On faithfully balancedness in functor categories

1. Synopsis

This is a generalization of some results of Ma-Sauter [3] from module categories over artin algebras to more general functor categories (and partly to exact categories). In particular, we generalize the definition of a faithfully balanced module to a *faithfully balanced subcategory* and find the generalizations of dualities and characterizations from Ma-Sauter.
What is new? This generality is new but the results for artin algebras can be found in the joint work [3].

2. Introduction

For an exact category $\mathcal{E}$ in the sense of Quillen and a full subcategory $\mathcal{M}$ we define categories $\mathrm{gen}_k^{\mathcal{E}}(\mathcal{M})$ (and $\mathrm{cogen}_{\mathcal{E}}^k(\mathcal{M})$) of $\mathcal{E}$ (consisting of objects admitting a k-presentation in $\mathcal{M}$ by successive $\mathcal{M}$-approximations, cf. section 4). We also consider the two functors
$\Phi(X) := \mathrm{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{M}}, \Psi(X) := \mathrm{Hom}_{\mathcal{E}}(X, -)|_{\mathcal{M}}$.

We give the relatively obvious but technical generalizations of results in [3] related to these categories and functors. If $\mathcal{E}$ is a functor category (of some sort) these functors have adjoints and therefore stronger results can be found. We state here two of these:
Let $\mathcal{P}$ be an essentially small additive category. We denote by $\mathrm{Mod}\,\mathcal{P}$ the category of contravariant additive functors $\mathcal{P} \to (Ab)$ (and we set $\mathcal{P}\,\mathrm{Mod} := \mathrm{Mod}\,\mathcal{P}^{op}$). We write $\mathrm{mod}_k\,\mathcal{P}$ for the full subcategory of objects which admit a k-presentation by finitely generated projectives. We denote by $h\colon \mathcal{P} \to \mathrm{Mod}\,\mathcal{P}$, $P \mapsto h_P = \mathrm{Hom}_{\mathcal{P}}(-, P)$ the Yoneda embedding.
Cogen^1-duality: Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and assume now $\mathcal{M} \subset \mathrm{mod}_k\,\mathcal{P}$. We shorten the notation $\mathrm{cogen}^k(\mathcal{M}) := \mathrm{cogen}^k_{\mathrm{mod}_k\,\mathcal{P}}(\mathcal{M}) \subset \mathrm{mod}_k\,\mathcal{P}$.
We say $\mathcal{M}$ is **faithfully balanced** if $h_P \in \mathrm{cogen}^1(\mathcal{M})$ for all $P \in \mathcal{P}$.

Lemma 2.1. *(cf. Lem. 4.11) (*cogen^1*-duality) If $\mathcal{M}$ is faithfully balanced, we denote by $\tilde{\mathcal{M}} = \Psi(h_{\mathcal{P}}) \subset \mathcal{M}\,\mathrm{mod}_k$. Then Ψ defines a contravariant equivalence*

$$\mathrm{cogen}^1_{\mathrm{mod}_1\,\mathcal{P}}(\mathcal{M}) \longleftrightarrow \mathrm{cogen}^1_{\mathcal{M}\,\mathrm{mod}_1}(\tilde{\mathcal{M}}).$$

The symmetry principle states as follows:

THEOREM 2.2. *(cf. Thm. 4.16, Symmetry principle). Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ and $k \geq 1$. The following two statements are equivalent:*

(1) $\mathcal{P} \subset \mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M})$ *and* $\Phi(I) = \mathrm{Hom}_{\mathcal{E}}(-, I)|_{\mathcal{M}} \in \mathrm{mod}_k\,\mathcal{M}$ *for every* $I \in \mathcal{I}$.
(2) $\mathcal{I} \subset \mathrm{gen}^{\mathcal{E}}_k(\mathcal{M})$ *and* $\Psi(P) = \mathrm{Hom}_{\mathcal{E}}(P, -)|_{\mathcal{M}} \in \mathcal{M}\,\mathrm{mod}_k$ *for every* $P \in \mathcal{P}$.

A nice special case: Assume additionally that $\mathcal{E}$ is a Hom-finite K-category for a field K and $\mathcal{M} = \mathrm{add}(M)$ for an object $M \in \mathcal{E}$. Then the following two statements are equivalent:

(1) $\mathcal{P} \subset \mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M})$.
(2) $\mathcal{I} \subset \mathrm{gen}^{\mathcal{E}}_k(\mathcal{M})$.

Since: If we set $\Lambda = \mathrm{End}_{\mathcal{E}}(M)$, then $\mathrm{mod}_k \mathcal{M}$, $\mathcal{M}\,\mathrm{mod}_k$ can be identified with finite-dimensional (left and right) modules over Λ and $\Phi(I) = \mathrm{Hom}_{\mathcal{E}}(M, I), \Psi(P) = \mathrm{Hom}_{\mathcal{E}}(P, M)$ are by assumption finite-dimensional Λ-modules.

3. In additive categories

Here we want to extend Yoneda's embedding to a bigger subcategory: Let $\mathcal{C}$ be an additive category and $\mathcal{M}$ an essentially small full additive subcategory. A right $\mathcal{M}$-module is a contravariant additive functor from $\mathcal{M}$ into abelian groups. We denote by $\mathrm{Mod}\,\mathcal{M}$ the category of all right $\mathcal{M}$-modules. This is an abelian category. We have the fully faithful (covariant) Yoneda embedding $\mathcal{M} \to \mathrm{Mod}\,\mathcal{M}$ defined by $M \mapsto \mathrm{Hom}_{\mathcal{M}}(-, M)$. Clearly, we can extend this functor to a functor $\Phi\colon \mathcal{C} \to \mathrm{Mod}\,\mathcal{M},\ \Phi(X) := \mathrm{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{M}} = (-, X)|_{\mathcal{M}}$ where the last notation is our abbreviation for the Hom functor. The aim of this section is to define a subcategory $\mathcal{M} \subset \mathcal{G} \subset \mathcal{C}$ such that $\Phi|_{\mathcal{G}}$ is fully faithful.

We define a full subcategory of $\mathcal{C}$ as follows

$$\mathrm{gen}_1^{\mathrm{add}}(\mathcal{M}) := \left\{ Z \in \mathcal{C} \mid \begin{array}{c} \exists M_1 \xrightarrow{f} M_0 \xrightarrow{g} Z,\ M_i \in \mathcal{M},\ g = \mathrm{coker}(f) \text{ is an epim.} \\ (M, M_1) \to (M, M_0) \to (M, Z) \to 0 \\ \text{exact sequence of abelian groups } \forall M \in \mathcal{M} \end{array} \right\}$$

We observe that $g = \mathrm{coker}(f)$ and g an epimorphism is equivalent to that we have an exact sequence of $\mathcal{C}^{op}$-modules

$$0 \to (Z, -) \to (M_0, -) \to (M_1, -).$$

Furthermore the second line in the definition is equivalent to an exact sequence in $\mathrm{Mod}\,\mathcal{M}$

$$(-, M_1) \to (-, M_0) \to (-, Z)|_{\mathcal{M}} \to 0.$$

Dually, we define $\mathrm{cogen}^1_{\mathrm{add}}(\mathcal{M}) := (\mathrm{gen}_1^{\mathrm{add}}(\mathcal{M}^{op}))^{op}$ where $\mathcal{M}^{op}$ is considered as a full additive subcategory of $\mathcal{C}^{op}$.

Lemma 3.1. (1) *The functor* $\mathrm{gen}_1^{\mathrm{add}}(\mathcal{M}) \to \mathrm{Mod}\,\mathcal{M}$ *defined by* $Z \mapsto (-, Z)|_{\mathcal{M}}$ *is fully faithful. We even have for every* $Z \in \mathrm{gen}_1^{\mathrm{add}}(\mathcal{M}), C \in \mathcal{C}$ *a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(Z, C) \to \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{M}}((-, Z)|_{\mathcal{M}}, (-, C)|_{\mathcal{M}})$$

(2) *The functor* $\mathrm{cogen}^1_{\mathrm{add}}(\mathcal{M}) \to \mathrm{Mod}\,\mathcal{M}^{op}$ *defined by* $Z \mapsto (Z, -)|_{\mathcal{M}}$ *is fully faithful. We even have for every* $Z \in \mathrm{cogen}^1_{\mathrm{add}}(\mathcal{M}), C \in \mathcal{C}$ *a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(C, Z) \to \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{M}^{op}}((Z, -)|_{\mathcal{M}}, (C, -)|_{\mathcal{M}})$$

PROOF. We only prove (1), the second statement follows by passing to opposite categories. We consider the functor $\Phi\colon \mathcal{C} \to \mathrm{Mod}\,\mathcal{M}$ defined by $\Phi(X) := (-, X)|_{\mathcal{M}}$. Since $Z \in \mathrm{gen}_1^{\mathrm{add}}(\mathcal{M})$ we an exact sequences

$$0 \to (Z, C) \to (M_0, C) \to (M_1, C) \quad \text{of ab. groups}$$

and $\Phi(M_1) \to \Phi(M_0) \to \Phi(Z) \to 0$ in $\mathrm{Mod}\,\mathcal{M}$. By applying $(-, \Phi(C))$ to the second exact sequence we obtain an exact sequence

$$0 \to (\Phi(Z), \Phi(C)) \to (\Phi(M_0), \Phi(C)) \to (\Phi(M_1), \Phi(C)) \quad \text{of ab. groups.}$$

Since Φ is a functor, we find a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & (Z, C) \arrow[r] \arrow[d] & (M_0, C) \arrow[r] \arrow[d] & (M_1, C) \arrow[d] \\ 0 \arrow[r] & (\Phi(Z), \Phi(C)) \arrow[r] & (\Phi(M_0), \Phi(C)) \arrow[r] & (\Phi(M_1), \Phi(C)) \end{tikzcd}$$

By the Lemma of Yoneda, we have for every $F \in \mathrm{Mod}\,\mathcal{M}$ and $M \in \mathcal{M}$ that $\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{M}}(\Phi(M), F) = F(M)$. This implies that the maps $(M_i, C) \to (\Phi(M_i), \Phi(C))$ are isomorphisms of groups. and therefore, the induced map on the kernels is an isomorphism. $\square$

Remark 3.2. If $\mathcal{M}$ is not essentially small, $\operatorname{Hom}_{\mathcal{M}\,\mathrm{Mod}}(F,G)$ is not necessarily a set. But if one passes to the full subcategory of finitely presented $\mathcal{M}$-modules $\operatorname{mod}_1 \mathcal{M}$, this set-theoretic issue does not arise: Observe that $Z \mapsto (-,Z)|_{\mathcal{M}}$ defines by definition a covariant functor

$$\Phi\colon\ \operatorname{gen}_1^{\mathrm{add}}(\mathcal{M}) \to \operatorname{mod}_1 \mathcal{M},$$

the same proof as before shows that this is fully faithful. Similarly, the functor $Z \mapsto (Z,-)|_{\mathcal{M}}$ defines a fully faithful contravariant functor

$$\Psi\colon\ \operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M}) \to \operatorname{mod}_1 \mathcal{M}^{op}.$$

4. In exact categories

This section is a generalization of results from [3]. For exact categories we have subcategories of $\operatorname{cogen}^1_{\mathrm{add}}$ such that Ψ induces isomorphisms on (some) extension groups (cf. Lemma 4.3).
Given an exact category $\mathcal{E}$ with a full additive subcategory $\mathcal{M}$, we define $\operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M}) \subset \mathcal{E}$ to be the full subcategory of all objects X such that there is an exact sequence

$$0 \to X \to M_0 \to \cdots \to M_k \to Z \to 0$$

with $M_i \in \mathcal{M}, 0 \leq i \leq k$ such that for every $M \in \mathcal{M}$ the sequence

$$\operatorname{Hom}_{\mathcal{E}}(M_k, M) \to \cdots \to \operatorname{Hom}_{\mathcal{E}}(M_0, M) \to \operatorname{Hom}_{\mathcal{E}}(X, M) \to 0$$

is an exact sequence of abelian groups.
We define $\operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$ to be the full additive category of $\mathcal{E}$ given by all X such that there is an exact sequence

$$0 \to Z \to M_k \to \cdots \to M_0 \to X \to 0$$

with $M_i \in \mathcal{M}, 0 \leq i \leq k$ such that for every $M \in \mathcal{M}$ we have an exact sequence

$$\operatorname{Hom}_{\mathcal{E}}(M, M_k) \to \cdots \to \operatorname{Hom}_{\mathcal{E}}(M, M_0) \to \operatorname{Hom}_{\mathcal{E}}(M, X) \to 0$$

of abelian groups.
If it is clear from the context in which exact category we are working, then we leave out the index $\mathcal{E}$ and just write $\operatorname{cogen}^k(\mathcal{M})$ and $\operatorname{gen}_k(\mathcal{M})$.

Remark 4.1. Observe that $\operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M}) \subset \operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M})$, $\operatorname{gen}^{\mathcal{E}}_k(\mathcal{M}) \subset \operatorname{gen}_1^{\mathrm{add}}(\mathcal{M})$ for $k \geq 1$ and therefore the functor $\Psi\colon X \mapsto (X,-)|_{\mathcal{M}}$ (resp. $\Phi\colon X \mapsto (-,X)|_{\mathcal{M}}$) is fully faithful on $\operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$ (resp. on $\operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$) by Lemma 3.1 and Remark 3.2.

Remark 4.2. Let $k \geq 1$. We denote by $\operatorname{mod}_k \mathcal{M}$ the category of $\mathcal{M}$-modules which admit a k-presentation (indexed from 0 to k) by finitely presented projectives. For $F \in \operatorname{mod}_k \mathcal{M}$ the Ext-groups $\operatorname{Ext}^i_{\mathcal{M}\,\mathrm{Mod}}(F,G)$ with $0 \leq i < k$ are sets.
If $X \in \operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$, then we have $\Psi(X) = (X,-)|_{\mathcal{M}} \in \operatorname{mod}_k \mathcal{M}^{op} (=: \mathcal{M} \operatorname{mod}_k)$.
If $Y \in \operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$, then we have $\Phi(Y) = (-,Y)|_{\mathcal{M}} \in \operatorname{mod}_k \mathcal{M}$.

Since we are now working in exact categories, we observe the following isomorphisms on extension groups:

Lemma 4.3. *Let $k \geq 1$.*

(a) *If $X \in \operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$, then the functor $Z \mapsto \Psi(Z) = (Z,-)|_{\mathcal{M}}$ induces a well-defined natural isomorphism of abelian groups*

$$\operatorname{Ext}^i_{\mathcal{E}}(Y,X) \to \operatorname{Ext}^i_{\mathcal{M}\,\mathrm{Mod}}(\Psi(X),\Psi(Y)), \quad 0 \leq i < k$$

for all $Y \in \bigcap_{1 \leq i < k} \ker \operatorname{Ext}^i_{\mathcal{E}}(-,\mathcal{M})$.

(b) *If $Y \in \operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$, then the functor $Z \mapsto \Phi(Z) = (-,Z)|_{\mathcal{M}}$ induces a well-defined natural isomorphism of abelian groups*

$$\operatorname{Ext}^i_{\mathcal{E}}(Y,X) \to \operatorname{Ext}^i_{\mathrm{Mod}\,\mathcal{M}}(\Phi(Y),\Phi(X)), \quad 0 \leq i < k$$

for all $X \in \bigcap_{1 \leq i < k} \ker \operatorname{Ext}^i_{\mathcal{E}}(\mathcal{M},-)$.

Proof. (a) the proof is a straight forward generalization of [3, Lemma 2.4, (2)](using Rem. 4.1) and (b) follows from (a) by passing to the opposite exact category $\mathcal{E}^{op}$. $\square$

We will later use the following simple observation:

Remark 4.4. Let $\mathcal{E}$ be an exact category, $\mathcal{X}$ be a fully exact category and $\mathcal{M} \subset \mathcal{X}$ an additive subcategory. We say $\mathcal{X}$ is *deflation-closed* if for any deflation $d\colon X \to X'$ in $\mathcal{E}$ with X, X' in $\mathcal{X}$ it follows $\ker d \in \mathcal{X}$. The dual notion is *inflation-closed*.
If $\mathcal{X}$ is deflation-closed then $\mathrm{gen}_k^{\mathcal{X}}(\mathcal{M}) = \mathrm{gen}_k^{\mathcal{E}}(\mathcal{M}) \cap \mathcal{X}$. If $\mathcal{X}$ is inflation-closed then $\mathrm{cogen}^k_{\mathcal{X}}(\mathcal{M}) = \mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M}) \cap \mathcal{X}$.

4.1. Inside functor categories. Let $\mathcal{P}$ be an essentially small additive category. We denote by $h\colon \mathcal{P} \to \mathrm{Mod}\,\mathcal{P}$, $P \mapsto h_P = \mathrm{Hom}_{\mathcal{P}}(-, P)$ the Yoneda embedding, we write $h_{\mathcal{P}}$ for the essential image of h.

4.1.1. *Adjoint functors.* Let now $\mathcal{M}$ be an essentially small full additive subcategory of $\mathrm{Mod}\,\mathcal{P}$. We consider the contravariant functor

$$\begin{aligned}\Psi\colon\ \mathrm{Mod}\,\mathcal{P} &\to \mathcal{M}\,\mathrm{Mod},\\ X &\mapsto \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X,-)|_{\mathcal{M}} = (X,-)|_{\mathcal{M}}\end{aligned}$$

We also consider the contravariant functor

$$\begin{aligned}\Psi'\colon \mathcal{M}\,\mathrm{Mod} &\to \mathrm{Mod}\,\mathcal{P}\\ Z &\mapsto (P \mapsto \mathrm{Hom}_{\mathcal{M}\,\mathrm{Mod}}(Z, \Psi(h_P)))\end{aligned}$$

We generalize [1, Lemma 3.3].

Lemma 4.5. *The functors Ψ and Ψ' are contravariant adjoint functors, i.e. the following is a (bi)natural isomorphim*

$$\chi\colon\ \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X, \Psi'(Z)) \to \mathrm{Hom}_{\mathcal{M}\,\mathrm{Mod}}(Z, \Psi(X))$$

defined as follows: A natural transformation $f \in \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X, \Psi'(Z))$, is determined by for every $P \in \mathcal{P}, x \in X(P), M \in \mathcal{M}$ a group homomorphism

$$f_{P,x}(M)\colon Z(M) \mapsto \Psi(h_P)(M) = M(P)$$

then, we define a natural transformation $\chi(f)\colon Z \to \Psi(X) = \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X,-)|_{\mathcal{M}}$ for $M \in \mathcal{M}$ as follows

$$\begin{aligned}\chi(f)(M)\colon Z(M) &\to \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X, M),\\ z &\mapsto (X(P) \xrightarrow{f_{P,-}(z)} M(P), x \mapsto f_{P,x}(M)(z))_{P \in \mathcal{P}}\end{aligned}$$

Proof. We define $\chi'\colon \mathrm{Hom}_{\mathcal{M}\,\mathrm{Mod}}(Z, \Psi(X)) \to \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X, \Psi'(Z))$ as follows: For $g\colon Z \to \Psi(X) = \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(X,-)|_{\mathcal{M}}$ we have for every $M \in \mathcal{M}, z \in Z(M)$ a natural transformation $g_{M,z}\colon X \to M$, i.e. for every $P \in \mathcal{P}$ a group homomorphism

$$g_{M,z}(P)\colon X(P) \to M(P), x \mapsto g_{M,z}(P)(x),$$

then we define $\chi'(g)(P)\colon X(P) \to \Psi'(Z)(P) = \mathrm{Hom}_{\mathcal{M}\,\mathrm{Mod}}(Z, (h_P,-)|_{\mathcal{M}})$ as follows

$$x \mapsto (Z(M) \to M(P), z \mapsto g_{M,z}(P)(x))_{M \in \mathcal{M}}.$$

Then χ' is the inverse map to χ. $\square$

Remark 4.6. Given an adjoint pair of contravariant functors Ψ and Ψ', the natural isomorphisms

$$\mathrm{Hom}(X, \Psi(Z)) \to \mathrm{Hom}(Z, \Psi'(X))$$

induce natural transformations $\alpha\colon\ \mathrm{id} \to \Psi'\Psi$ (and $\alpha'\colon\ \mathrm{id} \to \Psi\Psi'$) as follows

$$\mathrm{Hom}(X,X) \xrightarrow{\Psi(-)} \mathrm{Hom}(\Psi(X), \Psi(X)) \cong \mathrm{Hom}(X, \Psi'\Psi(X)), \quad \mathrm{id}_X \mapsto \alpha_X$$

in this case we have triangle identities

$$\mathrm{id}_{\Psi(X)} = (\Psi(X) \xrightarrow{\alpha'_{\Psi(X)}} \Psi\Psi'\Psi(X) \xrightarrow{\Psi(\alpha_X)} \Psi(X))$$
$$\mathrm{id}_{\Psi'(Z)} = (\Psi'(Z) \xrightarrow{\alpha_{\Psi'(Z)}} \Psi'\Psi\Psi'(Z) \xrightarrow{\Psi'(\alpha'_Z)} \Psi'(Z))$$

In [**4**, section4] a tensor bifunctor is introduced

$$- \otimes_{\mathcal{M}} - \colon \operatorname{Mod}\mathcal{M} \times \mathcal{M}\operatorname{Mod} \to (Ab), (F,G) \mapsto F \otimes_{\mathcal{M}} G$$

Now, we consider the covariant funtor

$$\Phi\colon \operatorname{Mod}\mathcal{P} \to \operatorname{Mod}\mathcal{M}, \quad X \mapsto \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(-,X)|_{\mathcal{M}} =: (-,X)|_{\mathcal{M}}$$

and the following covariant functor

$$\Phi'\colon \operatorname{Mod}\mathcal{M} \to \operatorname{Mod}\mathcal{P}, \quad Z \mapsto (P \mapsto Z \otimes_{\mathcal{M}} \Psi(h_P))$$

Lemma 4.7. *The functor Φ is right adjoint to Φ', i.e. we have a (bi)natural maps*

$$\operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(\Phi'(Z), X) \to \operatorname{Hom}_{\operatorname{Mod}\mathcal{M}}(Z, \Phi(X))$$

Remark 4.8. If $F\colon \mathcal{C} \leftrightarrow \mathcal{D}\colon G$ is an adjoint pair of functors (with F left adjoint to G), then we have a unit $u\colon 1_{\mathcal{C}} \to GF$ and a counit, $c\colon FG \to 1_{\mathcal{D}}$. Let $\mathcal{C}_u$ be the full subcategory of objects in X in $\mathcal{C}$ such that $u(X)$ is an isomorphism. Let $\mathcal{D}_c$ be the full subcategory of objects Y in $\mathcal{D}$ such that $c(Y)$ is an isomorphism. Then, the triangle identities show directly that F, G restrict to quasi-inverse equivalences $F\colon \mathcal{C}_u \leftrightarrow \mathcal{D}_c\colon G$.

4.1.2. $\boxed{\operatorname{cogen}^k}$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and assume now $\mathcal{M} \subset \operatorname{mod}_k \mathcal{P}$. In this subsection we study $\operatorname{cogen}^k(\mathcal{M}) := \operatorname{cogen}^k_{\operatorname{mod}_k \mathcal{P}}(\mathcal{M}) \subset \operatorname{mod}_k \mathcal{P}$.
Our aim is to give a different description of the categories $\operatorname{cogen}^k(\mathcal{M})$ (cf. Lemma 4.9) and to introduce *faithfully balancedness* which leads to the cogen^1 duality (cf. Lemma 4.11).

We have the contravariant functor

$$\Psi\colon \operatorname{Mod}\mathcal{P} \to \mathcal{M}\operatorname{Mod}, \quad X \mapsto \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}}$$

and $\Psi|_{\operatorname{cogen}^k(\mathcal{M})}\colon \operatorname{cogen}^k(\mathcal{M}) \to \mathcal{M}\operatorname{mod}_k$ is fully faithful for $1 \leq k < \infty$.
The natural transformation $\alpha\colon \mathrm{id}_{\operatorname{Mod}\mathcal{P}} \to \Psi'\Psi$, for $X \in \operatorname{Mod}\mathcal{P}$ is given by a morphism in $\operatorname{Mod}\mathcal{P}$, $\alpha_X\colon X \to \Psi'\Psi(X) = \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(\Psi(X), \Psi(h_-))$ which is defined at $P \in \mathcal{P}$ via

$$X(P) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P, X) \to \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(\operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}}, \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P,-)|_{\mathcal{M}})$$
$$f \mapsto [\operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-) \xrightarrow{-\circ f} \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P,-)]|_{\mathcal{M}}$$

We observe that α_M is an isomorphism for every $M \in \mathcal{M}$ (since

$$\begin{aligned}(\Psi'\Psi(M))(P) &= \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(\operatorname{Hom}_{\mathcal{M}}(M,-), \Psi(h_P)) \\ &= \Psi(h_P)(M) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P, M) = M(P)\end{aligned}$$

using Yoneda's Lemma twice).

Lemma 4.9. *For $1 \leq k \leq \infty$ we have $\operatorname{cogen}^k_{\operatorname{mod}_k \mathcal{P}}(\mathcal{M})$ equals*

$$\{X \in \operatorname{mod}_k -\mathcal{P} \mid \alpha_X \textit{ isom.}, \Psi(X) \in \mathcal{M}\operatorname{mod}_k, \operatorname{Ext}^i_{\mathcal{M}\operatorname{Mod}}(\Psi(X), \Psi(h_P)) = 0, 1 \leq i < k, \forall P \in \mathcal{P}\}$$

Proof. The proof is a straight forward generalization of [**3**, Lemma 2.2,(1)] (the functor $\operatorname{Hom}_\Gamma(-,M)$ has to be replaced by applying $\operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(-,\Psi(h_P))$ for all $P \in \mathcal{P}$). $\square$

Definition 4.10. We say $\mathcal{M}$ is **faithfully balanced** if $h_{\mathcal{P}} \subset \operatorname{cogen}^1(\mathcal{M})$.

Lemma 4.11. *(*cogen^1 *duality) If* $\mathcal{M}$ *is faithfully balanced, we denote by* $\tilde{\mathcal{M}} = \Psi(h_{\mathcal{P}}) \subset \mathcal{M}\,\mathrm{mod}_k$, *then* Ψ *defines a contravariant equivalence*

$$\mathrm{cogen}^1_{\mathrm{mod}_1 \mathcal{P}}(\mathcal{M}) \longleftrightarrow \mathrm{cogen}^1_{\mathcal{M}\,\mathrm{mod}_1}(\tilde{\mathcal{M}})$$

and contravariant equivalences

$$\mathrm{cogen}^k_{\mathrm{mod}_k \mathcal{P}}(\mathcal{M}) \longleftrightarrow \mathrm{cogen}^1_{\mathcal{M}\,\mathrm{mod}_1}(\tilde{\mathcal{M}}) \cap \bigcap_{1 \le i < k} \ker(\mathrm{Ext}^i_{\mathcal{M}\,\mathrm{mod}_k}(-, \tilde{\mathcal{M}}))$$

Proof. Let $k = 1$. Since we have an adjoint pair of contravariant functors Ψ, Ψ' it follows from the triangle identities (cf. Remark 4.6): If α_X is an isomorphism then also $\alpha'_{\Psi(X)}$ and if α'_Z is an isomorphism then also $\alpha_{\Psi'(Z)}$. Now, since $\mathcal{M}$ is faithfully balanced we have that Ψ induces an equivalence $\mathcal{P}^{op} \cong \tilde{\mathcal{M}} = \Psi(h_{\mathcal{P}})$ by Lemma 3.1. It follows from the definition of Ψ' and a right module version of Lemma 4.9 that $\mathrm{cogen}^1(\tilde{\mathcal{M}}) = \{Z \in \mathcal{M}\,\mathrm{mod}_1 \mid \alpha'_Z \text{ isom}\}$.
The rest is a straightforward generalization of the proof of [**3**, Lemma 2.9]. $\square$

4.1.3. $\boxed{\mathrm{gen}_k}$. We study $\mathrm{gen}_k(\mathcal{M}) = \mathrm{gen}_k^{\mathrm{Mod}\,\mathcal{P}}(\mathcal{M}) \subset \mathrm{Mod}\,\mathcal{P}$. We again give a different description of these categories using tensor products of $\mathcal{M}$-modules (cf. Lemma 4.13). This is the main ingredient in the proof of the symmetry principle in the next subsection.

We have the covariant functor

$$\Phi\colon \mathrm{Mod}\,\mathcal{P} \to \mathrm{Mod}\,\mathcal{M}, \quad X \mapsto \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(-, X)|_{\mathcal{M}}$$

and $\Phi|_{\mathrm{gen}_k(\mathcal{M})}\colon \mathrm{gen}_k(\mathcal{M}) \to \mathrm{mod}_k\,\mathcal{M}$ is fully faithful. We have an induced covariant functor

$$\varepsilon = \Phi' \circ \Phi\colon \mathrm{Mod}\,\mathcal{P} \to \mathrm{Mod}\,\mathcal{P}, \quad X \mapsto \varepsilon_X$$

defined for $P \in \mathcal{P}$ as

$$\varepsilon_X(P) := \Phi(X) \otimes_{\mathcal{M}} \Psi(h_P)$$

and a natural transformation $\varphi\colon \varepsilon \to \mathrm{id}_{\mathrm{Mod}\,\mathcal{P}}$, for $X \in \mathrm{Mod}\,\mathcal{P}$ this is given by a morphism $\varphi_X\colon \varepsilon_X \to X$ which is defined at $P \in \mathcal{P}$ via

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(-, X)|_{\mathcal{M}} \otimes_{\mathcal{M}} (\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(h_P, -)|_{\mathcal{M}}) &\to \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(h_P, X) = X(P) \\ \underbrace{g \otimes f}_{\in \mathrm{Hom}(M,X) \otimes_{\mathbb{Z}} \mathrm{Hom}(h_P, M)} &\mapsto g \circ f \end{aligned}$$

Remark 4.12. Φ and is right adjoint functor of Φ' between abelian categories therefore Φ is left exact and Φ' is right exact, φ is the counit of this adjunction. If $M \in \mathcal{M}$, then φ_M is an isomorphism.

Lemma 4.13. *For* $1 \le k \le \infty$ *we have*

$$\begin{aligned} &\mathrm{gen}_k^{\mathrm{Mod}\,\mathcal{P}}(\mathcal{M}) = \\ &\{X \in \mathrm{Mod}\,\mathcal{P} \mid \varphi_X \textit{ isom.}\ , \Phi(X) \in \mathrm{mod}_k\,\mathcal{M},\ \mathrm{Tor}^i_{\mathcal{M}}(\Phi(X), \Psi(h_P)) = 0, 1 \le i < k, \forall P \in \mathcal{P}\} \end{aligned}$$

Proof. Let $X \in \mathrm{gen}_k(\mathcal{M})$, then there exists an exact sequence $M_k \to \cdots \to M_0 \to X \to 0$ such that Φ preserves its exactness, this implies $\Phi(X) \in \mathrm{mod}_k\,\mathcal{M}$. Now, we apply $\varepsilon = \Phi'\Phi$ and consider the commutative diagram

$$\begin{tikzcd} M_k \arrow[r] & \cdots \arrow[r] & M_0 \arrow[r] & X \arrow[r] & 0 \\ \varepsilon_{M_k} \arrow[r] \arrow[u, "\varphi_{M_k}"] & \cdots \arrow[r] & \varepsilon_{M_0} \arrow[r] \arrow[u, "\varphi_{M_0}"] & \varepsilon_X \arrow[r] \arrow[u, "\varphi_X"] & 0 \end{tikzcd}$$

Now, since Φ' is right exact and φ_{M_i} is an isomorphism for $0 \le i \le k$, we conclude that φ_X is an isomorphism and the lower row is exact. This implies $\mathrm{Tor}^i_{\mathcal{M}}(\Phi(X), \Psi(h_P)) = 0, 1 \le i < k$.

Conversely, if we take $X \in \operatorname{Mod} \mathcal{P}$ fulfilling the assumptions in the set bracket of the lemma. We can apply Φ' to the projective k-presentation of $\Phi(X)$, then we can find a diagram as before but this time we know from the assumptions that the bottom row is exact. Furthermore, since φ_* is an isomorphism in all places of the diagram, we have that also the top row is exact. This implies $X \in \operatorname{gen}_k^{\operatorname{Mod} \mathcal{P}}(\mathcal{M})$. ☐

4.2. The symmetry principle. Now, we study these subcategories in more general exact categories. For an exact category $\mathcal{E}$ with enough projectives $\mathcal{P}$ and an exact category $\mathcal{F}$ with enough injectives $\mathcal{I}$, we consider the covariant, exact, fully faithful functors

$$\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_\infty \mathcal{P}, \quad X \mapsto \operatorname{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{P}}$$
$$\mathbb{I}\colon \mathcal{F}^{\mathrm{op}} \to \operatorname{mod}_\infty \mathcal{I}^{op}, \quad X \mapsto \operatorname{Hom}_{\mathcal{F}}(X, -)|_{\mathcal{I}^{op}}$$

cf. [**2**, Prop. 2.2.1, Prop.2.2.8]

Remark 4.14. For an additive category $\mathcal{M}$ of $\mathcal{E}$ (resp. of $\mathcal{F}$) we have:

$$\mathbb{P}(\operatorname{gen}_k^{\mathcal{E}}(\mathcal{M})) = \operatorname{Im} \mathbb{P} \cap \operatorname{gen}_k^{\operatorname{Mod} \mathcal{P}}(\mathbb{P}(\mathcal{M})),$$
$$\mathbb{I}((\operatorname{cogen}^k_{\mathcal{F}}(\mathcal{M}))^{op}) = \mathbb{I}(\operatorname{gen}_k^{\mathcal{F}^{op}}(\mathcal{M}^{op})) = \operatorname{Im} \mathbb{I} \cap \operatorname{gen}_k^{\operatorname{Mod} \mathcal{I}^{op}}(\mathbb{I}(\mathcal{M}^{op}))$$

This follows from remark 4.4 since $\mathbb{P}\colon \mathcal{E} \to \operatorname{Im} \mathbb{P}$ is an equivalence of exact categories and $\operatorname{Im} \mathbb{P}$ is deflation-closed in $\operatorname{mod}_\infty \mathcal{P}$ and $\operatorname{mod}_\infty \mathcal{P}$ is deflation-closed in $\operatorname{Mod} \mathcal{P}$. The second statement follows by passing to the opposite category.

As before, let $\Phi\colon \mathcal{E} \to \operatorname{Mod} \mathcal{M}, \Phi(X) = \operatorname{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{M}}$, $\Psi\colon \mathcal{E} \to \mathcal{M} \operatorname{Mod}, \Psi(X) = \operatorname{Hom}_{\mathcal{E}}(X, -)|_{\mathcal{M}}$. We have the immediate corollary:

Corollary 4.15. *(of Lem. 4.13 and Rem. 4.14) (1) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and $\mathcal{M}$ a full additive subcategory. Then the following are equivalent:*

(1) $X \in \operatorname{gen}_k^{\mathcal{E}}(\mathcal{M})$
(2) $\Phi(X) \in \operatorname{mod}_k \mathcal{M}$ *and for every* $P \in \mathcal{P}$*:*

$$\Phi(X) \otimes_{\mathcal{M}} \Psi(P) \to \operatorname{Hom}_{\mathcal{E}}(P, X), \quad g \otimes f \mapsto g \circ f$$

is an isomorphism, $\operatorname{Tor}^i_{\mathcal{M}}(\Phi(X), \Psi(P)) = 0$*,* $1 \leq i < k$*.*

(2) If $\mathcal{E}$ is an exact category with enough injectives $\mathcal{I}$ and $\mathcal{M}$ a full additive subcategory. Then the following are equivalent:

(1) $X \in \operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$
(2) $\Psi(X) \in \mathcal{M} \operatorname{mod}_k$ *and for every* $I \in \mathcal{I}$*:*

$$\Phi(I) \otimes_{\mathcal{M}} \Psi(X) \to \operatorname{Hom}_{\mathcal{F}}(X, I), \quad g \otimes f \mapsto g \circ f$$

is an isomorphism, $\operatorname{Tor}^i_{\mathcal{M}}(\Phi(I), \Psi(X)) = 0$*,* $1 \leq i < k$*.*

Theorem 4.16. *(Symmetry principle). Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ and $k \geq 1$. The following two statements are equivalent:*

(1) $\mathcal{P} \subset \operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$ *and* $\Phi(I) = \operatorname{Hom}_{\mathcal{E}}(-, I)|_{\mathcal{M}} \in \operatorname{mod}_k \mathcal{M}$ *for every* $I \in \mathcal{I}$
(2) $\mathcal{I} \subset \operatorname{gen}_k^{\mathcal{E}}(\mathcal{M})$ *and* $\Psi(P) = \operatorname{Hom}_{\mathcal{E}}(P, -)|_{\mathcal{M}} \in \mathcal{M} \operatorname{mod}_k$ *for every* $P \in \mathcal{P}$

Proof. We consider $\mathbb{P}, \mathbb{I}$ as before defined for the category $\mathcal{E}$. Then, it is straight forward from the previous Lemma to see that (1) and (2) are both equivalent to for all $P \in \mathcal{P}, I \in \mathcal{I}$, $\Psi(P) \in \mathcal{M} \operatorname{mod}_k, \Phi(I) \in \operatorname{mod}_k \mathcal{M}$ and

$$\Phi(I) \otimes_{\mathcal{M}} \Psi(P) \to \operatorname{Hom}_{\mathcal{E}}(P, I), \quad g \otimes f \mapsto g \circ f$$

is an isomorphism, $\operatorname{Tor}^i_{\mathcal{M}}(\Phi(I), \Psi(P)) = 0$, $1 \leq i < k$. Therefore (1) and (2) are equivalent. ☐

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