The posets of exact subcategories

1. Synopsis

This is the quest to extend known descriptions of the lattice of exact structures on a given additive category to the much bigger lattice of all exact subcategories.
What is new? This question is usually not considered, so everything in this chapter.

2. Introduction

Now, we fix one essentially small, idempotent complete exact category $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ and introduce the following four posets of exact subcategories (the first poset is just auxiliary):

$$\boxed{\begin{aligned} \mathrm{ADD}(\mathcal{A}) &= \text{ all additively closed subcategories } \mathcal{A}' \text{ of } \mathcal{A} \\ \mathrm{EX}(\mathcal{E}) &= \text{ exact categories } (\mathcal{A}', \mathcal{S}') \text{ such that } \mathcal{A}' \in \mathrm{ADD}(\mathcal{A}) \\ & \quad \text{with } (\mathcal{A}', \mathcal{S}') \subseteq (\mathcal{A}, \mathcal{S}) \text{ is an exact functor} \\ \mathrm{Ex}(\mathcal{E}) &= (\mathcal{A}', \mathcal{S}') \in \mathrm{EX}(\mathcal{E}) \text{ such that } \mathcal{A}' \text{ is extension-closed in } \mathcal{E} \\ \mathrm{ext}(\mathcal{E}) &= \text{ fully exact subcategories } \mathcal{A}' \text{ in } \mathrm{ADD}(\mathcal{A}) \\ \mathrm{ex}(\mathcal{E}) &= \text{ exact substructures of } \mathcal{E} \end{aligned}}$$

here, *fully exact* extension-closed means that the exact structure $\mathcal{S}'$ on $\mathcal{A}'$ is all kernel-cokernel pairs (i, p) in $\mathcal{A}'$ such that $(i, p) \in \mathcal{S}$.
Observe that $\mathrm{Ex}(\mathcal{E})$ contains $\mathrm{ext}(\mathcal{E})$ and $\mathrm{ex}(\mathcal{E})$.
We have the following operation on $\mathrm{EX}(\mathcal{E})$

$$(\mathcal{A}', \mathcal{S}') \wedge (\mathcal{A}'', \mathcal{S}'') := (\mathcal{A}' \cap \mathcal{A}'', \mathcal{R} = \{\sigma \in \mathcal{S}' \cap \mathcal{S}'' \mid \text{ all three objects are in } \mathcal{A}' \cap \mathcal{A}''\})$$

with respect to this operation, $\mathrm{EX}(\mathcal{E})$ becomes a complete meet-semilattice and all three $\mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E}), \mathrm{ex}(\mathcal{E})$ are closed under this operation (making them complete meet-subsemilattices).
If a complete meet-semilattice $(X, \leq, \wedge)$ has a unique maximal element then one can define a join such that it becomes a complete lattice, so given a subset $\{x_i \mid i \in I\}$ of X its join is given by

$$\bigvee_{i \in I} x_i := \bigwedge_{y \colon x_i \leq y \forall i \in I} y.$$

Observe that the joins obtained this way for $\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E})$ usually differ.

We first make the following easy observation.

THEOREM 2.1. *(cf. Thm 3.3)* $\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E})$ *are complete lattices and* $\mathrm{ex}(\mathcal{E})$ *is a complete sublattice of* $\mathrm{EX}(\mathcal{E})$ *and of* $\mathrm{Ex}(\mathcal{E})$.

Open question 2.2. Considering the bijection between Ziegler-closed subsets (containing a given closed set) and exact structures on an idempotent complete small additive category with weak cokernels (cf. Chapter 2), we ask: Are the opposite lattices frames? Are they even coherent frames? (see e.g. [**9**] for the definitions)

As a corollary of Rump (cf. in Chapter 1, Cor. **??**) we deduced that for every additive functor between small exact categories $f: \mathcal{F} \to \mathcal{E}$ there is a unique maximal exact substructure of $\mathcal{F}$ such that f becomes an exact functor on it.
Let now $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be a small exact category and $i: \mathcal{B} \subseteq \mathcal{A}$ a full additively closed subcategory. We write $\mathcal{S} \cap \mathcal{B}$ for the subset of $\mathcal{S}$ given by all short exact sequences such with all three objects in $\mathcal{B}$ (also denoted by $\mathcal{S}_i$ in Chapter 1).
Let $\mathcal{F} = \mathcal{F}_{max}$ the maximal exact structure on $\mathcal{B}$ and we look at the inclusion functor $i: \mathcal{F} \to \mathcal{E}$, then we denote by

$$\mathcal{F}_{\mathcal{B}} := (\mathcal{B}, \mathcal{S}^{\leq (\mathcal{S} \cap \mathcal{B})})$$

the maximal exact structure on $\mathcal{B}$ such that i is an exact functor on it.

Then our second result is the following simple corollary of this:

THEOREM 2.3. . *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an idempotent complete essentially small exact category. Then we have equalities of sets*

$$\mathrm{EX}(\mathcal{E}) = \bigsqcup_{\mathcal{B} \in \mathrm{ADD}(\mathcal{A})} \mathrm{ex}(\mathcal{F}_{\mathcal{B}})$$

$$\mathrm{Ex}(\mathcal{E}) = \bigsqcup_{\mathcal{E}' \in \mathrm{ext}(\mathcal{E})} \mathrm{ex}(\mathcal{E}')$$

To our knowledge, usually either only the lattice of exact structures is studied or a chosen subposet of $\mathrm{ext}(\mathcal{E})$ assuming extra-properties (such as Serre subcategories, torsion classes, thick subcategories, wide subcategories, resolving subcategories, tilting subcategories etc.).

We are interested in the following questions

(1) Explicit descriptions
(2) Lattice isomorphisms for $\mathrm{EX}(\mathcal{E})$
(3) Can we find homological properties which are preserved under forming meets?

In neither case we claim to have a good answer but we give some partial answers.
Representation-finite means *here*:
Krull-Schmidt, K-linear (for some field K), Hom-finite with only finitely many indecomposables.
The first task is even in the representation-finite complicated because of the size of the constructed lattice, see e.g. some enumerative example in the end. We remark that the Ziegler spectrum is not functorial and even in the representation-finite case we do not see how we can use it here. In this case the cogenerators (whose indecomposable summands) give Ziegler-closed subsets in the Ziegler spectrum of $\mathcal{B} \in \mathrm{ADD}(\mathcal{A})$ are those which contain the cogenerator from the dual of Lemma 6.1.

For the second task, we look again at Auslander's functorial point of view (cf. Chapter 2). We extend the three lattice isomorphisms from $\mathrm{ex}(\mathcal{E})$ to the whole lattice $\mathrm{EX}(\mathcal{E})$: Using the Auslander category, the tf Auslander category and the category of effaceble functors. This result is Theorem 4.15.

For the third question:
Is a given *homological*[1] condition preserved under taking meet in $\mathrm{EX}(\mathcal{E})$?
We have no systematic way of studying this, we just collect some answers (if you know more please let me know).
Some negative answers:

(1) homologically exactness (and also homologically faithfulness)
(2) gldim $= n$ (same fixed n)
(3) having enough projectives (or injectives)
(4) ((2) for $n = 1$ but more specifically:) hereditary exact substructures in an hereditary exact category

[1] i.e. defined by imposing conditions on the Ext-functors.

Also, the subposet of hereditary exact substructures may not have a unique maximal element.
Some positive answers:
This is what I found:

(1) 1-homologically exact (i.e. extension-closed).
Special case: If $\mathcal{E}$ is hereditary exact, all extension-closed subcategories are homologically exact (and also hereditary exact)
(2) n-rigid subcategories (all with the same n).
Special case: self-orthogonal subcategories (n-rigid for all n)
(3) resolving subcategories in an exact category with enough projectives
(4) (only wrt. finite meets!) exact substructures with enough projectives in an exact structure with enough projectives

There are probably many more. We also give the following examples (without further applications). Given an exact fully faithful functor $f\colon \mathcal{F} = (\mathcal{B}, \mathcal{S}') \to \mathcal{E}$ We say that the $f - \operatorname{gldim} \mathcal{F} \leq n$ if $\operatorname{Ext}^n_{\mathcal{E}}(f(X), f(\sigma))$ is right exact for all objects X in $\mathcal{B}$ and $\mathcal{F}$-exact short exact sequences σ.
This can be seen as a relative version of gldim, since for $f = \mathrm{id}\colon \mathcal{E} \to \mathcal{E}$ we have $f - \operatorname{gldim} \mathcal{E} \leq n$ iff $\operatorname{gldim} \mathcal{E} \leq n$ by Lemma 5.4.
We observe the following completely obvious:

Lemma 2.4. *Let $\mathcal{E}$ be essentially small idempotent complete exact category and $\mathcal{F}_j = (\mathcal{B}_j, \mathcal{S}_j)$ be in* $\mathrm{EX}(\mathcal{E})$*, $j \in J$. Then:*
If all $\mathcal{F}_j$ have relative global dimension $\leq n$ for all $j \in J$, then also $\bigwedge_{j \in J} \mathcal{F}_j$ has relative global dimension $\leq n$.

Here is another property one may consider: Let $\mathcal{E}$ be an exact category. Contravariantly-finite $\mathcal{E}$-generators $\mathcal{G}$ (i.e. for every object X in $\mathcal{E}$ there is an $\mathcal{E}$-deflation $d_X\colon G_X \to X$ which is also a right $\mathcal{G}$-approximation) which are also $\mathcal{E}$-subobject-closed (i.e. given an $\mathcal{E}$-exact sequence $X \rightarrowtail G \twoheadrightarrow Y$ with $G \in \mathcal{G}$, then X is also in $\mathcal{G}$) induce always hereditary exact substructures with enough projectives (by looking at the exact substructure such that $\operatorname{Hom}(G, -)$ becomes exact for all $G \in \mathcal{G}$). These give usually not all hereditary exact substructures. But we observe that $\mathcal{E}$-subobject-closedness is preserved under arbitrary intersections (but we do not know when such an intersection is contravariantly finite!).

Lemma 2.5. *Let $\mathcal{E}$ be an exact category such that the underlying additive category is representation-finite (see above). Then all generators are contravariantly finite. The set of all hereditary exact substructures with enough projectives given by an $\mathcal{E}$-subobject-closed generator is closed under taking arbitrary joins in* $\mathrm{EX}(\mathcal{E})$*.*

Proof. Given $\mathcal{G}_i$ subobject-closed generators with $\mathcal{E}_i$ corresponding exact substructures such that $\mathcal{P}(\mathcal{E}_i) = \mathcal{G}_i$. As we are in a finite-type situation the join is $\bigvee_i \mathcal{E}_i = \mathcal{F}$ with $\mathcal{P}(\mathcal{F}) = \bigcap \mathcal{G}_i =: \mathcal{G}$ which is again a subobject-closed generator. □

Example 2.6. Let $\mathcal{E} = \Lambda_n \operatorname{mod}$ with Λ_n the path algebra over $1 \to 2 \to \cdots \to n$. In this case, the subobject-closed generators are not only closed under intersections but also by taking direct sums and form a sublattice of $\mathrm{ex}(\mathcal{E})$. This is in bijection with subobject-closed subcategories in $\Lambda_{n-1} \operatorname{mod}$. In general, subobject-closed subcategories in $\Lambda \operatorname{mod}$ with Λ a Dynkin quiver has been studied, explicit bijections to the elements of the Weyl group of the corresponding (simply-laced) Dynkin-type have been found in [**10**].

What I do not know:
In the following we do not know the answer (cf. Chapter 1 for the definitions): Left (or right) cofinal, (co)resolving (in general exact category), partially (co)resolving

3. The lattice of exact subcategories in an exact category

Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category.
We will always assume $\mathcal{A}$ is idempotent complete We also will assume that $\mathcal{A}$ is essentially small (else we would have to generalize the notion *poset* from sets to classes).
Let $\mathrm{ADD}(\mathcal{A})$ be the collection of all additively closed subcategories of $\mathcal{A}$ (this is a complete lattice). Observe that additively closed subcategories are also idempotent complete.
For $\mathcal{A}' \in \mathrm{ADD}(\mathcal{A})$ we denote bu $i_{\mathcal{A}',\mathcal{A}}$ (or if there is no confusion, just i) for the inclusion $\mathcal{A}' \subseteq \mathcal{A}$.

Definition 3.1. Now we define $\mathrm{EX}(\mathcal{E})$ as the collection of exact categories $(\mathcal{A}', \mathcal{S}')$ with $\mathcal{A}' \in \mathrm{ADD}(\mathcal{A})$ and $i_{\mathcal{A}',\mathcal{A}}$ an exact functor (wrt. $(\mathcal{A}', \mathcal{S}')$ and $(\mathcal{A}, \mathcal{S})$). We call this the **poset of exact subcategories** of $\mathcal{E}$.
We also consider $\mathrm{Ex}(\mathcal{E}) \subseteq \mathrm{EX}(\mathcal{E})$ consisting of all $(\mathcal{A}', \mathcal{S}')$ such that $\mathcal{A}'$ is also extension-closed in $\mathcal{E}$. This implies that $(\mathcal{A}', \mathcal{S}')$ is an exact substructure on $(\mathcal{A}', \mathcal{S} \cap \mathcal{A}')$. We call this the **poset of extension-closed exact subcategories** of $\mathcal{E}$.

In the literature two subposets of $\mathrm{Ex}(\mathcal{E})$ are studied:

(1) $\mathrm{ext}(\mathcal{E})$ consisting of all additively closed **fully exact subcategories** $\mathcal{E}$.
(2) $\mathrm{ex}(\mathcal{E})$ consisting of all **exact substructures of** $\mathcal{E}$ (cf. [1, Thm 5.3] for the lattice structure).

We have by definition

$$\mathrm{Ex}(\mathcal{E}) = \bigsqcup_{\mathcal{E}' \in \mathrm{ext}(\mathcal{E})} \mathrm{ex}(\mathcal{E}')$$

Remark 3.2. If $f: \mathcal{E}' \to \mathcal{E}$ is a fully faithful exact functor, then the additive closure of the essential image equipped with the exact structure induced from $\mathcal{E}'$ gives an element in $\mathrm{EX}(\mathcal{E})$. It lies in $\mathrm{Ex}(\mathcal{E})$ if and only if the essential image is extension closed in $\mathcal{E}$.

We have the obvious poset structure on $\mathrm{EX}(\mathcal{E})$:

$$\mathcal{E}_1 \le \mathcal{E}_2 \quad \text{if} \quad \mathcal{E}_1 \in \mathrm{EX}(\mathcal{E}_2)$$

Theorem 3.3. $\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E})$ *are complete lattices and* $\mathrm{ex}(\mathcal{E})$ *is complete sublattice of* $\mathrm{EX}(\mathcal{E})$ *and of* $\mathrm{Ex}(\mathcal{E})$.

All four posets have a unique maximal element $\mathcal{E}$ and a unique minimal element (which is $\{0\}$ except for $\mathrm{ex}(\mathcal{E})$ where it is the split exact structure).

The main ingredient is the following observation

Proposition 3.4. *Let $\mathcal{A}$ be a small additive category and $i_j: \mathcal{A}_j \to \mathcal{A}$, $j \in I$ (for some set I) inclusions of full, additively closed subcategories (recall these also closed under isomorphism). Assume, we have exact structures $\mathcal{E} = (\mathcal{A}, \mathcal{S}), \mathcal{E}_j = (\mathcal{A}_j, \mathcal{S}_j)$ such that i_j are exact functors. Then:*

$$\bigwedge \mathcal{E}_j := (\mathcal{B} := \bigcap_{j \in I} \mathcal{A}_j,\ \mathcal{R} := \{X_1 \xrightarrow{i} X_2 \xrightarrow{p} X_3 \colon X_i \in \bigcap_{j \in I} \mathcal{A}_j, (i,p) \in \mathcal{S}_j \forall j \in I\})$$

is an exact category such that the inclusion $\bigwedge_{j \in I} \mathcal{E}_j \to \mathcal{E}_i$ is an exact functor for all $i \in I$.

Remark 3.5. $I = \{1, 2\}$, the exact category $\mathcal{E}_1 \wedge \mathcal{E}_2$ is an exact category, it will fulfill the universal property of a pullback in the category of exact categories with exact functors (warning: This will fulfill the universal property of the **strict** 2-pullback, we are not considering other versions of 2-pullbacks here!).

Proof. Clearly, $(i,p) \in \mathcal{R}$ implies that (i,p) is a kernel-cokernel pair in $\mathcal{B}$ as this is the case in the bigger categories $\mathcal{A}_j$ for each $j \in I$.

Assume we have a cospan of objects in $\mathcal{B}$: $Y \xrightarrow{p} Z \xleftarrow{f} C$ such that p is an $\mathcal{E}_j$-deflation for all $j \in I$.

Then by [2], Prop.2.12: We have an $\mathcal{E}_j$-exact sequence $K_j \xrightarrow{k_j} Y \oplus C \xrightarrow{(p,f)} Z$ for $j \in I$. But as i_j are exact functors, we find all k_j are $\mathcal{A}$-kernels of the same map, so they are isomorphic and therefore $K := K_j \in \mathcal{B}$. Now, as the $\mathcal{A}_j$-pullback of p coincides with the $\mathcal{A}_{j'}$-pullback for every $j, j' \in I$, it is an $\mathcal{R}$-deflation $K \to C$.
For $C = 0$, we also get the $\mathcal{A}_j$-kernel and the $\mathcal{A}_{j'}$-kernel of p are isomorphic for all $j, j' \in I$, then it is obvious that composition of $\mathcal{R}$-deflations are $\mathcal{R}$-deflations.

Now, assume we have a span in $\mathcal{B}$: $D \xleftarrow{g} X \xrightarrow{i} Y$ with i an $\mathcal{E}_j$-inflation for $j \in I$. As above we observe that we have $Q = \operatorname{coker}(X \to D \oplus Y) \in \mathcal{B}$ and then the $\mathcal{A}_j$-pushout of i coincides with the $\mathcal{A}_{j'}$-pushout of i for all $j, j' \in I$ and is an $\mathcal{R}$-inflation. We also easily see that composition of $\mathcal{R}$-inflations are $\mathcal{R}$-inflations. $\square$

Proof. (of Thm. 3.3) From the previous Proposition, we can conclude that $\mathrm{EX}(\mathcal{E})$ is a complete meet semi-lattice. As we have an obvious unique maximal element $\mathcal{E} \in \mathrm{EX}(\mathcal{E})$ it becomes a lattice via the following join described in the introduction. $\square$

Remark 3.6. In given $\mathcal{B} \in \mathrm{ADD}(\mathcal{A})$, and $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ an exact category. We may look at $\mathcal{B} \cap \mathcal{S}$, i.e. all short exact sequences with all three terms in $\mathcal{B}$. To characterise when this is an exact structure is technical (and not very enlightning), see Lemma below. Here are some easy positive answers:
If $\mathcal{B}$ is extension-closed it is and in this case $\mathrm{Ext}^1_{\mathcal{S}\cap\mathcal{B}} = (\mathrm{Ext}^1_{\mathcal{E}})|_{\mathcal{B}}$.
If $\mathcal{B}$ is deflation- and inflation-closed (i.e. closed under kernels of arbitary deflations and cokernels of arbitrary inflations between objects in $\mathcal{B}$) then it also is.
A negative answer is provided below.

Example 3.7. (A negative answer) Consider the abelian category of finite-dimensional representations (over some field) of the quiver $1 \to 2 \to 3$. The indecomposables are the projectives P_i, injectives I_i (with $I_3 = P_1$) and S_2. We consider $\mathcal{B} = \mathrm{add}(P_3 \oplus P_1 \oplus I_2 \oplus S_2)$. We have an exact sequence $0 \to P_3 \to P_1 \to I_2 \to 0$. Which in $\mathcal{E}$ has a pull-back along $S_2 \to I_2$ given by P_2. But in $\mathcal{B} \cap \mathcal{S}$ there does not exist a kernel-cokernel pair to which it could pullback.

Example 3.8. Consider $\mathcal{E} = \mathrm{Mod}\, A$ for a ring A. Let I be a two sided ideal, then $\mathcal{A}' = \{X \in \mathrm{Mod}-A \mid IX = 0\}$ is extension-closed if and only if $I^2 = 0$. In either case the restriction of scalars $\mathrm{Mod}-(A/I) \to \mathrm{Mod}-A$ is a fully faithful exact functor with essential image $\mathcal{A}'$ and $\mathcal{A}'$ is inflation- and deflation-closed. So this gives an element $(\mathcal{A}', \mathcal{S}') \in \mathrm{EX}(\mathcal{E})$. It is easily see that this exact structure is abelian (since it is equivalent to the one on $\mathrm{Mod}-A/I$).

Lemma 3.9. *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category and $\mathcal{B} \in \mathrm{ADD}(\mathcal{A})$. Then, the following are equivalent:*

(1) *$(\mathcal{B}, \mathcal{S} \cap \mathcal{B})$ is an exact category*
(1') *$\mathcal{S} \cap \mathcal{B}$ are closed under pull-back and $\mathcal{S} \cap \mathcal{B}$ are closed under push-out.*
(In particular, these pull-back and push-out have to exist in $\mathcal{B}$).
(2) *For every inflation $i\colon B \rightarrowtail B'$ in $\mathcal{S} \cap \mathcal{B}$ we have: If i factors in $\mathcal{A}$ as $i = ba$, $a\colon B \to C$ with $\operatorname{coker} a$ in $\mathcal{B}$, then $C \in \mathcal{B}$.*
For every deflation $d\colon B' \twoheadrightarrow B''$ in $\mathcal{S} \cap \mathcal{B}$ we have: If d factors in $\mathcal{A}$ as $d = ef$, $e\colon D \to B''$ with $\ker e$ in $\mathcal{B}$ then $D \in \mathcal{B}$.

In this case, $i_{\mathcal{B},\mathcal{A}}$ is an exact functor, i.e. $(\mathcal{B}, \mathcal{S} \cap \mathcal{B}) \in \mathrm{EX}(\mathcal{E})$.

Proof. The equivalence (1) to (1') follows from [4, Lem. 1.9, Prop.1.10]. The equivalence (1') to (2) follows from the strong Obscure axiom [2, Prop. 7.6]. $\square$

This is one of the few instances where more general exact subcategories than just extension-closed are considered:

Example 3.10. Not extension-closed exact subcategories in (the following exact categories) Hausdorff locally convex spaces, Frechet spaces and topological vector spaces respectively are studied in [3].

As a direct corollary of Chapter 1, Cor. **??** we have the following: Every $\mathcal{F} \in \mathrm{EX}(\mathcal{E})$ we have $i\colon \mathcal{F} \to \mathcal{E}$ can be factorized either as

$$\mathcal{F} \leq \mathcal{F}_{\mathcal{B}} \xrightarrow{i} \mathcal{E}$$

where $\mathcal{F}_{\mathcal{B}}$ is the maximal exact structure making i exact (and $\leq$ means we have an inclusion of a substructure). Or it can be factorized as

$$\mathcal{F} \xrightarrow{i} \mathcal{E}' \leq \mathcal{E}$$

where $\mathcal{E}' = \bigwedge \mathcal{E}''$ where $\mathcal{E}''$ runs through all exact substructures such that $i\colon \mathcal{F} \to \mathcal{E}''$ is exact.
From the first factorization we can conclude:

THEOREM 3.11. *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an i.c. small exact category then we have*

$$\mathrm{EX}(\mathcal{E}) = \bigsqcup_{\mathcal{B} \in \mathrm{ADD}(\mathcal{A})} \mathrm{ex}(\mathcal{F}_{\mathcal{B}})$$
$$\mathrm{Ex}(\mathcal{E}) = \bigsqcup_{\mathcal{E}' \in \mathrm{ext}(\mathcal{E})} \mathrm{ex}(\mathcal{E}')$$

There are many other posets one can define here. Since we think fully faithful exact functors with extension-closed images are interesting, these two posets are our main interest. But for example we could also look at

$$\mathrm{Ex}'(\mathcal{E}) := \bigsqcup_{\mathcal{E}' \in \mathrm{ex}(\mathcal{E})} \mathrm{ext}(\mathcal{E}')$$

Example 3.12. Let $\mathcal{T} \subseteq \mathcal{E}$ be a self-orthogonal category, then $\mathrm{pres}^{\mathcal{E}}(\mathcal{T}) = \{X \colon \exists T \twoheadrightarrow X, \text{ with } T \in \mathcal{T}\}$ is extension closed in $\mathcal{E}$. If $\mathcal{E}'$ is an exact substructure of $\mathcal{E}$, we have $\mathcal{T} \subseteq \mathcal{E}'$ is still self-orthogonal and $\mathrm{pres}^{\mathcal{E}'}(\mathcal{T}) \subseteq \mathrm{pres}^{\mathcal{E}}(\mathcal{T})$. We get a subposet of $\mathrm{Ex}'(\mathcal{E})$

$$\{\mathrm{pres}^{\mathcal{E}'}(\mathcal{T}) \mid \mathcal{E}' \in \mathrm{ex}(\mathcal{E})\}$$

It has a unique maximal element $\mathrm{pres}^{\mathcal{E}}(\mathcal{T})$ and a unique minimal element $\mathcal{T}$.

Remark 3.13. Let $\mathcal{A}' \in \mathrm{ADD}(\mathcal{A})$ and $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ an exact category. Let $\mathcal{I}_{\mathcal{A}'} \subseteq \mathrm{ex}(\mathcal{E})$ be the exact substructures of $\mathcal{E}$ such that $\mathcal{A}'$ is extension-closed in it. I do not know anything on maximal elements in $I_{\mathcal{A}'}$.

4. The functorial point of view

Let $\mathcal{E}$ be an essentially small exact category. We consider three classical assignments (which are all 2-functorial on the category of small exact category with exact functors) for $\mathcal{F} = (\mathcal{B}, \mathcal{S}') \in \mathrm{EX}(\mathcal{E})$ the **Auslander exact category, tf Auslander category** and **effaceable functors** respectively

$$\mathrm{AE}(\mathcal{F}) = \{\mathrm{coker}(\mathrm{Hom}_{\mathcal{B}}(-, f)) \mid f \ \mathcal{F}\text{-admissible}\}$$
$$\mathrm{H}(\mathcal{F}) := \{\mathrm{coker}\,\mathrm{Hom}_{\mathcal{B}}(-, i) \mid i \ \mathcal{F}\text{-inflation}\}$$
$$\mathrm{eff}(\mathcal{F}) := \{\mathrm{coker}(\mathrm{Hom}_{\mathcal{B}}(-, d) \mid d \ \mathcal{F}\text{-deflation}\}$$

All will be considered fully exact subcategories in $\mathrm{mod}_1\, \mathcal{A}$ (where $\mathcal{A}$ is the underlying additive category of $\mathcal{E}$). By a results of [7] and [6] (cf. Chapter 2), we have characterizations of the subcategories when we look only at exact substructures of $\mathcal{E}$. Our aim is to extend these to the whole lattice $\mathrm{EX}(\mathcal{E})$.

4.1. Partially resolving subcategories. We start with some background definitions.

Definition 4.1. Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Let $\mathcal{F} \subseteq \mathcal{E}$ be a fully exact subcategory. We say that $\mathcal{F}$ is **partially resolving** if

(PR1) $\mathcal{F} = \operatorname{add}(\mathcal{F})$ (i.e. $\mathcal{F}$ is closed under taking direct summands in $\mathcal{E}$)
(PR2) For every $F \in \mathcal{F}$ we find an $\mathcal{E}$-short exact sequence $\Omega F \to P \to F$ with $P \in \mathcal{P} \cap \mathcal{F}$ and $\Omega F \in \mathcal{F}$ (we call this property: $\mathcal{F}$ is *closed under taking syzygies*)

Then $\mathcal{F}$ also has enough projectives with $\mathcal{P}(\mathcal{F}) = \mathcal{Q} \subseteq \mathcal{P}(\mathcal{E})$ and we say $\mathcal{F}$ is partial resolving with respect to $\mathcal{Q} \subseteq \mathcal{P}$.

Remark 4.2. A partially resolving subcategory $\mathcal{F}$ in an exact category with enough projectives $\mathcal{P}$ is resolving if and only if $\mathcal{P} \subseteq \mathcal{F}$.

Lemma 4.3. *([5], Lem. 2.5) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Let $\mathcal{F}$ be a fully exact subcategory which is closed under taking summands in $\mathcal{E}$, then the following are equivalent:*

(1) *$\mathcal{F}$ is partially resolving*
(2) *$\mathcal{F}$ is deflation-closed with enough projectives $\mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P}$.*

The proof is completely analogue to the given reference, we leave it to the reader.

Remark 4.4. If we are in a Krull-Schmidt category (say we have minimal projective covers) then the syzygies in (PR2) can be always taken with respect to the minimal projective cover and in this case we can find intersections of arbitrary partially resolving subcategories are partially resolving.

Remark 4.5. If $\mathcal{F}$ is partially resolving in $\mathcal{E}$ then it is homologically exact, cf. Chapter 1.

For a small additive category $\mathcal{A}$, we denote by $\operatorname{mod}_1 \mathcal{A}$ the category of all additive functors $F\colon \mathcal{A}^{op} \to (Ab)$ with $F \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, f)$ for some morphism f in $\mathcal{A}$ ($f \in \operatorname{Mor} - \mathcal{A}$). We see this as a fully exact subcategory of the abelian category $\operatorname{Mod} -\mathcal{A}$ (all additive contravariant functors $\mathcal{A}^{op} \to (Ab)$).
For a full additive subcategory $\mathcal{B} \subseteq \mathcal{A}$ we define the full subcategory of $\operatorname{mod}_1 \mathcal{A}$

$$\operatorname{mod}_1(\mathcal{A}|\mathcal{B}) := \{F \in \operatorname{mod}_1 \mathcal{A} \mid F \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, f), f \in \operatorname{Mor} - \mathcal{B}\}$$

Then this is a fully exact subcategory of $\operatorname{mod}_1 \mathcal{A}$ by the horseshoe lemma.

Lemma 4.6. *The restriction functor*

$$\Phi\colon \operatorname{mod}_1(\mathcal{A}|\mathcal{B}) \to \operatorname{mod}_1 \mathcal{B}, \quad F \mapsto F|_{\mathcal{B}}$$

is an equivalence of additive categories which is also an exact functor.

The proof is straight-forward. This is usually not an equivalence of exact categories, the quasi-inverse functor is a not necessarily exact tensor functor - we see $\operatorname{mod}_1(\mathcal{A}|\mathcal{B})$ as an exact substructure of $\operatorname{mod}_1 \mathcal{B}$.
(Nevertheless it restricts to an exact equivalence of many smaller categories, e.g. on $\operatorname{mod}_2(\mathcal{A}|\mathcal{B}) \to \operatorname{mod}_2 \mathcal{B}$ it is already an exact equivalence, see other instances later).

Recall a Serre subcategory is a full additive subcategory $\mathcal{F}$ in an exact category $\mathcal{E}$ with the following property: For every $\mathcal{E}$-short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ we have $Y \in \mathcal{F}$ if and only if $X, Z \in \mathcal{F}$.

Definition 4.7. We denote by $\mathcal{P}^2(\mathcal{A})$ the full subcategory of $\operatorname{Mod} \mathcal{A}$ given by all functors F such that there exists an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(-, X) \to \operatorname{Hom}_{\mathcal{A}}(-, Y) \to \operatorname{Hom}_{\mathcal{A}}(-, Z) \to F \to 0$$

for some X, Y, Z in $\mathcal{A}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be a full additively closed subcategory. We write $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ for the full subcategory of $\mathcal{P}^2(\mathcal{A})$ given by all functors F such that there exists an exact sequence as above with X, Y, Z in $\mathcal{B}$.

It is an easy horse-shoe-lemma argument to see that $\mathcal{P}^2(\mathcal{A}|\mathcal{B}) \subseteq \mathcal{P}^2(\mathcal{A}) \subseteq \operatorname{Mod}\mathcal{A}$ are inclusions of extension-closed subcategories. From now on, we equip them with the fully exact structure.

Lemma 4.8. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a full additive subcategory. Then, $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ is a partially resolving subcategory of $\mathcal{P}^2(\mathcal{A})$. Furthermore the restriction functor*

$$\mathcal{P}^2(\mathcal{A}|\mathcal{B}) \to \mathcal{P}^2(\mathcal{B}), \quad F \mapsto F|_{\mathcal{B}}$$

is an equivalence of exact categories (i.e. an equivalence of categories which is homologically exact).

Proof. As $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ is by definition closed under taking syzygies in $\mathcal{P}^2(\mathcal{A})$, it follows that it is an exact category with enough projectives given by $\operatorname{Hom}_{\mathcal{A}}(-,B)$, $B \in \mathcal{B}$. This implies it is partially resolving.
Restriction functors are exact functors on functor categories, therefore their restrictions to fully exact subcategories are still exact. By definition this functor is essentially surjective. Using the projective presentations one can see that this is an equivalence of additive categories which restricts to an equivalence on the category of projetives. Now, both are exact categories with enough projectives and have gldim ≤ 2, therefore the derived functor is a triangle equivalence. This implies that the funtor is homologically exact. □

In particular, (using the quasi-inverse of the equivalence) we will consider $\mathcal{P}^2(\mathcal{B})$ from now on as a partially resolving subcategory in $\mathcal{P}^2(\mathcal{A})$.

4.2. Short recap of definitions from Chapter 2. The **grade** of $F \in \operatorname{Mod}\mathcal{A}$ is defined as the supremum of all natural numbers $i \geq 0$ such that
$\operatorname{Ext}^j_{\operatorname{Mod}\mathcal{A}}(F, \operatorname{Hom}_{\mathcal{A}}(-,A)) = 0 \forall A \in \mathcal{A}$ for all $j < i$ (of course, only if this exists, else we define it to be ∞). Let us denote by $\operatorname{KC}(\mathcal{A})$ the **collection of all kernel-cokernel pairs** in $\mathcal{A}$

$$\begin{aligned}
\mathcal{G}^2(\mathcal{A}) &= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d),(j,p) \in \operatorname{KC}(\mathcal{A}), F \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, j \circ d)\} \\
&\subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \operatorname{grade(F)} \in \{0,2\}\} \\
\mathcal{C}^2(\mathcal{A}) &= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d) \in \operatorname{KC}(\mathcal{A}), F \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,d)\} \\
&= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \operatorname{grade(F)} = 2\} \\
\mathcal{J}^1(\mathcal{A}) &= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (j,p) \in \operatorname{KC}(\mathcal{A}), F \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,j)\} \\
&\subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \operatorname{grade(F)} = 0\}
\end{aligned}$$

Enomoto's duality:

$$\begin{gathered}
E \colon \mathcal{C}^2(\mathcal{A})^{op} \to \mathcal{C}^2(\mathcal{A}^{op}) \\
E(\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,d)) \cong \operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(i,-)) \quad (i,d) \in \operatorname{KC}(\mathcal{A})
\end{gathered}$$

Auslander-Bridger transpose (also a duality):
The ideal quotient of $\operatorname{mod}_1\mathcal{A}$ with respect to the projectives is denoted by $\underline{\operatorname{mod}_1}\mathcal{A}$.

$$\begin{gathered}
\operatorname{Tr} \colon (\underline{\operatorname{mod}_1}\mathcal{A})^{\mathrm{op}} \to \underline{\operatorname{mod}_1}(\mathcal{A}^{op}) \\
\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,f) \mapsto \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(f,-).
\end{gathered}$$

Lemma 4.9. *([6, Prop. 2.8]) If there is an exact structure $\mathcal{E}$ and we have a short exact sequence (i,d) and a kernel-cokernel pair (j,p) such that* $\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,d) = F = \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,p)$, *then (j,p) is also an $\mathcal{E}$-short exact sequence.*

Following loc. cit. we say $\mathcal{E}$ short exact sequences are *closed under homotopy* (within kernel-cokernel presentations). In Chapter 4 we investigate homotopy-closedness more generally.

In a similar way, we showed in Lemma 4.9 that $\mathcal{E}$-inflations are closed under homotopy among all presentations.

4.3. Generalizations. Now, given a full additive subcategory $\iota\colon \mathcal{B} \subseteq \mathcal{A}$ and denote by $\mathrm{KC}(\mathcal{A}|\mathcal{B})$ the **collection of all relative kernel-cokernel pairs** in $\mathcal{A}$, these are kernel-cokernel pairs $(\iota(j), \iota(p))$ in $\mathcal{A}$ such that (j,p) is a kernel-cokernel pair in $\mathcal{B}$.

$$\begin{aligned}
\mathcal{G}^2(\mathcal{A}|\mathcal{B}) &= \{F \in \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \mid \exists (i,d),(j,p) \in \mathrm{KC}(\mathcal{A}|\mathcal{B}), F \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, j\circ d)\}\\
\mathcal{C}^2(\mathcal{A}|\mathcal{B}) &= \{F \in \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \mid \exists (i,d) \in \mathrm{KC}(\mathcal{A}|\mathcal{B}), F \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, d)\}\\
\mathcal{J}^1(\mathcal{A}|\mathcal{B}) &= \{F \in \mathcal{P}^1(\mathcal{A}|\mathcal{B}) \mid \exists (j,p) \in \mathrm{KC}(\mathcal{A}|\mathcal{B}), F \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, j)\}
\end{aligned}$$

For an exact category $\mathcal{F} = (\mathcal{B}, \mathcal{S}')$ in $\mathrm{EX}(\mathcal{E})$, we obviously have

$$\mathrm{AE}(\mathcal{F}) \subseteq \mathcal{G}^2(\mathcal{A}|\mathcal{B}), \quad \mathrm{eff}(\mathcal{F}) \subseteq \mathcal{C}^2(\mathcal{A}|\mathcal{B}), \quad \mathrm{H}(\mathcal{F}) \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B}).$$

The Auslander category $\mathrm{AE}(\mathcal{F}) \subseteq \mathcal{P}^2(\mathcal{B}) \cong \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \subseteq \mathrm{mod}_1\, \mathcal{A}$ is an extension-closed subcategory. As $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ is partially resolving in $\mathcal{P}^2(\mathcal{A})$, we have that $\mathrm{AE}(\mathcal{F})$ is partially resolving in $\mathcal{P}^2(\mathcal{A})$ (and also $\mathrm{H}(\mathcal{F})$ is partially resolving in $\mathcal{P}^1(\mathcal{A})$). By Auslander correspondence we expect $\mathcal{F} \mapsto \mathrm{AE}(\mathcal{F})$ to be a bijection with certain partially resolving subcategories of $\mathcal{P}^2(\mathcal{A})$. But we have to modify the definition of the transpose category (because it always adds all projectives).
On the other hand, for $\mathrm{eff}(\mathcal{F})(\subseteq \mathrm{AE}(\mathcal{F})) \subseteq \mathrm{mod}_1\, \mathcal{A}$ we have to expect to get very often the same subcategory (for example: The split exact categories $\mathcal{F}$ always have $\mathrm{eff}(\mathcal{F}) = 0$ no matter on which underlying category). In fact, the data which is lost, is given precisely by the additive category $\mathcal{B}$ on which the exact structure has to be defined.

Assume that we have a left exact functor $f\colon \mathcal{F} = (\mathcal{B}, \mathcal{T}) \to (\mathcal{A}, \mathcal{S}) = \mathcal{E}$. In [7], Thm 3.9, it is shown that the composition $\mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{X \mapsto \mathrm{Hom}(-,X)} \mathrm{AE}(\mathcal{E})$ is left exact and factors (uniquely up to isomorphism) over an exact functor $\mathrm{AE(f)}\colon \mathrm{AE}(\mathcal{F}) \to \mathrm{AE}(\mathcal{E})$.

Lemma 4.10. *Then the following are equivalent:*

(1) *f is fully faithful.*
(2) *$\mathrm{AE}(f)$ is homologically exact.*

Furthermore, if f is inclusion of an additively closed subcategory, we can identify $\mathrm{AE}(\mathcal{F})$ with the essential image of $\mathrm{AE(f)}$ which is a partial resolving subcategory in $\mathrm{AE}(\mathcal{E})$. In this subcategory all objects have either grade 2 ore 0. The grade 2-objects are precisely the effaceable functors $\mathrm{eff}(\mathcal{F})$, i.e. we have

$$\mathrm{AE}(\mathcal{F}) \cap \mathcal{C}^2(\mathcal{A}) = \mathrm{eff}(\mathcal{F})$$

Proof. The derived functor of $\mathrm{AE}(f)$ identifies with $K^b(f)\colon K^b(\mathcal{B}) \to K^b(\mathcal{A})$. Therefore f is fully faithful iff $K^b(f)$ is fully faithful iff $\mathrm{AE}(f)$ is homologically exact. The second claim follows from $\mathrm{AE}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}) = \mathrm{eff}(\mathcal{E})$ observed in [7], then $\mathrm{AE}(f)$ maps the torsion pair $(\mathrm{eff}(\mathcal{F}) = {}^{\perp}\mathcal{Q}, \mathrm{H}(\mathcal{F}) = \operatorname{copres}\mathcal{Q})$ where $\mathcal{Q} = \mathrm{AE}(\mathcal{F})$ to $(\mathrm{eff}(\mathcal{E}) = {}^{\perp}\mathcal{P}, \mathrm{H}(\mathcal{E}) = \operatorname{copres}\mathcal{P})$ where $\mathcal{P} = \mathrm{AE}(\mathcal{E})$ because it preserves projectives and is exact. Therefore the last claim follows. $\square$

Remark 4.11. With the same proof we also show for an inflation-preserving $f\colon \mathcal{E} \to \mathcal{F}$: The functor f is fully faithful if and only if $\mathrm{H}(f)$ is homologically exact.

Lemma 4.12. *Let $\mathcal{F} \in \mathrm{EX}(\mathcal{E})$ and consider $\mathrm{eff}(\mathcal{F})$ as a full subcategory of $\mathrm{eff}(\mathcal{E})$ (by applying $\Phi^{-1}_{\mathcal{A}|\mathcal{B}}$ to it), then we have*

$$\mathrm{eff}(\mathcal{F}) \subseteq \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$$

is a Serre subcategory.
Furthermore if $\mathcal{E}$ restricts to $\mathcal{F}$ on $\mathcal{B}$ (i.e. every ses in $\mathcal{E}$ which has all three objects in $\mathcal{B}$ is exact in $\mathcal{F}$), then we have $\mathrm{eff}(\mathcal{F}) = \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$.

PROOF. Clearly $\mathrm{eff}(\mathcal{F}) \subseteq \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$ extension-closed. As $\mathcal{F}$ is an exact structure, $\mathrm{eff}(\mathcal{F})$ is already Serre subcategory in $\mathcal{C}^2(\mathcal{B}) \cong \mathcal{C}^2(\mathcal{A}|\mathcal{B})$, this implies it also is a Serre subcategory in $\mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$.
Assume now that $\mathcal{E}$ restricts to $\mathcal{F}$ on $\mathcal{B}$, then we have to show the other inclusion. Let $F \in \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$, then we find two $\mathcal{A}$-kernel-cokernel pairs representing F and one is an $\mathcal{E}$-ses and the other one is an $(i,p) \in \mathrm{KC}(\mathcal{A}|\mathcal{B})$. As $\mathcal{E}$-short exact sequences are closed under *homotopy* (Lemma 4.9) it follows that (i,p) is also an $\mathcal{E}$-short exact sequence. By our assumption, the $\mathcal{E}$-short exact sequences with all three terms in $\mathcal{B}$ are just the $\mathcal{F}$-short exact sequences, it follows that $F \in \mathrm{eff}(\mathcal{F})$. □

We will also need the following definition.

Definition 4.13. Let $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A})$ an additive subcategory, we define $\mathcal{B} := \mathcal{B}_{\mathcal{X}} \subseteq \mathcal{A}$ to be the full (additive) subcategory of objects $B \in \mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(-, B) \in \mathcal{X}$.
We consider the composition $\mathcal{P}^2(\mathcal{A}^{op}|\mathcal{B}^{op}) \to \mathrm{mod}_1\, \mathcal{A} \to \underline{\mathrm{mod}}_1 \mathcal{A}^{op}$ as the identity on objects. In this case we define the **relative transposed category** $\mathrm{Tr}_{\mathrm{rel}}(\mathcal{X})$ to be the full subcategory of objects X in $\mathcal{P}^2(\mathcal{A}^{op}|\mathcal{B}^{op})$ such that $X \cong \mathrm{Tr}(\mathrm{X}')$ in $\underline{\mathrm{mod}}_1 \mathcal{A}^{op}$ for some $X' \in \mathcal{X}$.

Remark 4.14. By definition $\mathcal{B}_{\mathcal{G}^2(\mathcal{A}|\mathcal{B})} = \mathcal{B} = \mathcal{B}_{\mathcal{J}^1(\mathcal{A}|\mathcal{B})}$ and

$$\mathrm{Tr}_{\mathrm{rel}}(\mathcal{G}^2(\mathcal{A}|\mathcal{B})) = \mathcal{G}^2(\mathcal{A}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}}), \quad \Omega\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}^1(\mathcal{A}|\mathcal{B})) = \mathcal{J}^1(\mathcal{A}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}})$$

We also remark that $\mathcal{B}_{\mathcal{C}^2(\mathcal{A}|\mathcal{B})} = \{0\}$.

We fix an exact structure $\mathcal{E}$ on $\mathcal{A}$. For every $\mathcal{B} \subseteq \mathcal{A}$ full additively closed subcategory let $\mathcal{C}_{\mathcal{B},max} \subseteq \mathrm{mod}_1\, \mathcal{B}$ be the Serre subcategory corresponding to the maximal exact structure $\mathcal{F}$ on $\mathcal{B}$ such that the inclusion $\mathcal{F} \to \mathcal{E}$ is an exact functor. Recall that we have an equivalence $\Phi_{\mathcal{A}|\mathcal{B}} \colon \mathcal{C}^2(\mathcal{A}|\mathcal{B}) \to \mathcal{C}^2(\mathcal{B}), F \mapsto F|_{\mathcal{B}}$ of exact categories.

THEOREM 4.15. *Let $\mathcal{A}$ be an idempotent complete, small additive category and $\mathcal{E} = \mathcal{E}_{max}$ the maximal exact structure on it. Then the assignments*

$$\mathcal{F} \mapsto \mathrm{AE}(\mathcal{F}), \quad \mathcal{F} \mapsto \inf(\mathcal{F}), \quad \mathcal{F} = (\mathcal{B}, \mathcal{S}) \mapsto (\mathcal{B}, \mathrm{eff}(\mathcal{F}))$$

give bijections between $\mathrm{EX}(\mathcal{E})$ *and (1), (2) and (3) respectively.*

(1) *Partially resolving subcategories $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A})$ such that $\mathcal{X} \subseteq \mathcal{G}^2(\mathcal{A}|\mathcal{B})$ for $\mathcal{B} = \mathcal{B}_{\mathcal{X}}$ and $\mathrm{Tr}_{\mathrm{rel}}(\mathcal{X}) \subseteq \mathcal{P}^2(\mathcal{A}^{\mathrm{op}})$ is also partially resolving.*
(2) *Partially resolving subcategories $\mathcal{J} \subseteq \mathcal{P}^1(\mathcal{A})$ such that $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B})$ for $\mathcal{B} = \mathcal{B}_{\mathcal{J}}$ and $\Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}) \subseteq \mathcal{P}^1(\mathcal{A}^{\mathrm{op}})$ is also partially resolving.*
(3) *pairs of categories $(\mathcal{B}, \mathcal{C})$ with*
  (*) *$\mathcal{B} \subseteq \mathcal{A}$ a full additively closed subcategory and*
  (*) *$\mathcal{C} \subseteq \mathcal{C}^2(\mathcal{A}|\mathcal{B})$ a full additively closed subcategory such that $\Phi_{\mathcal{A}|\mathcal{B}}(\mathcal{C})$ is a Serre subcategory in $\mathcal{C}_{\mathcal{B},max}$.*

PROOF. We observe that (3) is just a trivial consequence of Enomoto's bijection, Chapter 2, Theorem **??**, we just state it here for completeness sake.
Let us turn to (1) and (2) and show the assignments are well-defined. Let $\mathcal{F} = (\mathcal{B}, \mathcal{S}) \in \mathrm{EX}(\mathcal{E})$.

(1) We already observed that $\mathcal{X} := \mathrm{AE}(\mathcal{F})$ is partially resolving in $\mathcal{P}^2(\mathcal{A})$. We want to see $\mathrm{AE}(\mathcal{F}^{op}) = \mathrm{Tr}_{\mathrm{rel}}(\mathcal{X})$. We denote by $\underline{\mathcal{G}^2(\mathcal{A})}$ the essential image of $\mathcal{G}^2(\mathcal{A}) \to \underline{\mathrm{mod}_1}\mathcal{A}$. When we restrict the functor $\mathrm{Tr}_{\mathcal{A}}$ we a commutative diagram (*)

$$\begin{array}{ccc} \underline{\mathcal{G}^2(\mathcal{A}|\mathcal{B})} & \xrightarrow{\text{incl}} & \underline{\mathcal{G}^2(\mathcal{A})} \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathcal{B}}} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathcal{A}}} \\ \underline{\mathcal{G}^2(\mathcal{A}^{op}|\mathcal{B}^{op})} & \xrightarrow{\text{incl}} & \underline{\mathcal{G}^2(\mathcal{A}^{op})} \end{array}$$

The underline on the right hand side can either be seen as the essential image in $\underline{\mathrm{mod}_1}\mathcal{A}^{(op)}$ or the essential image in $\underline{\mathrm{mod}_1}\mathcal{B}^{(op)}$, the functors $\mathrm{Tr}_{\mathcal{A}}$ and $\mathrm{Tr}_{\mathcal{B}}$ coincide on this subcategory. This means that the relative trace category $\mathrm{Tr}_{rel}(\mathcal{X}) = \mathrm{Tr}_{\mathcal{B}}(\mathcal{X}) = \mathrm{AE}(\mathcal{F}^{op})$ and this is partially resolving in $\mathcal{P}^2(\mathcal{A}^{op})$.

(2) We already know that $\mathcal{J} := \mathrm{H}(\mathcal{F})$ is partially resolving in $\mathcal{P}^1(\mathcal{A})$ and we want to see $\mathrm{H}(\mathcal{F}^{op}) = \Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J})$. We look at the diagram (*). Not just trace also $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{B}}$ identify on these subcategories, so $\Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}) = \Omega_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\mathcal{J}) = \mathrm{H}(\mathcal{F}^{\mathrm{op}})$ and this is partially resolving in $\mathcal{P}^1(\mathcal{A}^{op})$.

Now, we define the inverse assignments:

(1) Assume we have $\mathcal{X}$ as in (1) and define $\mathcal{B} = \mathcal{B}_{\mathcal{X}}$, then we see that $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \cong \mathcal{P}^2(\mathcal{B})$ is resolving and as we have $\mathcal{X} \subseteq \mathcal{G}^2(\mathcal{A}|\mathcal{B})$, it follows as above $\mathrm{Tr}_{rel}(\mathcal{X}) = \mathrm{Tr}_{\mathcal{B}}(\mathcal{X}) \subseteq \mathcal{G}^2(\mathcal{A}^{op}|\mathcal{B}^{op})$ is resolving in $\mathcal{P}^2(\mathcal{A}^{op}|\mathcal{B}^{op}) \cong \mathcal{P}^2(\mathcal{B}^{op})$. By Theorem **??** it follows that $\mathcal{X} = \mathrm{AE}(\mathcal{F})$ for an exact structure $\mathcal{F}$ on $\mathcal{B}$. The inclusion $\mathcal{X} \subseteq \mathrm{AE}(\mathcal{E})$ apriori corresponds to a fully faithful left exact functor $f\colon \mathcal{F} \to \mathcal{E}$. But the inclusion $\mathrm{Tr}_{rel}(\mathcal{X}) \to \mathrm{AE}(\mathcal{E}^{op})$ corresponds to f^{op} also being left exact, we conclude that $f\colon \mathcal{E} \to \mathcal{F}$ is exact and so $\mathcal{F} \in \mathrm{EX}(\mathcal{E})$. This gives the inverse map.

(2) Assume we have $\mathcal{J}$ as in (2) and define $\mathcal{B} = \mathcal{B}_{\mathcal{J}}$, then $\mathcal{J}$ is resolving in $\mathcal{P}^1(\mathcal{A}|\mathcal{B}) \cong \mathcal{P}^1(\mathcal{B})$. As $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B})$ we conclude $\Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}) = \Omega_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\mathcal{J})$ is resolving in $\mathcal{P}^1(\mathcal{A}^{op}|\mathcal{B}^{op})$. By Theorem **??** it follows that $\mathcal{J} = \mathrm{H}(\mathcal{F})$ for an exact structure on $\mathcal{B}$ and as we have $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B})$ we conclude (using Lemma 4.9) hat all short exact sequences in $\mathcal{F}$ are mapped to kernel-cokernel pairs in $\mathcal{A}$. Then we look at the $H(\mathcal{F}) \subseteq H(\mathcal{E})$, by Appendix B, Chapter 2, it corresponds to a fully faithful inflation-preserving functor $\mathcal{E} \to \mathcal{F}$, but as this functor also maps short exact sequences to kernel-cokernel pairs it is exact.

□

5. Meet-preserving of homological conditions

5.1. Examples for negative answers. We collected these negative answers:

(1) homologically exactness and also homologically faithfulness
(2) gldim $= n$
(3) having enough projectives (or injectives)
(4) hereditary exact substructures in an hereditary exact category

Also, the subposet of hereditary exact substructures may not have a unique maximal element. The following is an example for (1).

Example 5.1. Intersections of homologically exact subcategories may not be homologically exact subcategories. We give an example of a resolving and a coresolving subcategory in an abelian category whose intersection is a semi-simple subcategory which is not homologically exact in the abelian category. Take $\mathcal{E} = \Lambda\,\mathrm{mod}$, with $\Lambda = KQ/I$ where (Q, I) is the following bound quiver

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd} \qquad I = (ba)$$

Then this has nine indecomposable representations (the projectives, the injectives S_2 and $\mathrm{rad(P_1)}$). Then $\mathcal{R} = \mathrm{add}(\Lambda \oplus \mathrm{rad(P_1)} \oplus \mathrm{I_1})$ is resolving and $\mathcal{C} = \mathrm{add}(\mathrm{D}\Lambda \oplus \mathrm{S_2} \oplus \mathrm{P_3})$ is coresolving. Their intersection is $\mathrm{add}(P_1 \oplus I_1 \oplus P_3)$. This is a semi-simple extension-closed subcategory (observe $P_1 = I_4$). Since $\mathrm{Ext}^2_\Lambda(I_1, P_3) \neq 0$ it is not homologically exact in $\mathcal{E}$.

For (3), one just has to observe that given infinitely many contravariantly finite generators $\mathcal{P}_i$ in an exact category, it is generally not true that $\bigvee \mathcal{P}_i$ (i.e. the smallest generator containing all $\mathcal{P}_i$) is contravariantly finite.

Example 5.2. Let $\mathcal{E} = \Lambda \operatorname{mod}$ with Λ the path algebra of the Kronecker quiver. We find finite dimensional preprojective generators $G_n = \Lambda \oplus \tau^- \Lambda \oplus \cdots \oplus \tau^{-n}\Lambda$, $n \in \mathbb{N}$ such that $\bigvee_{n\in\mathbb{N}} \mathrm{add}(G_n)$ is the preprojective component. Then we observe that the preprojective component is not contravariantly finite.

The following gives an example for (2), (4) and shows that homologically faithfulness is not preserved under forming meet.

Example 5.3. We give an example of an hereditary exact category with two hereditary exact substructures such that the intersection is no longer hereditary exact. Let $\mathcal{E} = \Lambda \operatorname{mod}$ where Λ is the path algebra of $1 \to 2 \to 3 \to 4$. We choose two generators $G_1 = \Lambda \oplus I_3 \oplus S_3 \oplus S_1$ and $G_2 = \Lambda \oplus S_2$. Let $\mathcal{E}_i = (\Lambda \operatorname{mod}, F_{G_i})$ be the exact substructure such that $\mathrm{Hom}(G_i, -)$ is exact on short exact sequences, then $\mathcal{E}_1 \cap \mathcal{E}_2 = (\Lambda \operatorname{mod}, F_{G_1 \oplus G_2})$. By looking at projective resolutions of non-projective indecomposables one easily see $\operatorname{gldim}(\Lambda \operatorname{mod}, F_{G_i}) = 1$ for $i = 1, 2$ and $\operatorname{gldim}(\Lambda \operatorname{mod}, F_{G_1 \oplus G_2}) = 2$.

5.2. Positive answers.

Are already discussed in the introduction, we just prove here this missing Lemma:

Lemma 5.4. *Let $\mathcal{E}$ be an exact category and $n \in \mathbb{N}$. Then the following are equivalent*

(1) $\operatorname{gldim} \mathcal{E} \leq n$
(2) $\mathrm{Ext}^n(X, \sigma)$ *is right exact for all objects X and $\mathcal{E}$-short exact sequences σ*
(3) $\mathrm{Ext}^n(\sigma, X)$ *is right exact for all objects X and $\mathcal{E}$-short exact sequences σ*

Proof. We only show the equivalence of (1) and (2) (the other equivalence follows from passing to the opposite exact category). Clearly (1) implies (2) follows from the long exact sequence on the Ext-groups. So assume (2) and take $\sigma \in \mathrm{Ext}^{n+1}_{\mathcal{E}}(X, Y)$. Write σ as a concatenation $\sigma_1\sigma_2$ with $\sigma_1 \colon Y \rightarrowtail V \twoheadrightarrow W$ and apply $\mathrm{Hom}(X, -)$ to σ_1. We look at the connecting morphism

$$\mathrm{Ext}^n_{\mathcal{E}}(X, W) \to \mathrm{Ext}^{n+1}_{\mathcal{E}}(X, Y)$$

By [8, Cor. 4.2.12] this is given by concatenation with σ_1. In particular $\sigma_2 \mapsto \sigma$ and so σ is in the image. But as $\mathrm{Ext}^n_{\mathcal{E}}(X, \sigma_1)$ is right exact, this is zero. $\square$

We say a bifunctor which is middle exact and fulfills the (corresponding) condition (2) and (3) from the previous lemma is **right exact**. So we can identify hereditary exact substructures with right exact subfunctors of $\mathrm{Ext}^1_{\mathcal{E}_{max}}$. These are usually not closed under intersection (see: negative answers). There also can not exist a structure as a complete lattice on hereditary exact substructures.

Example 5.5. This is an example with two maximal hereditary substructures. Let Λ be the path algebra of $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ bound by the relation $ba = 0$. We consider the abelian category $\mathcal{E} = \Lambda \operatorname{mod}$. Let $\mathcal{E}_i$ be the exact substructure with $\mathcal{P}(\mathcal{E}_i) = \mathrm{add}(\Lambda \oplus S_i)$, $i = 1, 2$. Both are hereditary exact and maximal wrt being hereditary.

6. Representation-finiteness

Let K be a field. Now, we assume that $\mathcal{A}$ is Krull-Schmidt K-linear category of finite representation-type (i.e. only finitely many indecomposable objects in $\mathcal{A}$). Given a full additively closed subcategory $\mathcal{M}$, then it is covariantly finite in a category $\mathcal{A}$ and we denote for any object A in $\mathcal{A}$ by $f_A \colon A \to M_A$ with M_A in $\mathcal{M}$ a left $\mathcal{M}$-approximation of A.

Lemma 6.1. *Let K be a field. Let $\mathcal{A}$ be a small K-linear additive, Hom-finite Kull-Schmidt category of finite type and $\mathcal{M} = \operatorname{add}(M)$ an additively closed subcategory. Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$ with enough projectives with $\mathcal{P}(\mathcal{E}) = \operatorname{add}(G)$ and let $\mathcal{F}$ be the maximal exact structure on $\mathcal{M}$ making the inclusion an exact functor. Then we have $\mathcal{P}(\mathcal{F}) = \operatorname{add}(M_G \oplus M_{\operatorname{coker} f_C})$ where we define* $\operatorname{copres}_{\mathcal{E}}(\mathcal{M}) := \{X \in \mathcal{E} \mid \exists \mathcal{E}\text{-}ses\ X \rightarrowtail M' \twoheadrightarrow Y\}$ *and since this is of finite type, we assume it is* $\operatorname{add}(C)$ *for an object C in $\mathcal{A}$.*

Proof. We first show that (1) M_G and (2) $M_{\operatorname{coker} f_C}$ as in the lemma lie in $\mathcal{P}(\mathcal{F})$:
(1) We show that f_G is an $\mathcal{E}$-deflation: Take $g\colon G_0 \twoheadrightarrow M_G$ be a $\mathcal{E}$-deflation with $\operatorname{add} G_0 \in \operatorname{add} G$. We may assume $G = G_0$. Then g factors over f, i.e. there exists an endomorphism $h \in \operatorname{End}(M_G)$ such that $g = fh$. By the obscure axiom h is an $\mathcal{E}$-deflation. It is an easy observation that then $\dim \operatorname{Hom}(G, \ker h) = 0$ and therefore h an isomorphism. This implies f is also an $\mathcal{E}$-deflation. In particular $\operatorname{Hom}(f, M)\colon \operatorname{Hom}(M_G, M) \to \operatorname{Hom}(G, M)$ is an isomorphism. This shows $M_G \in \mathcal{P}(\mathcal{F})$.
(2) The left $\operatorname{add}(M)$-approximation $f_C\colon C \to M_C$ is an $\mathcal{E}$-inflation, let $D := \operatorname{coker}(f_C)$. This implies we have a a left exact sequence $0 \to \operatorname{Ext}^1_{\mathcal{E}}(D, M) \to \operatorname{Ext}^1_{\mathcal{E}}(M_C, M) \to \operatorname{Ext}^1_{\mathcal{E}}(C, M)$. Now $f_D\colon D \to M_D$ together with the composition $M_C \to M_D$ and passing looking at the inclusion of the subfunctor $\operatorname{Ext}^1_{\mathcal{F}} \subseteq \operatorname{Ext}^1_{\mathcal{E}}|_{\mathcal{M}}$ we look at the following commutative diagramm

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_{\mathcal{E}}(D, M) \arrow[r] & \operatorname{Ext}^1_{\mathcal{E}}(M_C, M) \arrow[r] & \operatorname{Ext}^1_{\mathcal{E}}(C, M) \\
& \operatorname{Ext}^1_{\mathcal{E}}(M_D, M) \arrow[u] & & \\
& \operatorname{Ext}^1_{\mathcal{F}}(M_D, M) \arrow[u] \arrow[r] & \operatorname{Ext}^1_{\mathcal{F}}(M_C, M) \arrow[uu] &
\end{tikzcd}$$

Now, we claim $\operatorname{Ext}^1_{\mathcal{F}}(M_A, M) \to \operatorname{Ext}^1_{\mathcal{E}}(A, M)$ is injective for all objects A (then applied in the diagramm for $A = D$ and $A = C$ implies first that $\operatorname{Ext}^1_{\mathcal{F}}(M_D, M) \to \operatorname{Ext}^1_{\mathcal{F}}(M_C, M)$ in a monomorphism and then that $\operatorname{Ext}^1_{\mathcal{F}}(M_D, M) = 0$)).
Let us prove the claim: The map is given by pull-back $\mathcal{F}$-short exact sequences ending in M_A along the map f_A, assume the lower line pulls back to a split exact sequence. Then we find the dashed morphism $f\colon A \to M_1$ such that the triangle commutes

$$\begin{tikzcd}
& & A \arrow[dl, dashed] \arrow[d, "f_A"] \\
M \arrow[r, tail] & M_1 \arrow[r, two heads] & M_A
\end{tikzcd}$$

As $M_1 \in \mathcal{M}$ using that f_A is a left $\mathcal{M}$-approximation we find splitting $M_A \to M_1$ of the lower exact sequence. This show the injectivity.
Secondly, we need to see that $\mathcal{P}(\mathcal{F}) \subseteq \operatorname{add}(M_G \oplus M_D)$.
We claim (call this (*)): For every $\mathcal{E}$-exact sequence $(i, d)\colon X \rightarrowtail M_{G_1} \twoheadrightarrow Y$ with $G_2' \in \operatorname{add}(G)$ and $Y \in \operatorname{add}(M)$ we have i is a left $\operatorname{add}(M)$ approximation, i.e. we may assume $i = f_X$.
Before, we proof the claim, let us explain its consequence. Let $Q \in \mathcal{P}(\mathcal{F})$ and take an $\mathcal{E}$-deflation $q\colon G' \twoheadrightarrow Q$ with $G' \in \operatorname{add}(G)$. It factors over an $\mathcal{E}$-deflation $q'\colon M_{G'} \to Q$. Let $X = \ker q'$, then by claim (*), we have $Q = \operatorname{coker} f_X = M_{\operatorname{coker} f_X} \in \operatorname{add}(M_G \oplus M_D)$.
Proof of claim (*): Let $G = \bigoplus_{i \in I} G^i$ be a direct sum decomposition into indecomposables. We assume wlog $M_G = \bigoplus_{i \in I} M_{G^i}$. We use the horse-shoe lemma to produce split exact sequence $H_1 \rightarrowtail H_2 \twoheadrightarrow H_3$, $H_i \in \operatorname{add}(G)$ such that there is a morphism (p_1, p_2, p_3) to the short exact sequence (i, d) with all p_j $\mathcal{E}$-deflations. Then p_2, p_3 have to factor over f_{H_2}, f_{H_3} respectively and therefore also p_1 has to factor over f_{H_1} (all of these are $\mathcal{E}$-deflations). We find another split exact sequence $M_{H_1} \rightarrowtail M_{H_2} \twoheadrightarrow M_{H_3}$ such that there exists a morphism (h_1, h_2, h_3) of ses to (i, d) with all h_i are $\mathcal{E}$-deflations. As $M_{G_2'} \in \operatorname{add}(M_G)$ it follows that h_2 is a split epimorphism, i.e. we find another split exact sequence $M_{G_2} \rightarrowtail M_{H_2} \twoheadrightarrow M_{G_2'}$. Now, we define G_1 to be the largest common summand of the two summands H_1 and G_2 in H_2 and we set $G_1' = H_1/G_1$. As M_{G_1} is mapped under q_1 to zero, we get a commutative diagram

$$\begin{tikzcd} M_{G'_1} \arrow[r, hook, "split"'] \arrow[d, two heads, "q'_1"] & M_{G'_2} \arrow[d, "="] \\ X \arrow[r, "i"] & M_{G'_2} \end{tikzcd}$$

with q'_1 an $\mathcal{E}$-deflation. Right, now after all this affords we produced a split monomorphism $j\colon G'_1 \to G'_2$ with $r\colon G'_2 \to G'_1$, $rj = \mathrm{id}_{G'_1}$ such that we find a commutative diagram

$$\begin{tikzcd} G'_1 \arrow[r, hook, "j"'] \arrow[d, two heads, "g_1"] & G'_2 \arrow[d, two heads, "f_{G'_2}"] \\ X \arrow[r, "i"] & M_{G'_2} \end{tikzcd}$$

with g_1 also an $\mathcal{E}$-deflation. Then we look at a morphism $t\colon X \to M$ and we want to see it factors over i. We have there exists $m\colon M_{G'_2} \to M$ such that

$$tg_1 r = m f_{G'_2} \quad \Rightarrow \quad tg_1 = m f_{G'_2} j = m i g_1$$

and since g_1 is an epimorphism it follows $t = mi$. $\square$

Remark 6.2. Even in the representation-finite case: Exact subcategories can have more, less or equal number of indecomposable projectives to the exact category in which they are embedded into.

By a result of Enomoto, [6, Prop. 3.14, Cor. 3.15], every exact structure on $\mathcal{A}$ has enough projectives and enough injectives and is an Auslander-Reiten category.
Exact structures on $\mathcal{A}$ is the boolean lattice of generators. The lattice of all exact subcategories has for every additively closed subcategory $\mathcal{B}$ a boolean sublattice of all generators containing the generator constructed in the previous lemma (for $\mathcal{E} = \mathcal{E}_{max}$ the maximal exact structure on $\mathcal{A}$). The disjoint union of all these sublattices contains all exact subcategories in $\mathcal{A}$.
We also easily find:

Lemma 6.3. *Let $\mathcal{A}$ as through-out in this subsection. Then $\mathrm{EX}(\mathcal{A})$ is a finite poset, let $\mathcal{F}_i$ be an exact structure on $\mathcal{B}_i \subseteq \mathcal{A}$, $i = 1, 2$. Let $\mathcal{F}_1 \to \mathcal{F}_2$ be an arrow in the Hasse diagramm, then:*
(1) If the underlying additive categories are equal, then it is an arrow in the Boolean lattice corresponding to this subcategory.
(2) If they are not equal we have $|\mathcal{B}_1| < |\mathcal{B}_2|$ and $\mathcal{F}_1$ is the maximal exact structure making the inclusion $\mathcal{B}_1 \to \mathcal{F}_2$ exact and for all proper intermediate $\mathcal{B}_1 \subsetneq \mathcal{B} \subsetneq \mathcal{B}_2$, if $\mathcal{F}_{\mathcal{B}}$ is the maximal exact structure making the inclusion $\mathcal{B} \to \mathcal{F}_2$ exact then the inclusion $\mathcal{F}_1 \to \mathcal{F}_{\mathcal{B}}$ is not exact.

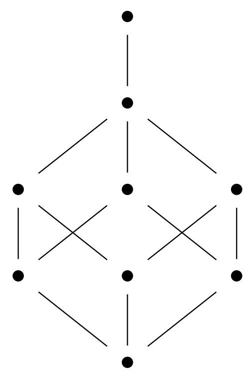
The proof is obvious.
With the previous two Lemmata we can theoretically compute these lattices. Instead we just look at the easiest non-trivial case and count how many objects we have (that already takes some time).

Exact structures on $\mathcal{A}$ is a graded poset by the map: $\mathcal{E} \mapsto |\mathcal{P}(\mathcal{E})| \in \mathbb{N}$ where the last one is the number of indecomposable projectives (up to isomorphism). Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$ and $E(\mathcal{E}) \in \{\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E}), \mathrm{ex}(\mathcal{E})\}$. As these lattices are notoriously big, we look instead at the following simple generating function

$$\mu_{E(\mathcal{E})}(X, Y, T) := \sum_{\mathcal{E}' \in E(\mathcal{E}), \mathrm{gldim}\, \mathcal{E}' < \infty} X^{|\mathcal{P}(\mathcal{E}')|} Y^{|\mathcal{E}'|} T^{\mathrm{gldim}\, \mathcal{E}'}$$

e.g. if we set a^{E}_{ijk} to be the multiplicity of $X^i Y^j T^k$. For example, let Q be a Dynkin quiver, then $a^{\mathrm{ext}(KQ\,\mathrm{mod})}_{nn0}$ with $n = |Q_0|$ is the number of basic tilting KQ-modules.
Now, let us look at the simplest cases.
Let $\mathcal{E} = \Lambda$-mod with $\Lambda = K(1 \to 2 \to 3 \to \cdots \to n)$ and let G be basic module given by the direct sum of all indecomposable non-projectives. As a lattice $\mathrm{ex}(\mathcal{E})$ is a cube given by all summands of G, the meet of $\mathrm{add}(G')$ and $\mathrm{add}(G'')$ is given by $\mathrm{add}(G' \oplus G'')$, and the join is given by $\mathrm{add}(G) \cap \mathrm{add}(G')$.

Example 6.4. $n = 2$, then $\mathrm{EX}(\mathcal{E})$ has 9 elements, the cube on the bottom is $\mathrm{ADD}(\mathcal{A})$ (i.e. all split exact structures) and the maximal element is the abelian structure on $\mathcal{E}$, i.e. the Hasse diagramm looks like



Example 6.5. $n = 3$, we have 8 generators, so $\mathrm{ex}(\mathcal{E})$ has 8 elements, we have $2^6 = 64$ additively closed subcategories in $\mathrm{ADD}(\mathcal{A})$ of whom 34 are extension-closed, then $\mathrm{Ex}(\mathcal{E})$ has 56 elements and $\mathrm{EX}(\mathcal{E})$ has 95.

$$\begin{aligned}
\mu_{\mathrm{ex}(\mathbb{A}_3)}(X,Y,T) &= Y^6[X^6 + (X^3 + 2X^4 + 3X^5)T + X^4T^2] \\
\mu_{\mathrm{ext}(\mathbb{A}_3)}(X,Y,T) &= 1 + 6(XY) + 10(XY)^2 + 5(XY)^3 \\
&\quad + [4Y^3X^2 + (5Y^4 + 2Y^5 + Y^6)X^3]T \\
\mu_{\mathrm{Ex}(\mathbb{A}_3)}(X,Y,T) &= 1 + 6(XY) + 10(XY)^2 + 9(XY)^3 + 5(XY)^4 + 2(XY)^5 + (XY)^6 \\
&\quad + [4Y^3X^2 + (5Y^4 + 2Y^5 + Y^6)X^3 + (4Y^5 + 2Y^6)X^4 + 3Y^6X^5]T \\
&\quad + Y^6X^4T^2 \\
\mu_{\mathrm{EX}(\mathbb{A}_3)}(X,Y,T) &= 1 + 6XY + 15(XY)^2 + 20(XY)^3 + 15(XY)^4 + 6(XY)^5 + (XY)^6 \\
&\quad + +[4Y^3X^2 + (9Y^4 + 2Y^5 + Y^6)X^3 + (8Y^5 + 2Y^6)X^4 + 3Y^6X^5]T \\
&\quad + (Y^4X^3 + Y^6X^4)T^2
\end{aligned}$$

Example 6.6. $n = 4$, $E(\mathcal{E}) = \mathrm{ex}(\mathcal{E})$ has $2^6 = 64$ elements. Then one can calculate

$$\begin{aligned}
\mu_{\mathrm{ex}(\mathbb{A}_4)}(X,Y,T) = Y^{10}[X^{10} &+ (X^4 + 3X^5 + 7X^6 + 14X^7 + 12X^8 + 6X^9)T \\
&+ (3X^5 + 7X^6 + 5X^7 + 3X^8)T^2 + (X^6 + X^7)T^3]
\end{aligned}$$

Observe, that we have 42 exact substructures of gldim = 1, this means the poset of hereditary exact substructures has 43 elements, which is substantially more then the 24 submodule-closed generators.

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