Exact categories represented by morphisms

1. Synopsis

For every module over a ring there is the associated module category over the endomorphism ring, i.e. a construction of a new module category over a ring.
This has the following generalization to exact categories: For an exact category $\mathcal{E}$ and a full subcategory $\mathcal{M}$ we look at all the $\mathcal{E}$-admissible morphisms S in $\mathcal{M}$ and define the category of S-presented functors $F\colon \mathcal{M}^{op} \to (\mathrm{Ab})$, this is a fully exact category of all finitely presented functors. Interestingly, we can also choose other classes of morphisms (for example deflations or inflations) and still obtain an exact category. Of course this topic deserves a systematic study that I can not give (due to time constraints).
We have already seen in Chapter 2, the Auslander exact category as an instance of this construction, we explain its generalizations which lead to the generality of a contravariantly finite generator $\mathcal{M}$ in an arbitrary exact category. We give a short history of ideas in the next section.
Furthermore, we have an exact category with enough projectives always presented by its admissible morphism between the projectives. Tilting subcategories are a generalization of the subcategory of projectives and we start using this construction in tilting theory for exact categories (cf. Chapter 10).
What is new? Functor categories represented by (general) classes of morphisms (in the literature you find either admissible morphisms or deflations). The generator correspondence for exact categories.

2. A short history of ideas

Of course, one would like to have an *endo-dictionary* translating properties into each other. Related to this is the question: Can one *reconstruct* the module category/exact category from this endomorphism ring/admissibly presented functor category?
This question has been answered in many different situations, we quickly survey the history of ideas here: We start recalling two result of M. Auslander.

THEOREM 2.1. ***(Auslander correspondence, 1971**, [2]) There exists a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras Λ and that of finite-dimensional algebras Γ with* $\operatorname{gldim}\Gamma \leq 2 \leq \operatorname{domdim}\Gamma$. *It is given by* $\Lambda \mapsto \Gamma = \operatorname{End}_\Lambda(M)$ *where* $\operatorname{mod}\Lambda = \operatorname{add}(M)$

This has further generalizations to (from special to more general, some predate Auslander's result)

(*) The higher Auslander correspondence [10]
(*) The Morita-Tachikawa correspondence [14], [16]
(*) The generator correspondence [15], [3]

They are all instances of faithfully balancedness which we explain by considering an assignment of Morita equivalence classes of pairs of rings and modules and the assignment

$$\mathbb{E}\colon [\Lambda, {}_\Lambda M] \mapsto [\Gamma = \operatorname{End}_\Lambda(M), {}_\Gamma M]$$

Then we call a module ${}_\Lambda M$ **faithfully balanced** if $\mathbb{E}^2[\Lambda, M] = [\Lambda, M]$. All *correspondences* (for rings and modules) using this assignment $\mathbb{E}$ are instances of faithfully balanced modules. As a feature,

$\mathrm{Hom}(-, M)$ then always gives a duality between certain subcategories of the module categories [**12**, Lem 2.9] and all dualities given by such a Hom-functor arise from faithfully balanced modules (e.g. Matlis duality - [**11**]). The best studied examples apart from (co)generators are (co)tilting (cf. [**13**]) in which the duality becomes the Theorem of Brenner and Butler ([**4**]). There are many more correspondences of faithfully balanced modules, we recommend to read the introduction in [**12**].
In loc. cit. we generalized faithfully balanced to Auslander-Solberg exact structures of finite type on f.d. module categories of f.d. algebras. In Chapter 5 we look at faithfully balancedness in functor categories (because we need some of the results in tilting theory for exact categories).
(It is unclear how general faithfully balancedness can be defined - but all ambient exact categories are with enough projectives and we think that having enough projectives will always be an assumption for the set-up.)
Then, secondly

Theorem 2.2. *(**Auslander's formula, 1966**, [**1**]) Let $\mathcal{C}$ be a small abelian category and $\mathrm{mod}_1\,\mathcal{C}$ the category of finitely presented additive functors $\mathcal{C}^{op} \to (Ab)$. Then this is an abelian category and there exists a left adjoint exact functor $L\colon \mathrm{mod}_1\,\mathcal{C} \to \mathcal{C}$ to the Yoneda embedding. Its kernel is a Serre subcategory, called the* effaceable functors*, $\ker L = \mathrm{eff}(\mathcal{C})$ and there is an induced equivalence*

$$\mathrm{mod}_1\,\mathcal{C}/\mathrm{eff}(\mathcal{C}) \to \mathcal{C}$$

This suggest a different way of reconstructing the category $\mathcal{C}$ from its module category $\mathrm{mod}_1\,\mathcal{C}$, namely as localization with respect to the subcategory of effaceable functors. We call this approach *reconstruction using Auslander's formula.* Using this the following has been generalized to arbitrary exact categories

(*) The Auslander correspondence [**9**]
(*) The higher Auslander correspondence [**7**]
(*) The Morita-Tachikawa correspondence [**8**]
(*) The generator correspondence, cf. Theorem 3.28

Nevertheless, the assignment considered in all cases is the following:
To an exact category $\mathcal{E}$ and an additively closed subcategory $\mathcal{M}$, we assign the category $\mathrm{mod}_S\,\mathcal{M}$ of additive functors $F\colon \mathcal{M}^{op} \to (Ab)$ such that there exists an $\mathcal{E}$-admissible morphism $s\colon M_1 \to M_0$, $M_i \in \mathcal{M}$ such that $F = \mathrm{coker}\,\mathrm{Hom}_{\mathcal{M}}(-, s)$. We write this as assignment of (exact equivalence classes of) pairs of exact categories together with a subcategory.

$$\mathbb{E}'\colon [\mathcal{E}, \mathcal{M}] \mapsto [\mathrm{mod}_S\,\mathcal{M}, \mathrm{eff}(\mathcal{M})]$$

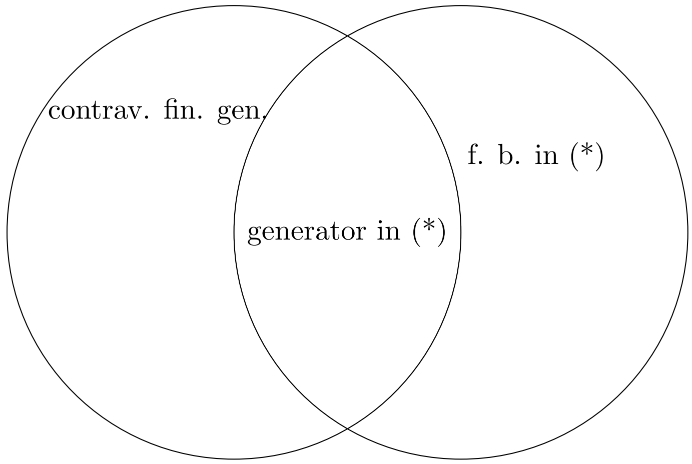
We will consider $\mathbb{E}'$ and $\mathbb{E}$ at least on the first entry as the same assignment.
The question is: "Can we find a localization sequence as in Auslanders formula which reconstructs $\mathcal{E}$ and $\mathcal{M}$ and therefore gives an inverse assignment to $\mathbb{E}'$?".

The obvious general open question

Open question 2.3. How does the first and the second type correspondence fit together? Both are crucially using adjoint pairs of functors, is there a joint generalization?

Let us look at an idempotent recollement on an endomorphism ring of a generator. In this situation we can study this generclosedator as a faithfully balanced module or we can look at the recollement and can recover the right hand side abelian category as a localization. Yet for other faithfully balanced modules this is not known to be true, so a description with an Auslander formula can not really be expected.

always small exact categories with subcategory $\mathcal{M}$:



where (*) is an exact category in which faithfully balanced (f.b.) is defined, so far only for Auslander-Solberg exact structures of finite type [**12**] and for categories of all additive functors $\operatorname{Mod}\mathcal{P}$ with $\mathcal{P}$ essentially small, cf. Chapter 5.

3. Presentations of exact categories

We start with a study of *exact* categories of the form $\operatorname{mod}_S \mathcal{M}$ (called exactly presented by $(\mathcal{M}, S)$) for some class of morphisms S in the first section. Then we translate properties of S into properties of $\operatorname{mod}_S \mathcal{M}$. The most important: S has weak kernels in S translates into $\operatorname{mod}_S \mathcal{M}$ is has enough projectives given by the presentables.

Definition 3.1. Given an additive category $\mathcal{M}$ and a class of morphisms S in $\mathcal{M}$. We say it is **closed under homotopy** if for two morphisms s, t in $\mathcal{M}$ with $\operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, s) \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, t)$ in $\operatorname{mod}_1 \mathcal{M}$ we have $s \in S$ if and only if $t \in S$.

Being homotopy-closed is often useful, such as:

Lemma 3.2. *If S is closed under homotopy and direct sums and summands of morphisms (i.e. $s, t \in S$ iff $s \oplus t \in S$) then $\operatorname{mod}_S \mathcal{M}$ is an additively closed subcategory in $\operatorname{Mod}\mathcal{M}$.*

The proof is obvious.

Remark 3.3. If S is homotopy closed then it contains all split epimorphisms. If every representable $\operatorname{Hom}(-, M)$ is of the form $\operatorname{coker}\operatorname{Hom}(-, s)$ for some $s \in S$ and if S is homotopy closed then all split admissible morphisms are contained in it.

Lemma 3.4. *If $S \subseteq \operatorname{Mor}\mathcal{M}$ is a class of morphisms which is closed under direct sums and summands. If S contains all split epimorphisms then S is homotopy-closed.*

Proof. Assume $F = \operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, s) = \operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, t)$. We look at the projective presentations of F in $\operatorname{Mod}\mathcal{M}$, say $s\colon M_1 \to M_0$, $t\colon N_1 \to N_0$. Then we have $s \oplus \mathrm{id}_{N_0} \oplus (N_1 \to 0) \cong t \oplus \mathrm{id}_{M_0} \oplus (M_1 \to 0)$ and by assumption $s \in S$ if and only if $t \in S$. □

Definition 3.5. Let $\mathcal{M}$ be an additive category and S a class of morphisms in $\mathcal{M}$. We define the **category of S-represented $\mathcal{M}$-modules** $\operatorname{mod}_S \mathcal{M}$ to be the full subcategory of $\operatorname{Mod}\mathcal{M}$ consisting of the functors $F\colon \mathcal{M}^{op} \to (\mathrm{Ab})$ such that there exists an exact sequence

$$\operatorname{Hom}(-, M_1) \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}(-, M_0) \to F \to 0$$

Definition 3.6. We say that a class of morphisms S on an additive category $\mathcal{M}$ (or the pair $(\mathcal{M}, S)$) is an **exact presentation** if $\mathcal{E} = \operatorname{mod}_S \mathcal{M}$ is an extension-closed subcategory in $\operatorname{Mod} \mathcal{M}$ (we will always equip it with this exact structure). In this case, we also say the exact category $\mathcal{E}$ is **exactly represented** by $(\mathcal{M}, S)$.

We will from now on assume that S is homotopy-closed and closed under direct sums.

Example 3.7. Exactly presented exact categories are precisely all fully exact subcategories of $\operatorname{mod}_1 \mathcal{M}$ for some additive category $\mathcal{M}$. If $\mathcal{E}$ is a fully exact subcategory of $\operatorname{mod}_1 \mathcal{M}$, take S to be the class of all morphisms s in $\mathcal{M}$ such that $\operatorname{coker} \operatorname{Hom}_{\mathcal{M}}(-, s)$ lies in $\mathcal{E}$, then obviously $\mathcal{E} = \operatorname{mod}_S \mathcal{M}$.

Definition 3.8. We say an exact category $\mathcal{E}$ is **projectively determined** (by $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{E})$) if a kernel-cokernel pair (i, p) is an exact sequence in $\mathcal{E}$ if and only if $(\operatorname{Hom}(Q, i), \operatorname{Hom}(Q, p))$ is an exact sequence of abelian groups for all $Q \in \mathcal{Q}$.
More generally, given an exact subcategory $i \colon \mathcal{E}' \subseteq \mathcal{E}$, we say that $\mathcal{E}'$ is **projectively determined** by $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{E}')$ **inside** $\mathcal{E}$ if: An $\mathcal{E}$-exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ with X, Y, Z in $\mathcal{E}'$ is $\mathcal{E}'$-exact if and only if $(\operatorname{Hom}(Q, i), \operatorname{Hom}(Q, p))$ is an exact sequence of abelian groups for all $Q \in \mathcal{Q}$.

In particular if $\mathcal{E}$ is projectively determined by $\mathcal{Q}$ than it is projectively determined by $\mathcal{Q}$ inside the maximal exact structure on the underlying additive category.

Example 3.9. Given an exact category $\mathcal{E}$ with enough projectives then $\mathcal{E}$ is projectively determined.

Example 3.10. Let $\mathcal{E}$ be an exact category. Every additive subcategory $\mathcal{M}$ of $\mathcal{E}$ gives an exact substructure $\mathcal{E}_{\mathcal{M}} \leq \mathcal{E}$ defined as follows: A $\mathcal{E}$-exact sequence (i, p) is an exact sequence in $\mathcal{E}_{\mathcal{M}}$ if and only if $(\operatorname{Hom}(Q, i), \operatorname{Hom}(Q, p))$ is an exact sequence of abelian groups for all $Q \in \mathcal{M}$.
These exact substructures are called Auslander-Solberg structures. By definition they are projectively determined by $\mathcal{M}$ inside $\mathcal{E}$. In particular, if $\mathcal{E}$ is projectively determined by $\mathcal{Q}$, then $\mathcal{E} = \mathcal{E}^{max}_{\mathcal{Q}}$ is the Auslander-Solberg structure given by $\mathcal{Q}$ of the maximal exact structure on the underlying additive category.

Example 3.11. Given an exact category $\mathcal{E}$ which is projectively determined by $\mathcal{Q}$. If $\mathcal{E}'$ is a fully exact subcategory which also contains $\mathcal{Q}$ then $\mathcal{E}'$ is also projectively declosedtermined by $\mathcal{Q}$ and also projectively determined by $\mathcal{Q}$ inside $\mathcal{E}$.

Definition 3.12. Let S be a class of morphisms in an additive category $\mathcal{M}$, then we say S has **weak kernels** in S if for every morphism $s \colon M \to N$ in S there exists another morphism $t \colon L \to M$ such that

$$\operatorname{Hom}_{\mathcal{M}}(-, L) \xrightarrow{\operatorname{Hom}(-,t)} \operatorname{Hom}_{\mathcal{M}}(-, M) \xrightarrow{\operatorname{Hom}(-,s)} \operatorname{Hom}_{\mathcal{M}}(-, N)$$

is exact in the middle (in $\operatorname{Mod} \mathcal{M}$).
Dually weak cokernels in S if S^{op} has weak kernels in S^{op}.

Lemma 3.13. *Let S be a class of morphisms closed under isomorphism, direct sums and summands and contains all split admissible morphisms. Let $\mathcal{E}$ be exactly presented by $(\mathcal{M}, S)$. Let $\widetilde{\mathcal{M}} = \{\operatorname{Hom}(-, M) \in \operatorname{mod}_1 \mathcal{M} \colon M \in \mathcal{M}\}$ and $\widetilde{S} := \{\operatorname{Hom}(-, s) \mid s \in S\}$.*
The following are equivalent

(a) *S has weak kernels in S*
(b) *$\widetilde{S}$ equals all $\mathcal{E}$-admissible morphisms in $\widetilde{\mathcal{M}}$*
(c) *$\mathcal{E}$ has enough projectives given by* $\operatorname{add}(\widetilde{\mathcal{M}})$.

Proof. Very easy. We leave it to the reader. $\square$

Definition 3.14. We call a class of morphisms S in a category $\mathcal{M}$ **suitable** if $(\mathcal{M}, S)$ is an exact presentation and S has weak kernels in S and is closed under homotopy.
In this case we say that the exact category $\operatorname{mod}_S \mathcal{M}$ is **suitably presented** by $(\mathcal{M}, S)$.

We also observe that $\widetilde{\mathcal{M}} \subseteq \mathcal{E}$ implies $\mathcal{E}$ is projectively determined by $\widetilde{\mathcal{M}}$ (as $\operatorname{Mod}\mathcal{M}$ fulfills this and $\mathcal{E}$ is fully exact in it).

Lemma 3.15. *Let $\mathcal{E}$ be an exact category and $\mathcal{M}$ a full additively closed subcategory. Let S be either*

(1) *S_{adm} all $\mathcal{E}$-admissible morphisms in $\mathcal{M}$, or*
(2) *S_{infl} all $\mathcal{E}$-inflations in $\mathcal{M}$, or*
(3) *S_{defl} all $\mathcal{E}$-deflations in $\mathcal{M}$,*

then S is an exact presentation. If the ambient exact category $\mathcal{E}$ is clear, we will use the following notation:

$$(1)\ \operatorname{mod}_{\text{adm}}\mathcal{M} = \operatorname{mod}_{S_{\text{adm}}}\mathcal{M}, \quad (2)\ \mathrm{H}_{\mathcal{M}} := \operatorname{mod}_{S_{\text{infl}}}\mathcal{M}, \quad (3)\ \operatorname{eff}_{\mathcal{M}} = \operatorname{mod}_{\mathrm{S}_{\text{defl}}}\mathcal{M}$$

Proof. (1) The proof is an easy adaptation of [9, Prop. 3.5].
(2), (3) We just need to see that $H_{\mathcal{M}}$ and $\operatorname{eff}_{\mathcal{M}}$ are extension-closed in $\operatorname{mod}_{\text{adm}}\mathcal{M}$ but this follows from the horseshoe lemma and [5, Cor. 3.2]. □

Corollary 3.16. *If $\mathcal{E}$ is an exact category with enough projectives $\mathcal{P}$. Then*

$$\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_{\text{adm}}\mathcal{P}, \quad E \mapsto \operatorname{Hom}(-,E)|_{\mathcal{P}}$$

is an exact equivalence (i.e. equivalence which is an exact functor and its quasi-inverse is also an exact functor). If $\mathcal{P}$ is idempotent complete, then $\operatorname{mod}_{\text{adm}}\mathcal{P}$ is a resolving subcategory in $\operatorname{mod}_{\infty}\mathcal{P}$.

Now, we need the following observation:

Lemma 3.17. *([6, Lemma 21, 22]) Let $\mathcal{E}$ be an exact category.*

(a) *$\mathcal{E}$-inflations is closed under direct summands iff*
$\mathcal{E}$ is weakly idempotent complete iff
$\mathcal{E}$-deflations are closed under direct summands.
(b) *$\mathcal{E}$-admissible morphisms are closed under direct summands if and only if $\mathcal{E}$ is idempotent complete.*

This can be used to:

Example 3.18. If $\mathcal{M} \subseteq \mathcal{E}$ is a contravariantly finite generating subcategory and S_{adm} the class of admissible morphisms on $\mathcal{M}$. Then S_{adm} has weak kernels in S_{adm} and $\operatorname{mod}_{\text{adm}}\mathcal{M}$ has enough projectives given by $\operatorname{add}(\widetilde{\mathcal{M}})$. We have that the functor $E \mapsto \operatorname{Hom}(-,E)|_{\mathcal{M}}$ restricts to a fully faithful functor $\Phi\colon \mathcal{E} \to \operatorname{mod}_{\text{adm}}\mathcal{M}$.
By the previous Lemma and Lemma 3.4: If $\mathcal{E}$ is idempotent complete then $(\mathcal{M}, S_{\text{adm}})$ is a suitable presentation.

Example 3.19. If the exact structure of $\mathcal{E}$ restricts to $\mathcal{M}$ to an abelian structure, then every $\mathcal{E}$-admissible has a kernel in $\mathcal{M}$ which is given by an $\mathcal{E}$-inflation and so the class of $\mathcal{E}$-admissible morphism on $\mathcal{M}$ coincides with all morphisms in $\mathcal{M}$ and this is suitable (this presents the abelian category $\operatorname{mod}_1\mathcal{M}$).

Here is a little warning.

Remark 3.20. Given a fully exact subcategory $\mathcal{F} \subseteq \mathcal{E}$, then there might be many more $\mathcal{E}$-admissible morphisms on $\mathcal{F}$ than there are $\mathcal{F}$-admissible ones. Even if $\mathcal{F}$ is homologically exact- just consider the case above: $\mathcal{F} = \mathcal{P}(\mathcal{E})$ is semi-simple, only projections onto summands are $\mathcal{F}$-admissible.

Remark 3.21. Let us summarize the discussion from before:
We have 1) ⇒ 2) ⇒ 3) ⇒ 4) with

1) Exact categories with enough projectives

2) Exactly presented categories, projectively determined by $\widetilde{\mathcal{M}}$
3) Exactly presented categories
4) Exact categories

We do not know a small exact category which is not exactly presented.
An example fulfilling 3) but not 2) is given by categories of effaceable functors on an exact category.
An example fulfilling 2) but not 1) is given by $\operatorname{mod}_1 \mathcal{M}$ (i.e. finitely presented additive functors in $\operatorname{Mod} \mathcal{M}$) where $\mathcal{M}$ does not have weak kernels (then this category has not enough projectives).

3.0.1. *Universal Property.* Now, we take $\mathcal{F}$ exactly presented by $(\mathcal{M}, S)$ and an additive functor $f \colon \mathcal{M} \to \mathcal{B}$ such that $f(s)$ has a cokernel for all $s \in S$. Then we can define a functor

$$\overline{f} \colon \mathcal{F} \to \mathcal{A}, \quad \overline{f}(\operatorname{coker} \operatorname{Hom}(-, s)) = \operatorname{coker} f(s)$$

Lemma 3.22. *In the above situation*

(1) *Assume that $\mathcal{A}$ is weakly idempotent complete. If $f \colon \mathcal{M} \to \mathcal{E}$ with $\mathcal{E}$ an exact structure on $\mathcal{A}$ and if $f(s)$ admissible for all $s \in S$, the $\overline{f} \colon \mathcal{F} \to \mathcal{E}$ is right exact.*
(2) *In the situation of (1); If all morphisms in $s \in S$ there exist a weak kernel $t \in S$ such that $(f(t), f(s))$ is exact in the middle then $\overline{f} \colon \mathcal{F} \to \mathcal{E}$ is exact.*

Proof. (1) Assume $F_1 \rightarrowtail F_2 \twoheadrightarrow F_3$ is exact in $\mathcal{F}$. We pick $s_i \in S$ such that $\operatorname{coker} \operatorname{Hom}_{\mathcal{M}}(-, s_i) = F_i$, $i = 1, 2, 3$. We now consider the $\operatorname{Hom}(-, s_i)$ as projective presentations of F_i in $\operatorname{Mod} \mathcal{M}$. We can assume by the horseshoe Lemma (using that we assume S is homotopy-closed) that these projective presentations are degree-wise split. As f is an additive functor, we obtain that $(f(s_1), f(s_2), f(s_3))$ is a morphism of two split exact sequences in $\mathcal{E}$. As $\mathcal{A}$ is weakly idempotent complete and $f(s_i)$ are $\mathcal{E}$-admissible we can apply the snake lemma and we obtain a right exact sequence $\overline{f}(F_1) \to \overline{f}(F_2) \twoheadrightarrow \overline{f}(F_3)$ on the cokernels.
(2) We repeat the same steps as in (1) but now with one longer projective presentations. The exactness of the outer sequences implies the exactness of the middle sequence in $\mathcal{F}$. Then apply the snake lemma.

□

Lemma 3.23. ***(Universal property)*** *Let $\mathcal{F}$ be suitably presented by $(\mathcal{M}, S)$. For every functor $f \colon \mathcal{M} \to \mathcal{E}$ which maps S to $\mathcal{E}$-admissible morphisms, the right exact functor $\overline{f} \colon \mathcal{F} \to \mathcal{E}$ with $\overline{f}(\operatorname{coker}(\operatorname{Hom}_{\mathcal{M}}(-, s))) = \operatorname{coker} f(s)$ for all $s \in S$ is (up to isomorphism of functors) the unique right exact functor F with $F \circ \mathbb{Y} = f$ where $\mathbb{Y} \colon \mathcal{M} \to \operatorname{mod}_{\mathrm{adm}} \mathcal{M}$, $M \mapsto \operatorname{Hom}_{\mathcal{M}}(-, M)$ is the Yoneda embedding.*

Proof. Let F be a right exact functor with $F \circ \mathbb{Y} = f$. Then as F is right exact and $\operatorname{Hom}(-, s)$ admissible for all $s \in S$ (because $\mathcal{F}$ is suitably presented). It follows that $F(\operatorname{coker} \operatorname{Hom}(-, s)) = \operatorname{coker} f(s) = \overline{f}(\operatorname{coker} \operatorname{Hom}(-, s))$. □

Remark 3.24. Observe that we do not need that $\mathcal{F}$ is suitably presented by $(\mathcal{M}, S)$ to show that $\overline{f}$ is right exact. But for the unique characterization we assume (even though it is a bit stronger than necessary).

3.1. Results for admissible morphisms. Let us come to the Yoneda embedding, recall from Lemma 3.13.

Remark 3.25. Let $\mathcal{M}$ be an additively closed subcategory in an idempotent complete exact category $\mathcal{E}$, then the Yoneda embedding $\mathbb{Y} \colon \mathcal{M} \to \operatorname{mod}_{\mathrm{adm}} \mathcal{M}$ reflects admissibility if and only if S_{adm} is suitable.

Definition 3.26. Let $\mathcal{E}$ be an exact category and $\mathcal{M} \subseteq \mathcal{E}$ be an additively closed subcategory and we denote by S the class of $\mathcal{E}$-admissible morphisms in $\mathcal{M}$. Then we define the following full subcategory of $\operatorname{mod}_S \mathcal{M}$

$$\operatorname{eff}_{\mathcal{M}} = \{F = \operatorname{coker} \operatorname{Hom}_{\mathcal{E}}(-, d)|_{\mathcal{M}} \mid d \text{ deflation } \}$$

and call this the subcategory of $\mathcal{M}$-effaceable functors.
We say that $\mathcal{M}$ **satisfies the Auslander formular** if $\operatorname{eff}_{\mathcal{M}}$ is a two-sided percolating subcategory (definition of [9]) and the quotient $\operatorname{mod}_S \mathcal{M}/\operatorname{eff}_{\mathcal{M}}$ is equivalent as an exact category to $\mathcal{E}$.

Lemma 3.27. *Let $\mathcal{M}$ be a contravariantly finite generator in $\mathcal{E}$, let S_{adm} be all $\mathcal{E}$-admissible morphisms in $\mathcal{M}$. Let $\operatorname{inc}\colon \mathcal{M} \subseteq \mathcal{E}$ be the inclusion functor and $L = \overline{\operatorname{inc}}\colon \operatorname{mod}_{\text{adm}} \mathcal{M} \to \mathcal{E}$ defined by $L(\operatorname{coker} \operatorname{Hom}_{\mathcal{M}}(-, s)) = \operatorname{coker} s$. We denote by $\Phi\colon \mathcal{E} \to \operatorname{mod}_{\text{adm}} \mathcal{M}$ the functor $\Phi(E) = \operatorname{Hom}_{\mathcal{A}}(-, E)|_{\mathcal{M}}$.*

(i) *L is exact, Φ is left exact and $\ker L = \operatorname{eff}_{\mathcal{M}}$*
(ii) *(L, Φ) are an adjoint pair, Φ is fully faithful, L is essentially surjective and $L \circ \Phi \cong \operatorname{id}_{\mathcal{E}}$*
(iii) *The subcategory $\operatorname{eff}_{\mathcal{M}}$ is percolating in $\operatorname{mod}_{\text{adm}} \mathcal{M}$ and the functor L factors over an equivalence of exact categories*

$$L'\colon \operatorname{mod}_{\text{adm}} \mathcal{M}/\operatorname{eff}_{\mathcal{M}} \to \mathcal{E}$$

Proof. (i) To see that L is exact we observe that we can find weak kernel of morphisms in $\mathcal{M}$ which give middle exact sequences and so Lemma 3.22 (2) applies. Clearly, for an $s \in S_{\text{adm}}$, we have: $L(\operatorname{coker} \operatorname{Hom}(-, s)) = \operatorname{coker}(s) = 0$ if and only if s is an $\mathcal{E}$-deflation.

We need to see that Φ maps deflations to admissible morphisms. So given an exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ we have a left exact sequence of functors $0 \to \Phi(X) \to \Phi(Y) \to \Phi(Z)$. So we need to see that if a is an inflation or a deflation then $\operatorname{coker} \Phi(a)$ in $\operatorname{Mod} \mathcal{M}$ is already in $\operatorname{mod}_{\text{adm}} \mathcal{M}$. Let $a\colon X \to Y$, let $p\colon M_Y \twoheadrightarrow Y$ be a right $\mathcal{M}$-approximation, we pull back p along a, i.e. we have a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "b"] \arrow[d, two heads] & M_Y \arrow[d, two heads, "p"] \\
X \arrow[r, "a"'] & Y
\end{tikzcd}$$

We claim in both cases (a inflation or deflation) we have a commutative diagram with right exact rows

$$\begin{tikzcd}
\Phi(R) \arrow[r, "\Phi(b)"] \arrow[d, two heads] & \Phi(M_Y) \arrow[r, two heads] \arrow[d, two heads, "p"] & F \arrow[d, "="] \\
\Phi(X) \arrow[r, "\Phi(a)"'] & \Phi(Y) \arrow[r, two heads] & F
\end{tikzcd}$$

Then let $r\colon M_R \twoheadrightarrow R$ be a right $\mathcal{M}$-approximation, it follows $br \in S_{\text{adm}}$ and $F = \operatorname{coker} \Phi(br)$.

(ii) As $\mathcal{M}$ is a contravariantly finite generator, L is surjective and Φ is well-defined and fully faithful. Furthermore, we have $L \circ \mathbb{Y} = \operatorname{inc}\colon \mathcal{M} \to \mathcal{E}$ and $\Phi|_{\mathcal{M}} = \mathbb{Y}$. For E in $\mathcal{E}$ we find an $\mathcal{E}$-admissible $s\colon M_1 \to M_0$ in S_{adm} such that $E = \operatorname{coker}(s)$ and $M_0 \to E$, $M_1 \to \operatorname{Im} s$ are $\mathcal{M}$-approximations, this implies that we have a right exact sequence $\Phi(M_1) \xrightarrow{\Phi(s)} \Phi(M_0) \twoheadrightarrow \Phi(E)$, the apply L to conclude $L\Phi(E) \cong E$.

For the adjunction, let $F \in \operatorname{mod}_{\text{adm}} \mathcal{M}$ and $E \in \mathcal{E}$, we claim $\operatorname{Hom}_{\operatorname{mod}_{\text{adm}} \mathcal{M}}(F, \Phi(E)) \cong \operatorname{Hom}_{\mathcal{E}}(L(F), E)$.

Choose $s \in S_{\text{adm}}$ such that $\Phi(M_1) \xrightarrow{\Phi(s)} \Phi(M_0) \twoheadrightarrow F$ is exact in $\operatorname{mod}_{\text{adm}} \mathcal{M}$. Now given a morphism $F \to \Phi(E)$, then the composition $\Phi(M_0) \to \Phi(M_1) \to \Phi(E)$ is zero. As Φ is fully faithful, there is a unique morphism $L(F) = \operatorname{coker}(s) \to E$. Conversely, as F is the cokernel of $\Phi(s)$, we have a unique morphism $c\colon F \to \Phi(\operatorname{coker}(s)) = \Phi L(F)$, so given an $a\colon L(F) \to E$ we just map it to $\Phi(a)c$.

(iii) The argument that $\operatorname{eff}_{\mathcal{M}}$ is percolating is just the observation that [9, Prop 3.6, 3.17] generalize to this set-up. By [9, Thm 2.12] we have the induced functor L' such that $L' \circ Q = L$. As Q, L are exact, the same is true for L'. By the same argument as in [9, Thm3.11], L' is an additive equivalence. As $L\Phi = \mathrm{id}_{\mathcal{E}}$ and $L = L'Q$, we see that $Q\Phi$ is the quasi-inverse of L'. We want to see that $Q\Phi$ is exact, as it is already left exact, it is enough to show that it preserves deflations. Let $f\colon X \twoheadrightarrow Y$ be an $\mathcal{E}$-deflation. We take a right $\mathcal{M}$-approximation $M_Y \twoheadrightarrow Y$ and pull back f along it, to an object Z. Then we take the $\mathcal{M}$-approximation of $M_Z \twoheadrightarrow Z$. Now, we have a commutative diagramm

$$\begin{tikzcd} M_Z \arrow[r, two heads] \arrow[d, two heads] & M_Y \arrow[d, two heads] \\ X \arrow[r, two heads] & Y \end{tikzcd}$$

with all arrows are $\mathcal{E}$-deflations. Now, Φ maps the right $\mathcal{M}$-approximations to deflations and $Q\Phi$ maps the $\mathcal{E}$-deflation $M_Z \to M_Y$ to one in $\operatorname{mod}_{\mathrm{adm}} \mathcal{M}/\operatorname{eff}_{\mathcal{M}}$. This implies by the diagonal $M_Z \to Y$ is mapped under $Q\Phi$ to a deflation and then by the obscure axiom it follows that $Q\Phi(f)$ is a deflation.

□

We look at the assignment of (exact equivalence classes of) pairs of exact categories together with a subcategory.

$$\mathbb{E}'\colon [\mathcal{E}, \mathcal{M}] \mapsto [\operatorname{mod}_S \mathcal{M}, \operatorname{eff}(\mathcal{M})]$$

As a reformulation we of (3) and (4) one gets:

THEOREM 3.28. *(**Generator correspondence for exact categories**) The assignment $\mathbb{E}'$ gives a bijection between*

(1) *Pairs of an exact category $\mathcal{E}$ together with a contravariantly finite generator $\mathcal{M}$*
(2) *Pairs of an exact category $\mathcal{F}$ with a percolating subcategory* eff *satisfying*
  (i) *$\mathcal{F}$ has enough projectives $\mathcal{P}$*
  (ii) $\operatorname{eff} \subseteq {}^{\perp}\mathcal{P} \quad (= \{X \in \mathcal{F} \mid \operatorname{Hom}(X, P) = 0 \; \forall P \in \mathcal{P}\})$ *and* eff *is a torsion class*
  (iii) $\operatorname{Ext}^1(\operatorname{eff}, \mathcal{P}) = 0$

PROOF. First of all, we need to see that $\mathbb{E}'$ is well-defined, by Lemma 3.27, (3) we have that $\operatorname{eff}_{\mathcal{M}}$ is percolating and by Ex. 3.18 we have that $\mathcal{F} = \operatorname{mod}_S \mathcal{M}$ has enough projectives. For the properties (ii) and (iii), we leave the reader to check that the proofs of [9, Prop. 3.6, Prop. 3.17 (3)] generalize to this more general situation.
Define $\mathbb{F}[\mathcal{F}, \operatorname{eff}] := [\mathcal{E} = \mathcal{F}/\operatorname{eff}, Q(\mathcal{P}(\mathcal{F}))]$ where $Q\colon \mathcal{F} \to \mathcal{F}/\operatorname{eff}$ is the localization with respect to the percolating subcategory and $\mathcal{P} := \mathcal{P}(\mathcal{F})$ are the projectives in $\mathcal{F}$. We first show: Condition (ii) and (iii) ensure that we always have that $\mathcal{P} \subseteq \mathcal{F} \xrightarrow{Q} \mathcal{E}$ is fully faithful.
A morphism $P' \to P$ in $\mathcal{E}$ with $P, P' \in \mathcal{P}$ is given by an equivalence class of pairs $[f, s]$ with $f\colon X \to P$ and $s\colon X \to P'$ is $\mathcal{F}$-admissible such that $\ker(s), \operatorname{coker}(s) \in \operatorname{eff}$.
First we use that we have a torsion pair $(\operatorname{eff}, \mathcal{G})$ and therefore we find an exact sequence $E \rightarrowtail X \twoheadrightarrow G$ with $E \in \operatorname{eff}$, $G \in \mathcal{G}$. As $\operatorname{eff} \subseteq {}^{\perp}\mathcal{P}$, we find a morphism $g\colon G \to P$ such that $f\colon X \twoheadrightarrow G \to P$. By definition $[f, s] = [g, i]$ where $i\colon G \rightarrowtail P'$ is the induced inflation, observe that $\operatorname{coker}(i) = \operatorname{coker}(s) =: E' \in \operatorname{eff}$. So we look at the short exact sequence $G \rightarrowtail P' \twoheadrightarrow E'$ and apply $\operatorname{Hom}_{\mathcal{A}}(-, P)$. Using also (iii) we conclude that $\operatorname{Hom}_{\mathcal{A}}(P', P) \cong \operatorname{Hom}_{\mathcal{A}}(G, P)$, this means that $g\colon G \to P$ factors over $i\colon P' \to P$ uniquely. This implies the functor is full and also faithful because assume that a morphism $p\colon P' \to P$ fulfills $Q(p) = 0$, then there exists some $s\colon X \to P'$ with $\ker(s), \operatorname{coker}(s)$ in eff such that $ps = 0$, now, as s is admissible we have $pi = 0$ with $i\colon X/\operatorname{coker}(s) =: G \rightarrowtail P'$. But now $pi\colon G \to P$ is in the image of the isomorphism from before and therefore $p = 0$.

As localization with respect to percolating subcategory reflects admissibility (cf. [9, Thm 2.16] we get: $\mathcal{F}$-admissible morphism in $\mathcal{P}$ are precisely $\mathcal{E}$-admissible morphisms in $Q(\mathcal{P}) =: \mathcal{M}$. This bijection restricts to the following two classes:

(a) $\mathcal{F}$-admissible morphisms $p\colon P' \to P$ such that $\operatorname{coker}(p) \in \operatorname{eff}$
(b) $\mathcal{E}$-deflations $P' \to P$ with P, P' in $\mathcal{M}$

That is clear by definition as the bijection follows from applying Q and Q is an exact functor. Therefore we conclude that $\mathcal{F} = \operatorname{mod}_{\mathcal{F}-adm} \mathcal{P} \cong \operatorname{mod}_S \mathcal{M}$ where S are the $\mathcal{E}$-admissible morphisms in $\mathcal{M}$. Under this equivalence, we have the objects in eff, i.e. the objects represented by morphisms in class (a) are mapped to the objects represented by morphisms in $\mathcal{M}$ in class (b) which is the category $\operatorname{eff}_{\mathcal{M}}$. This shows $\mathbb{E}' \circ \mathbb{F}$ is the identity.
Now, we look at $\mathbb{F} \circ \mathbb{E}'$. By the previous Lemma we have $\mathcal{E} \cong \operatorname{mod}_S \mathcal{A}/\operatorname{eff}_{\mathcal{M}}$. We just need to see that $\mathcal{M} \cong Q(\mathcal{P}(\operatorname{mod}_S \mathcal{A}/\operatorname{eff}_{\mathcal{M}}))$. We define $\mathcal{F} := \operatorname{mod}_S \mathcal{A}$ and $\mathcal{P} = \mathcal{P}(\mathcal{F})$. As the S are suitable morphisms for $\mathcal{M}$ (cf. Example 3.18) we have that the Yoneda embedding $\mathbb{Y}\colon \mathcal{M} \to \mathcal{F}$ identifies $\mathcal{M}$ with the projectives $\mathcal{P}$ and S with the $\mathcal{F}$-admissible morphisms between the projectives. But as the composition $\mathcal{P} \subseteq \mathcal{F} \xrightarrow{Q} \mathcal{E}$ is fully faithful (see before), the claim follows. $\square$

All other instances of $\mathbb{E}'$ in the history of idea section are specializations of the generator correspondence.

Open question 3.29. If $\mathcal{M}$ is a contravariantly finite generator and $\mathcal{E}$ has enough projectives, do we have an adjoint triple $(j_!, L, \Phi)$ defining the right half of a recollement of exact categories (cf. in [9], this exists for $\mathcal{M} = \mathcal{E}$). Then this should be used to find the right definition of faithfully balancedness in this situation.

3.2. Using other morphisms. In joint work in progress (with Janina Letz, Marianne Lawson) we conjecture the following:
Let $\mathcal{E} = (\mathcal{A}, S)$ be an exact category. Let S_m be one of the following

(1) Let S_m be the $\mathcal{F}$-inflations for a supstructure $\mathcal{E} \leq \mathcal{F}$ on $\mathcal{A}$
(2) Let S_m be the class of all $\mathcal{A}$-monomorphisms.

Then we define $S \subseteq \operatorname{Mor}(\mathcal{A})$ to be the class of all morphisms s which factor as $s = ip$ with i in S_m and p an $\mathcal{E}$-deflation.
As all morphisms in S have a kernel in $\mathcal{A}$ which is an $\mathcal{E}$-inflation and therefore in S again: In particular S has weak kernels. As S contains all split admissible morphisms, by Lemma 3.4, we conclude that S is closed under homotopy. It is also straightforward to see that the proof in [9, Prop. 3.5] generalizes to show that $\operatorname{mod}_S \mathcal{A}$ is extension-closed in $\operatorname{mod}_1 \mathcal{A}$. This means we have S is suitable.

Conjecture 3.30. *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category and S be a class of morphisms just described.*

(1) $\operatorname{eff}(\mathcal{E})$ *is a percolating subcategory in* $\operatorname{mod}_S \mathcal{A}$
(2) $\mathcal{E} \cong \operatorname{mod}_{\mathrm{adm}} \mathcal{A}/\operatorname{eff}(\mathcal{E}) \to \operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E})$ *is a fully exact subcategory.*

We look at the commutative diagramm

$$\begin{tikzcd} \operatorname{mod}_S \mathcal{A} \arrow[r, "i"] \arrow[d] & K^b(\mathcal{A}) \arrow[d] \\ \operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E}) \arrow[r] & D^b(\mathcal{E}) \end{tikzcd}$$

where the functor i is maps $F = \operatorname{coker} \operatorname{Hom}(-, s)$ to the 3-term complex with $X_{-2} \to X_{-1} \to X_0$ defined as $\ker s \to X \xrightarrow{s} Y$.

We claim that $\operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E})$ *is an admissible exact subcategory in* $D^b(\mathcal{E})$ *(i.e. the exact structure of the localization coincides with all composable morphisms which are part of a triangle). Then we look at* $\mathcal{E} \subseteq \operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E}) \subseteq D^b(\mathcal{E})$ *and conjecture that* $D^b(\mathcal{E}) \to D^b(\operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E}))$ *is a triangle equivalence.*

This would provide a useful tool to embed an exact category within its derived equivalence class into another exact category.

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