{E}}$ gives an equivalence

$$\mathrm{Ex}_{fc}(\overrightarrow{\mathcal{E}}, \mathcal{F}) \to \mathrm{Ex}(\mathcal{E}, \mathcal{F})$$

THEOREM 10.33. *(equivalence of 2-categories)*
The assignments $\mathcal{E} \mapsto \overrightarrow{\mathcal{E}}$ and $\mathcal{F} \mapsto \mathrm{fp}(\mathcal{F})$ are functorial and give rise to an equivalence of $(2-)$categories between

(1) *essentially small, idempotent complete exact categories $\mathcal{E}$ with exact functors*
(2) *Locally coherent exact categories $\mathcal{F}$ with exact functors that preserve arbitrary filtered colimits and restrict to the subcategories of finitely presented functors.*

We remark that a functor that preserves filtered colimits on objects, also preserves filtered colimits on morphism categories and this implies it also preserves filtered colimits of short exact sequences. This implies that as exact functor between essentially small exact categories $F\colon \mathcal{E}_1 \to \mathcal{E}_2$ extends with the universal property of the ind-completion uniquely to an exact functor $\overrightarrow{F}\colon \overrightarrow{\mathcal{E}_1} \to \overrightarrow{\mathcal{E}_2}$, this is a consequence of Thm 10.28. Recall: F fully faithful if and only $\overrightarrow{F}$ is fully faithful by Prop. 10.3.

Ignoring set-theory for a moment: Let $\mathcal{F}$ be an exact category, we consider $\mathrm{EX}(\mathcal{F})$ the lattice of all exact subcategories. For $\mathcal{F}$ locally coherent exact, we define $\mathrm{EX}_{fc}(\mathcal{F})$ to be the exact subcategories in the category (2) above, i.e. exact functors $i\colon \mathcal{F}' \to \mathcal{F}$ such that i is fully faithful, $\mathcal{F}'$ is locally coherent exact and i preserves filtered colimits.

Corollary 10.34. *Let $\mathcal{E}$ be an essentially small category. We have mutually inverse, isomorphisms of posets*

$$\overrightarrow{(-)} \quad : \mathrm{EX}(\mathcal{E}) \leftrightarrow \mathrm{EX}_{fc}(\overrightarrow{\mathcal{E}}): \quad \mathrm{fp}(-)$$

It restricts to all the usual subposets such as extension-closed, exact substructures etc.

Positselski found the maximal and minimal locally coherent exact structure on a locally finitely presented category particular interesting. The minimal exact structure is the ind-completion of the split exact structure (on an essentially small additive category) and is called **pure exact structure** on a locally finitely presented category.

Example 10.35. Let R be a ring. The abelian exact structure on $R\,\mathrm{Mod}$ is the maximal locally coherent exact structure, it corresponds to the left abelian structure on $R\,\mathrm{Mod}_1$ (fp R-modules). The category of flat R-modules $R\,\mathrm{Mod}_{fl}$ is extension-closed in $R\,\mathrm{Mod}$. Its subcategory of finitely presented objects is $(\mathrm{add}(R))^{ic}$ with the split exact structure is the fully exact substructure. By a Thm of Govorov-Lazard, $\overrightarrow{\mathrm{add}(R)} = R\,\mathrm{Mod}_{fl}$. In this case: The fully exact structure is the pure exact structure.

Stovicek generalized the notion of a Grothendieck category to an exact category of Grothendieck type.

THEOREM 10.36. *([**29**, Cor. 5.4]) Locally coherent exact categories are exact categories of Grothendieck type (in the sense of Stovicek).*

In particular, all established properties of exact categories of Grothendieck type hold true.

Corollary 10.37. *(also [**29**, Cor. 5.4]) $\overrightarrow{\mathcal{E}}$ has enough injectives.*

This implies that the unbounded derived category $\mathrm{D}(\overrightarrow{\mathcal{E}})$ is locally small (i.e. has Hom-sets), cf. Chapter 6.

11. Open problems

Here is my personal (naive) list of problems

(0) Describe $\overrightarrow{\mathcal{E}}$ for exact categories of the form $\mathrm{mod}_S\,\mathcal{M}$ and for Auslander-Soberg exact structures.
(1) If $\mathcal{E}$ has enough projectives/injectives what are the corresponding properties in $\overrightarrow{\mathcal{E}}$?
(2) Which conditions on the exact category imply $\mathrm{gldim}(\mathcal{E}) = \mathrm{gldim}(\overrightarrow{\mathcal{E}})$?

(3) When is $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\overrightarrow{\mathcal{E}})$ fully faithful? (for some answers, cf [**30**])
(4) Are there situations when derived equivalence is preserved/reflected by ind-completion of exact categories?

11.1. Literature.

11.1.1. *For ind-completion.* Ind-categories have been introduced for arbitrary categories by Grothendieck in [**21**], and more thoroughly studied by Grothendieck-Verdier in [**2**], Expose I. The concept of *Finitely presented/presentable categories* is due to [**19**].
In [**14**] it had been observed that in the *additive category*-setup, ind-completion for small additive functors can be realized as categories of flat functors.
A (multiply) more general approach can be found in [**1**], where the more general analogue of finitely presented categories is called finitely accessable categories.

11.1.2. *For the Gabriel-Quillen embedding.* References: Bühler *exact categories*, Appendix A contains a historical discussion of the origins. Further reference [**27**] and [**29**], section 5.

11.1.3. *For locally coherent exact categories.* This has been introduced in [**29**]. In the special case that $\mathcal{C}$ has weak cokernels an alternative construction is given using the embedding into the purity category by [**36**].

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