Derived categories and functors for exact categories

This includes a joint result with Juan Omar Gomez.

1. Synopsis

By now, derived categories and derived functors for abelian categories are standard topics in a course on homological algebra. This is a an introduction to derived categories of exact categories assuming that the reader is familiar with the theory for abelian categories. Other sources which include exact categories are [**12**], [**14**], [**5**].
Our treatment of derived functors is only shortly summarizing the results in [**12**].
What is new? We characterize when derived categories of exact categories are locally small (i.e. Hom-classes are sets).

2. Why derived categories?

This is an attempt in trying to explain in a nutshell why derived categories have homological algebra as their heart.
An exact category (in the sense of Quillen) is an additive category together with a collection of kernel-cokernel pairs called short exact sequences fulfilling axioms such that $\mathrm{Ext}^1_{\mathcal{E}} =$ (taking equivalence classes of short exact sequences) becomes an additive bifunctor. Using longer exact sequences one can find higher Ext-functors Ext^n - for the moment we assume that these are all set-valued functors (cf. next section).
Homological algebra for exact categories is the study of the bifunctors $\mathrm{Ext}^n_{\mathcal{E}}$ and in particular the conversion of short exact sequences into long exact sequences using higher Ext-groups.
Philosophically: Can we enlarge $\mathcal{E}$ to a category $\mathcal{D}$ such that these bifunctors become restrictions of the Hom-functor and are connected by an auto-equivalence $\Sigma\colon \mathcal{D} \to \mathcal{D}$ as
$\mathrm{Ext}^n_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{D}}(X, \Sigma^n Y)$?
For every short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ representing $\sigma \in \mathrm{Ext}^1_{\mathcal{E}}(Z,X) = \mathrm{Hom}_{\mathcal{D}}(Z, \Sigma X)$ we look at the sequences in $\mathcal{D}$

$$X \to Y \to Z \xrightarrow{\sigma} \Sigma X$$

and call these 'distinguished triangles with three objects in $\mathcal{E}$'. This *wish list* on such a category $\mathcal{D}$ has been formalized in the notion of a triangulated category with initial data an additive category $\mathcal{D}$ with an auto-equivalence Σ, called *suspension*, and a collection of distingished triangles such that a list of axioms is fulfilled (cf. TR0-TR5 in [**9**, Def. 10.1.6]). As the long exact sequences have no negative parts, we find another condition which the bounded derived category has to fulfill

$$\mathrm{Ext}^{-n}_{\mathcal{E}}(X,Y) = \mathrm{Hom}(X, \Sigma^{-n} Y) = 0 \quad \forall X, Y \in \mathcal{E},\ n \geq 1.$$

The structure preserving functors between triangulated categories are called *triangle functors*.
'Structure preserving functors' from exact categories into triangulated categories are called δ-functors (in the sense of Keller [**11**] or one can view them as extriangulated functors in the sense of [**3**]):

Definition 2.1. ([**11**]) Let $\mathcal{E}$ be an exact category and $\mathcal{D}$ be a triangulated category. A δ-functor $\mathcal{E} \to \mathcal{D}$ consists of a pair (F, δ) consisting of an additive functor $F\colon \mathcal{E} \to \mathcal{D}$ and an assignment δ

mapping short exact sequences $\sigma = (i, d)\colon X \rightarrowtail Y \twoheadrightarrow Z$ to a morphism $\delta_\sigma\colon F(Z) \to \Sigma F(X)$ fitting into a distinguished triangle

$$F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(d)} F(Z) \xrightarrow{\delta_\sigma} \Sigma F(X).$$

We call a δ-functor $\mathcal{E} \to \mathcal{D}$ *homological* if δ induces natural isomorphisms $\mathrm{Ext}^n_{\mathcal{E}}(Z, X) \to \mathrm{Hom}_{\mathcal{D}}(F(Z), \Sigma^n F(X))$ for every $n \in \mathbb{Z}$.

So, we hope for the following naive definition:
The bounded derived category should be the *universal homological δ-functor into triangulated categories*. This means we have an homological δ-functor $\mathcal{E} \to \mathrm{D}^b(\mathcal{E})$ such that every homological δ-functor $\mathcal{E} \to \mathcal{D}$ factors as $\mathcal{E} \to \mathrm{D}^b(\mathcal{E}) \xrightarrow{R} \mathcal{D}$. We also want R to be unique up to natural isomorphism. Unfortunately, uniqueness is in general unknown. We call it a **realization functor** for $\mathcal{E}$ (if it exists).
The existence of R can be proven when restricting to suitably *enhanced* triangulated categories (filtered derived [**4**], Neeman enhanced [**18**] or algebraic due to [**13**], proven in [**15**]). Uniqueness is usually not discussed, it requires restriction to triangle functors which are preserving the fixed enhancement (for algebraic triangulated categories it follows from the construction, cf. [**15**]).

Remark 2.2. Why would one also look at $\mathrm{D}^+(\mathcal{E})$, $\mathrm{D}^-(\mathcal{E})$, $\mathrm{D}(\mathcal{E})$?
The reason is that we can usually not define right and left derived functors on $\mathrm{D}^b(\mathcal{E})$ but if we extend our derived category we (often) can.
At the level of positive (resp. negative) derived categories we have an explicit method to calculate (at least partially) right (resp. left) derived functor using Deligne's right (resp. left) acyclic objects. For example, injective objects are always right acyclic and injective coresolutions (if they exist) can then be used to calculate right derived functors (as in the abelian case). We come back to this in detail in the last section of this chapter.
Our motivation to look at $\mathrm{D}(\mathcal{E})$ is not so strong but if you want for example a triangulated category with arbitrary set-valued coproducts then you would look at $\mathrm{D}(\mathcal{E})$ where $\mathcal{E}$ has arbitrary coproducts.

2.1. Explicit construction(s). Let us start with an exact category $\mathcal{E}$ with underlying additive category $\mathcal{A}$ and $* \in \{\emptyset, +, -, b\}$.

2.1.1. Variant 1*: As Verdier quotient.* Given a full triangulated subcategory in a triangulated category $\mathcal{U} \subseteq \mathcal{T}$ there exists a triangle functor $Q_{\mathcal{U}}\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$, called the **Verdier localization**, which fulfills the following universal property: Every triangle functor $\mathcal{T} \to \mathcal{R}$ which annihilates the objects of $\mathcal{U}$ factors uniquely over a triangle functor $\mathcal{T}/\mathcal{U} \to \mathcal{R}$.
There are two well-known *issues*

1. $\mathcal{T}/\mathcal{U}$ is defined by a localization and the Hom-classes may not always be sets.
2. Let $\mathcal{U} \subset \overline{\mathcal{U}} \subseteq \mathcal{U}^{\oplus}$ with $\overline{\mathcal{U}}$ the saturation (i.e. the closure of $\mathcal{U}$ under isomorphism in $\mathcal{T}$) and $\mathcal{U}^{\oplus}$ the thick closure (i.e. closure under direct summands and isomorphism in $\mathcal{T}$). Then clearly these larger categories fulfill the same universal property and $\mathcal{T}/\mathcal{U} = \mathcal{T}/\overline{\mathcal{U}} = \mathcal{T}/\mathcal{U}^{\oplus}$. Therefore, some authors consider Verdier quotients with respect to thick subcategories.

Then take the homotopy category of the additive category $\mathrm{K}^*(\mathcal{A})$. This is a triangulated category (cf. [**9**, Thm 11.3.8]).
The subcategory of $\mathcal{E}$-acyclic complexes $\underline{\mathrm{Ac}}^*(\mathcal{E})$ is a full triangulated subcategory (cf. [**17**], 1.1).

Definition 2.3. The ($*$-)**derived category** of $\mathcal{E}$ is defined as the Verdier quotient

$$\mathrm{D}^*(\mathcal{E}) := \mathrm{K}^*(\mathcal{A})/\underline{\mathrm{Ac}}^*(\mathcal{E})$$

Then problem (2) can be fully answered when looking at properties of the underlying additive category. We say an additive category $\mathcal{A}$ is **weakly idempotent complete (wic) (resp. impotent complete (ic))** if every idempotent endomorphism $e\colon A \to A$ has a kernel (resp. has kernel and image and gives rise to a split exact sequence $\ker e \to A \to \mathrm{Im}\, e$).

In [5], exact structure on an exact category is extended functorially to an exact structure on an idempotent completion of the underlying additive category. This gives an exact functor

$$\mathcal{E} \to \mathcal{E}^{ic}$$

Similarly a weakly idempotent completion can be constructed. This gives exact functors

$$\mathcal{E} \to \mathcal{E}^{wic} \to \mathcal{E}^{ic}$$

In [17], the following (2c) has been proven (and (2a,b) is partly attributed to Thomason):

(2a) For $* = b$.
$\underline{\mathrm{Ac}}^b(\mathcal{E})$ is saturated if and only it is thick if an only if $\mathcal{E}$ is weakly idempotent complete.
For every exact category $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}^{wic})$ is a triangle equivalence.

(2b) For $* = \pm$.
In this case we have both triangle equivalences $\mathrm{D}^{\pm}(\mathcal{E}) \to \mathrm{D}^{\pm}(\mathcal{E}^{wic}) \to \mathrm{D}^{\pm}(\mathcal{E}^{ic})$.

(2c) For $* = \emptyset$.
$\underline{\mathrm{Ac}}(\mathcal{E})$ is saturated if and only it is thick if an only if $\mathcal{E}$ is idempotent complete.
Then $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\mathcal{E}^{ic})$ is a triangle equivalence.

Remark 2.4. Then $\mathrm{D}^b(\mathcal{E})$ may not be idempotent complete: Balmer-Schlichting [2] showed that the idempotent completion of a triangulated category has a natural triangulated structure and $\mathrm{D}^b(\mathcal{E}^{ic})$ is triangle equivalent to $(\mathrm{D}^b(\mathcal{E}))^{ic}$.

2.1.2. Variant 2*: As Localization of exact category of complexes with respect to a biresolving subcategory.* This is a recent construction of Rump [23, Thm 5], cf. also [24].
A full additive subcategory $\mathcal{C}$ in an exact category $\mathcal{F}$ is a **biresolving subcategory** if it is thick (i.e. closed under direct summands and every $\mathcal{C}$ satisfies the 2-out of 3-property for short exact sequences (i.e. for every $\mathcal{F}$-short exact sequence $X \to Y \to Z$ with two out of X, Y, Z in $\mathcal{C}$ the third is also in $\mathcal{C}$) and it is generating-cogenerating (i.e. for every X in $\mathcal{F}$ there is a $\mathcal{F}$-deflation $d\colon C^0 \to X$ and an $\mathcal{F}$-inflation $i\colon X \to C^1$ with $C^0, C^1 \in \mathcal{C}$). (We slightly differ with this definition from loc. cit, as there it is not assumed that $\mathcal{C}$ is closed under direct summands.)
Then the localization $\mathcal{F}/\mathcal{C}$ is defined as follows: First consider $[\mathcal{C}]$ to be the ideal (i.e. subfunctor of Hom) given by all morphisms factoring through $\mathcal{C}$. We take $\Sigma(\mathcal{C}) \subseteq (\mathcal{F}, \ )$ the class of all morphisms which become in the ideal quotient category $\mathcal{F}/[\mathcal{C}]$ a monomorphism and also an epimorphism. Then in loc. cit. it is shown that there exists a left and right calculus of fractions and

$$\mathcal{F}/\mathcal{C} := \Sigma(\mathcal{C})^{-1}\mathcal{F}$$

admits a structure of triangulated category, loc. cit Theorem 5.
Now we apply this as follows: We assume that $\mathcal{E}$ is weakly idempotent complete (in loc. cit this is not assumed).
We take *-chain complexes $\mathrm{Ch}^*(\mathcal{E})$ in $\mathcal{A}$. We see this as an exact category with short exact sequences are degree-wise $\mathcal{E}$-short exact sequences (cf. [5, Lem. 9.1]). We look at the full subcategory $\mathrm{Ac}^*(\mathcal{E})$ of $\mathcal{E}$-acyclic-complexes. This is a biresolving subcategory of the exact category $\mathrm{Ch}^*(\mathcal{E})$, cf. [23, Example 2]. Then this gives the second definition of the derived category (cf. [24])

$$\mathrm{D}^*(\mathcal{E}) := \mathrm{Ch}^*(\mathcal{E})/\mathrm{Ac}^*(\mathcal{E})$$

Then one can show the following Lemma as a corollary, the canonical functor of the localization is also the composition $L\colon \mathrm{Ch}^*(\mathcal{E}) \to \mathrm{K}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{E})$. In [25, Tag 014Z], Lemma 13.12.1, it has been explained how to construct a δ such that (L, δ) is a δ-functor (even though in loc. cit. the category is abelian, the same arguments work for general exact categories). The construction goes as follows: Given a short exact sequence $\sigma\colon A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet$ in $\mathrm{Ch}^*(\mathcal{E})$, one constructs a quasi-isomorphism (i.e. morphism with acyclic cone) $q\colon C(a) \to C$ where $C(a)$ is the cone of a, then one has a standard triangle in $\mathrm{K}^*(\mathcal{A})$ for the morphism a, and this gives a distinguished triangle in $\mathrm{D}^*(\mathcal{E})$. In particular, we have in this triangle a morphism $p\colon C(a) \to \Sigma A^\bullet$, so we have a well-defined

$$\delta_\sigma := p \circ q^{-1}\colon C^\bullet \to \Sigma A^\bullet \quad \in \mathrm{D}^*(\mathcal{E})$$

Recall that a complex is numbered as follows by the integers $\cdots \to X^n \to X^{n+1} \to \cdots$. We also have a shift on $T\colon \mathrm{Ch}(\mathcal{E}) \to \mathrm{Ch}(\mathcal{E})$, where $T(X)$ is the shift of complexes to the right, i.e. $T(X^\bullet)^n := X^{n+1}$.
For a complex X concentrated in degree 0 we have the short exact sequence $\eta_X\colon TX \rightarrowtail (X \xrightarrow{1_X} X) \twoheadrightarrow X$ with $(X \xrightarrow{1_X} X)$ is the 2-term complex concentrated in degrees $0, 1$. We have $L \circ T \cong \Sigma \circ L$ (we say L is **shift invariant**). Also, by definition $\delta_{\eta_X} = 1_{L(X)}$ (we say that L is **normed**).

Lemma 2.5. *Let $\mathcal{E}$ be weakly idempotent complete exact category. Then*

(1) *the canonical functor $L\colon \mathrm{Ch}^*(\mathcal{E}) \to \mathrm{D}^*(\mathcal{E})$ is a δ-functor which is shift invariant and normed.*
(2) *Every shift invariant, normed δ-functor $G\colon \mathrm{Ch}^{\mathrm{b}}(\mathcal{E}) \to \mathcal{D}$ factors uniquely as $\overline{G} \circ L$ with $\overline{G}\colon \mathrm{D}^b(\mathcal{E}) \to \mathcal{D}$ a triangle functor.*

Proof. (1) has already been explained before the lemma.
(2) Let $G = (G, \delta^G)\colon \mathrm{Ch}^{\mathrm{b}}(\mathcal{E}) \to \mathcal{D}$ be a shift invariant δ-functor.
The property being *normed* can be replace by mapping split acyclic complexes to zero, this means that G factors over a triangle functor $\mathrm{K}^b(\mathcal{A}) \to \mathcal{D}$. As it maps $\mathcal{E}$-short exact sequences to triangles, the triangle functor maps acyclic complexes to zero (cf. same argument in [**15**, Lem 3.5]) and therfore factors over a triangle functor $\overline{G}\colon \mathrm{D}^b(\mathcal{E}) \to \mathcal{D}$. Assume now, we have a second triangle functor $H\colon \mathrm{D}^b(\mathcal{E}) \to \mathcal{D}$ with $H \circ L = \overline{G} \circ L$. Now, L factors as $L_V \circ L_K$ with $L_K\colon \mathrm{Ch}^b(\mathcal{E}) \to \mathrm{K}^b(\mathcal{A})$ is just the ideal quotient and $L_V\colon \mathrm{K}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{E})$ is the Verdier quotient. By the universal property of the Verdier quotient it is enough to see that $H \circ L_V = \overline{G} \circ L_V$. But L_K is full, so every morphism is of the form $L_{\mathrm{K}}(f)$ for some morphism and we see that $HL_V = \overline{G}L_V$. $\square$

Remark 2.6. One can also see the derived categories of exact categories as homotopy categories associated to certain model categories (different choices might lead to the same homotopy categories). Also derived functors can be constructed in this more general set-up. For example look into [**6**] and [**8**].

2.2. Completions.

2.2.1. *Countable envelope.* Exact categories $\mathcal{E}$ always have a countable envelope $\tilde{\mathcal{E}}$ constructed in [**10**], Appendix B:

First construct $\mathcal{FE}$ with objects are sequences $X = (X^0 \xrightarrow{i_X^0} X^1 \xrightarrow{i_X^1} X^2 \to \cdots)$ of consecutively composable inflations, and morphisms $X \to Y$ are sequences of $(f^p\colon X^p \to Y^p)_{p \in \mathbb{N}_0}$ such that $i_Y^p f^p = f^{p+1} i_X^p$ for all $p \geq 0$. Then $\mathcal{FE}$ is an exact category with a sequence (j, e) is a short exact sequence if and only if (j^p, e^p) are short exact sequences in $\mathcal{E}$ for all $p \geq 0$. Then we define $\tilde{\mathcal{E}}$ as the category with the same objects and morphisms $\mathrm{Hom}(X, Y) = \lim_p \operatorname{colim}_q \mathrm{Hom}(X^p, Y^q)$. By construction we have a functor $\mathcal{FE} \to \tilde{\mathcal{E}}$, we call a sequence $(\tilde{j}, \tilde{e})$ an exact sequence if there exists an exact sequence (j, e) in $\mathcal{FE}$ which maps via the natural functor to it. Observe that the underlying additive category of $\tilde{\mathcal{E}}$ has countable coproducts (by taking only split inflations in a sequence of inflations as before). We recall Keller's results.

Theorem 2.7. *([**10**, Appendix B])*

(a) *$\tilde{\mathcal{E}}$ is an exact category wrt the exact sequences described before. The constant functor $E\colon \mathcal{E} \to \tilde{\mathcal{E}}$, $X \to (X = X = X \cdots)$ is a homologically exact functor (i.e. exact and inducing isomorphisms on all Ext-groups).*
(b) *The following are equivalent:*
  (b1) *$\mathcal{E}$ is locally small (i.e. has Hom-sets and not classes) and $\mathrm{Ext}^n_{\mathcal{E}}$ are set-valued for all $n \geq 1$*
  (b2) *$\tilde{\mathcal{E}}$ is locally small and $\mathrm{Ext}^n_{\tilde{\mathcal{E}}}$ are set-valued for all $n \geq 1$*

Proof. or where to find it: For (b) observe that (a) already implies (b2) $\Rightarrow$ (b1). The other implication is using that (b1) implies that $\mathcal{F}\mathcal{E}$ has set-valued Ext^n for all $n \geq 1$ and then use [10, Lemma in B.3]. $\square$

Remark 2.8. By the previous Lemma we have an induced fully faithful triangle functor $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\tilde{\mathcal{E}})$ but we do not know if $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\tilde{\mathcal{E}})$ is faithful.

2.2.2. *Completion of small exact categories.* Small exact categories $\mathcal{E}$ have a completion $\overrightarrow{\mathcal{E}}$ (called a locally coherent exact category) with respect to filtered colimits, cf. [20] and Appendix in Chapter 2.

$$\mathcal{E} \to \overrightarrow{\mathcal{E}}$$

Then $\mathcal{E}$ is a full homologically exact category of $\overrightarrow{\mathcal{E}}$. Also, in loc. cit, the author shows that $\overrightarrow{\mathcal{E}}$ is a so-called exact category of Grothendieck type which implies that it has enough injectives.

Remark 2.9. $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\overrightarrow{\mathcal{E}})$ might be not full (examples are given by [21]), a characterization when this triangle functor is faithful is unknown.

The difference to the countable envelope is that we assume here that $\mathcal{E}$ is (essentially) small. If we drop this assumption then we do not know much about the Ind-completion (but it can have not enough injectives).

2.3. Passing between different boundedness levels.

Lemma 2.10. *([12]) By construction we have triangle functors* $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^{+/-}(\mathcal{E}) \to D(\mathcal{E})$*. They are all fully faithful.*

Let us consider two fixed exact categories $\mathcal{E}, \mathcal{E}'$ and the following three statements

(D^b) $\mathrm{D}^b(\mathcal{E})$ and $\mathrm{D}^b(\mathcal{E}')$ are triangle equivalent.
(D^+) $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^+(\mathcal{E}')$ are triangle equivalent.
(D) $\mathrm{D}(\mathcal{E})$ and $\mathrm{D}(\mathcal{E}')$ are triangle equivalent.

We are now looking at situations where one holds and another one not.

(a) If $\mathcal{E} \neq \mathcal{E}^{ic} = \mathcal{E}'$ then (D), (D^+) and not (D^b) holds (cp. [17]).
(b) If $\mathcal{E} \subseteq \mathcal{E}'$ is a coresolving subcategory of $\mathcal{E}'$, then we have (D^+). It is finitely coresolving if and only if (D^b) holds. It is n-coresolving for some $n \geq 0$ if and only if (D) holds.
This way, we find an instance where (D^+) and not (D^b) and also not (D) holds. Reference [7]. Also: an instance where (D^b) and not (D) holds.

This leaves only the following open: Does (D^b) imply (D^+)? I do not know.

Remark 2.11. When restricting to exact categories with certain similar properties, there are still interdependences. For module categories of rings (cf. main theorem in [22]). This has been generalized to functor categories [1]. The best explanation for this is given by Neeman's theory of approximable triangulated categories, cf. [19].

3. When is it a category?

In this text, we will work in the framework of ZFCU (Zermelo-Frankel, the axiom of choice and the axiom of the universe). We fix an infinite universe $\mathcal{U}$.

Let X be a set, we will say that X is: $\mathcal{U}$*-small* if $X \in \mathcal{U}$, $\mathcal{U}$*-class* if $X \subset \mathcal{U}$ or $\mathcal{U}$*-large set* if $X \not\subseteq \mathcal{U}$. In general, we will drop $\mathcal{U}$ from the notation.

We allow a category to have a class of objects and to have a class as morphisms between any two objects. We say that a category is *locally small* if $\mathrm{Hom}(X,Y)$ is a small set for every two objects X, Y - usually in the literature: locally small categories are called categories.

As usual, we let Set denote the category of small sets with morphisms given by maps between them. We will need a *big version* of this category, namely $\widehat{\mathrm{Set}}$, where we allow the objects to be classes as well as the morphisms between them. We denote $\widehat{\mathrm{Gps}}, \widehat{\mathrm{Ab}} \ldots$ denote the 'big' versions the categories of groups, abelian groups, etc.

Let $\mathcal{E}$ be a (locally small Quillen-) exact category.

THEOREM 3.1. *The following are equivalent:*

(1) $\mathrm{Ext}^n_{\mathcal{E}}$ *is set-valued for all* $n \geq 0$
(2) $\mathrm{D}^b(\mathcal{E})$ *is a locally small.*

If $\mathcal{E}$ *has countable coproducts then they are also equivalent to the following conditions:*

(3) $\mathrm{D}^-(\mathcal{E})$ *is locally small*
(3') $\mathrm{D}^+(\mathcal{E})$ *is locally small.*
(4) $\mathrm{D}(\mathcal{E})$ *is locally small.*

Remark 3.2. If we can proof that the inclusion of $\mathcal{E}$ into its countable envelope $\tilde{\mathcal{E}}$ induces a faithful functor $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\tilde{\mathcal{E}})$, then we can drop the assumption that $\mathcal{E}$ has to have countable coproducts because we could just replace $\mathcal{E}$ by its countable envelop (cp. subsubsection 2.2.1) .

Let us state a very easy corollary

Corollary 3.3. *If* $\mathcal{E}$ *is an exact category. We denote by* $\mathcal{E}^{ic}$ *its idempotent completion. Then:* $\mathrm{D}^b(\mathcal{E})$ *is a locally small if and only if* $\mathrm{D}^b(\mathcal{E}^{ic})$ *is locally small.*

PROOF. As $\mathcal{E}$ is homologically exact in $\mathcal{E}^{ic}$ (cf. [**2**]), we conclude that every $\mathrm{Ext}^n_{\mathcal{E}^{ic}}((X,e_1),(Y,e_2))$ is a summand of $\mathrm{Ext}^n_{\mathcal{E}}(X,Y)$. Then use Theorem 3.1. □

Corollary 3.4. *For every locally coherent exact category* $\mathcal{E}$*, the derived categories* $\mathrm{D}^*(\mathcal{E})$ *with* $* \in \{b, \pm, \emptyset\}$ *are locally small.*

PROOF. Locally coherent exact categories have enough injectives therefore all Ext^n are set-valued. They have countable coproducts therefore the theorem directly applies. □

For the proof of Theorem 3.1 we observe the following:

Lemma 3.5. *Let* $\mathcal{G}$ *be a triangulated category together with a homological functor* $F\colon \mathcal{G} \to \widehat{\mathrm{Ab}}$*. Consider the full subcategory* $\mathcal{C}$ *of* $\mathcal{G}$ *on the objects* X *such that* $F(\Sigma^n X) \in \mathrm{Ab}$ *for all* $n \in \mathbb{Z}$*. Then* $\mathcal{C}$ *is a thick subcategory of* $\mathcal{G}$*.*

PROOF. This is an immediate consequence of the fact that Ab is a closed under extensions. □

Then we have this easy corollary:

Corollary 3.6. *Let* $\mathcal{T}$ *be a triangulated category, and* $\mathcal{C} \subseteq \mathcal{T}$ *be a full locally small subcategory and assume that* $\mathcal{C}$ *is closed under all shifts. Then* $\mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$ *is also locally small.*

PROOF. Let $\mathcal{D}$ be the full subcategory of $\mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$ on the objects X such that $\mathrm{Hom}_{\mathcal{T}}(X,C)$ and $\mathrm{Hom}_{\mathcal{T}}(C,X)$ are small sets for any $C \in \mathcal{C}$. Note that $\mathcal{D}$ is closed under arbitrary shift by the hypothesis on $\mathcal{C}$. By Lemma 3.5, it follows that $\mathcal{D}$ is thick, and since it contains $\mathcal{C}$ we deduce that $\mathcal{D} = \mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$. □

Proof of Theorem 3.1. (n) implies (1) for $n \in \{2,3,3',4\}$: Follows directly since for all X, Y in $\mathcal{E}$ we have $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(X, \Sigma^n Y)$ for all $n \in \mathbb{Z}$ where $\mathrm{Ext}^0_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{E}}(X,Y)$ and $\mathrm{Ext}^{<0}_{\mathcal{E}}(X,Y) = 0$, cp [**14**, Prop. 4.2.11]. As $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^*(\mathcal{E})$ is fully faithful for $* = \pm, \emptyset$, the other implications also follow.
(1) implies (2): Just take $\mathcal{T} = \mathrm{D}^b(\mathcal{E})$ and $\mathcal{C} = \mathrm{add}(\mathcal{E}[n], n \in \mathbb{Z})$. Then, by definition $\mathcal{C}$ is locally small if and only if (1) in the Theorem 3.1 is fulfilled. Since $\mathrm{Thick}_{\mathrm{D}^b(\mathcal{E})}(\mathcal{C}) = \mathrm{D}^b(\mathcal{E})$, it follows (2) from the previous Corollary.
Now assume that $\mathcal{E}$ has countable coproducts, let $\mathcal{A}$ be the underlying additive category. Then, every object X in $\mathrm{K}^-(\mathcal{A})$ fits into a triangle

$$\bigoplus_{n \leq 0} \sigma_{\geq n} X \to \bigoplus_{n \leq 0} \sigma_{n \geq 0} X \to X \xrightarrow{+1}$$

where $\sigma_{\geq n} X$ is the (brutal) truncation of a complex X is defined as $\cdots 0 \to X^n \to X^{n+1} \to \cdots$ (see e.g. [**14**, Ex. 4.2.2]). By [**14**, Lemma 1.1.8], the Verdier quotient functor commutes with countable coproducts and maps this distinguished triangle to a distinguished triangle in $\mathrm{D}^-(\mathcal{E})$.
(2) implies ((3) and (3')): This means the extension-closure of $\mathrm{D}^b(\mathcal{E})$ in $\mathrm{D}^-(\mathcal{E})$ is the whole triangulated category and therefore Corollary 3.6 implies the claim. The argument for $\mathrm{D}^+(\mathcal{E})$ is analogue using brutal truncation in the other direction.
((3) and (3')) implies (4): Now, we look at the unbounded homotopy category $\mathrm{K}(\mathcal{A})$. Brutal truncation yields a distinguished triangle

$$\sigma_{\geq 0} X \to X \to \sigma_{<0} X \xrightarrow{+1}$$

Then passing to the Verdier quotient we can look at the smallest additive subcategory of $\mathrm{D}(\mathcal{E})$ containing $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^-(\mathcal{E})$ and call this category $\mathcal{C} = \mathrm{D}^-(\mathcal{E}) \vee \mathrm{D}^+(\mathcal{E})$. If $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^-(\mathcal{E})$ are locally small, then $\mathcal{C}$ is also locally small by Lemma 3.7. Clearly $\mathcal{C}$ is closed under all shifts. The previous triangle shows that $\mathcal{C}$ is a thick generator for $D(\mathcal{E})$ and therefore $\mathrm{D}(\mathcal{E})$ is also a locally small category.

□

Lemma 3.7. *Let $\mathcal{E}$ be an exact category with countable coproducts. Assume that $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^-(\mathcal{E})$ are locally small. Then for every $X \in \mathrm{D}^-(\mathcal{E}), Y \in \mathrm{D}^+(\mathcal{E})$ we have that $\mathrm{Hom}_{\mathrm{D}(\mathcal{E})}(X,Y)$ and $\mathrm{Hom}_{\mathrm{D}(\mathcal{E})}(Y,X)$ are sets.*

Proof. For X in $\mathrm{D}^-(\mathcal{E})$ we find a distinguished triangle (see above):

$$\bigoplus_{n \leq 0} \sigma_{\geq n} X \to \bigoplus_{n \leq 0} \sigma_{n \geq 0} X \to X \xrightarrow{+1}$$

Then apply $\mathrm{Hom}(-,Y)$ and apply $\mathrm{Hom}(Y,-)$ with $Y \in \mathrm{D}^+(\mathcal{E})$. The rest is obvious (use the five terms of the long exact sequences with $\mathrm{Hom}(X,Y)$ and resp $\mathrm{Hom}(Y,X)$ in the middle, $\mathrm{Hom}(\bigoplus X_i, Y) \cong \prod_i \mathrm{Hom}(X_i, Y)$ and $\mathrm{Hom}(Y, \bigoplus_i X_i) \cong \mathrm{Hom}(Y, X_i)$ implies that the other four terms are small abelian groups). □

Remark 3.8. As far as we know there is no characterization of all higher Ext-functors in an exact category being set-valued. Nevertheless there is the following list of examples where this is fulfilled.

(a) If $\mathcal{E}$ is essentially small. One reference for this [**14**], Lemma 4.2.17 together with Prop. 4.3.4.
(b) If $\mathcal{E}$ has enough projectives or enough injectives (more generally of $\mathrm{K}^b(\mathcal{E})$ has enough K-projectives or enough K-injectives). A reference for this [**14**], Cor. 4.3.2, p.123.
(c) If $\mathcal{E}$ has a small generator or a cogenerator.

We give some examples of categories to see that (1) can not be weakened.

Example 3.9. We will look at representations of *quivers* where we allow the vertices and arrows to be a proper classes and representations are to be understood as in vector spaces over some field K and of finite total dimension. Furthermore, we will always impose the relations that the composition of any two composable arrows is zero. These are abelian categories. Let's construct some examples. We fix a proper class M (i.e. this is not a set).

(1) We look at Q with two vertices 1 and 2 and arrows $a_m\colon 1 \to 2$ for each m in M. Then this gives an abelian category with Hom-sets but $\mathrm{Ext}^1(S_1, S_2)$ is not a set since we find for every m in M a short exact sequence

$$0 \to S_2 \to I_m \to S_1 \to 0$$

with I_m is the representation with $a_m = \mathrm{id}_{\mathrm{K}}$ and $a_n = 0$ for all $n \neq m$. These are pairwise non-isomorphic.

(2) Now we look at a quiver with vertices $1, 2$ and $v_m, m \in M$ and arrows $b_m\colon 1 \to v_m, c_m\colon v_m \to 2$ for every $m \in M$ and the relations $c_m b_m = 0$ for all $m \in M$. Then this gives an abelian category with Hom and Ext^1 are set-valued. But for every $m \in M$ we have an exact sequence

$$0 \to S_2 \to J_m \to L_m \to S_1 \to 0$$

with J_m 2-dimensional given by $b_n = 0$ for all $n \in M$ and $c_m = \mathrm{id}_K$, $c_n = 0$ for $n \neq m$ and L_m 2-dimensional given by $b_m = \mathrm{id}_{\mathrm{K}}$, $b_n = 0$ for $n \neq m$, $c_n = 0$ for all $n \in M$. Again these are pairwise non-isomorphic, so $\mathrm{Ext}^2(S_1, S_2)$ is not a set.

(3) Fix an integer $t \geq 1$. Now look at the quiver with vertices $1, 2$ and $v_{1,m}, \ldots, v_{t,m}$ for every $m \in M$ and arrows $a_{1,m}\colon 1 \to v_{1,m}, a_{2,m}\colon v_{1,m} \to v_{2,m}, \ldots, a_{t,m}\colon v_{t-1,m} \to v_{t,m}, a_{t+1,m}\colon v_{t,m} \to 2$. Again we impose $a_{i+1,m} a_{i,m} = 0$ for $1 \leq i \leq t$ and $m \in M$. With a similar argument as in the previous cases one shows that $\mathrm{Ext}^{t+1}(S_1, S_2)$ is not a set. But Hom and Ext^i are set-valued for $1 \leq i \leq t$.

4. Keller's approach to deriving additive functors between exact categories

Or, as much as we could verify of it. We summarize the construction of [**12**], in a simple language. If $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ and $\mathcal{F} = (\mathcal{B}, \mathcal{S}')$ we look at the class $\mathcal{S}_f := \{(i, d) \in \mathcal{S} \mid (f(i), f(\mathrm{D})) \in \mathcal{S}'\}$ which we call f-exact sequences. If $(i, d) \in \mathcal{S}_f$, we call i an f-inflation and d an f-deflation.

Definition 4.1. Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ an exact category. Let $\mathcal{X}$ be a class of kernel-cokernel pairs on $\mathcal{A}$. We call an object $X \in \mathcal{A}$ **right $\mathcal{X}-$acyclic** if every $X \xrightarrow{i} Y \xrightarrow{d} Z$ with $(i, d) \in \mathcal{S}$ fulfills $(i, d) \in \mathcal{X}$. We call $\mathcal{C}_{\mathcal{X}}$ the full subcategory of $\mathcal{A}$ of right $\mathcal{X}$-acyclic objects.

Definition 4.2. Let $\mathcal{S}_f^{pb}$ be the class of all $\sigma \in \mathcal{S}$ such that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "="'] & Y' \arrow[r, two heads] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, tail] & Y \arrow[r, two heads] & Z \end{tikzcd}$$

with $\sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0)$ in the upper row and an f-exact short exact sequence $\sigma_f \in \mathcal{S}_f$ in lower row.

We define the category of **right f-acyclics** as the full subcategory $\mathcal{C} := \mathcal{C}_{\mathcal{S}_f^{pb}}$ of $\mathcal{A}$.

This will later be used to show that this notion is the one mentioned in the literature.

Lemma 4.3. *The following are equivalent for $X \in \mathcal{E} = (\mathcal{A}, \mathcal{S})$.*

(1) *X is right f-acyclic*
(2) *Every morphism $M \to X$ in $\mathrm{K}^+(\mathcal{A})$ with M an $\mathcal{E}$-acyclic complex factors as $M \to M' \to X$ with M' an $\mathcal{E}$-acyclic and f-exact complex (i.e. the corresponding short exact sequences are all in $\mathcal{S}_f$).*

Proof. Assume $f\colon M \to X$ is a chain map with $M = (M^n, d^n)$ acyclic and X in degree 0. Then such a map is given by a morphism $h\colon N = \operatorname{Im} d^0 \to X$. Therefore, it always factors over the push-out of $\operatorname{Im} d^0 \rightarrowtail M^1 \twoheadrightarrow \operatorname{Im} d^1$ along h. This means we may assume wlog that M is a short exact sequence $\sigma\colon X \rightarrowtail Y \twoheadrightarrow Z$ (in degrees $0,1,2$) and the morphism $M \to X$ is the identity on X in degree 0 and zero in all other degrees.
Assume (2): If this now factors over an f-exact short exact sequence $\sigma\colon X' \rightarrowtail Y' \twoheadrightarrow Z'$ then we find that σ' pulls back to $\sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0)$.
Assume (1): If $\sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0)$ is a pull back of $\sigma_f \in \mathcal{S}_f$, then we have that $M \to X$ factors over $\sigma \to \sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0) \to \sigma_f =: M'$. $\square$

Remark 4.4. Recall in Chapter 1 we showed: If f is left or right exact or if f is fully faithful with extension-closed essential image, then $\mathcal{S}_f$ is already an exact structure.

Lemma 4.5. *Assume that $\mathcal{E}_f = (\mathcal{A}, \mathcal{S}_f)$ is an exact substructure of $\mathcal{E}$. Then an object X in $\mathcal{A}$ is right f-acyclic if and only if the natural morphism*

$$\operatorname{Ext}^1_{\mathcal{E}_f}(Y,X) \to \operatorname{Ext}^1_{\mathcal{E}}(Y,X)$$

is an isomorphism for all $Y \in \mathcal{A}$. In particular, $\mathcal{C}$ is extension- and inflation-closed in $\mathcal{E}$. Furthermore, Given an $\mathcal{E}$-short exact sequence $X \xrightarrow{i} Y \xrightarrow{d} Z$ with X,Y,Z in $\mathcal{C}$ then it is f-exact (i.e $(i,d) \in \mathcal{S}_f$)

Proof. This is easy to see. $\square$

Definition 4.6. If $\mathcal{E}_f = (\mathcal{A}, \mathcal{S}_f)$ is an exact substructure of $\mathcal{E}$, we say f has enough right f-acyclics if $\mathcal{C}$ is cogenerating in $\mathcal{E}$ (i.e. every object $X \in \mathcal{E}$ there exists an inflation $X \to C$ with $C \in \mathcal{C}$). Then it is a coresolving subcategory and $i\colon \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{E})$ is a triangle equivalence and we have a triangle functor

$$rf\colon \mathrm{D}^+(\mathcal{E}) \xrightarrow{i^{-1}} \mathrm{D}^+(\mathcal{C}) \xrightarrow{f} \mathrm{D}^+(\mathcal{F})$$

In general, we do not see why $\mathcal{C}$ should be extension-closed in $\mathcal{E}$ nor why it should satisfy condition (C2) from [12] (this is claimed in [12, Lemma 15.3]). As this is not proven in loc. cit. one should treat it as a conjecture.
There are two strategies how one can pass to an extension-closed subcategory $\mathcal{C}_2 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}$ $i = 1,2$. Either $X \in \mathcal{C}_1$ are all objects such that every $\mathcal{E}$-short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ is also f-exact, or one looks at the maximal exact substructure $\mathcal{E}_{f,max} \le \mathcal{E}$ making the functor f exact and then $X \in \mathcal{C}_2$ are all objects such that all $\mathcal{E}$-short exact sequences $X \rightarrowtail Y \twoheadrightarrow Z$ are already $\mathcal{E}_{f,max}$-short exact sequences. The advantage of $\mathcal{C}_2$ is that we can generalize the previous Lemma imediately and conclude that $\mathcal{C}_2$ is extension- and inflation-closed.

Example 4.7. Observe that the full subcategory on $\mathcal{E}$-injectives $\mathcal{I} = \mathcal{I}(\mathcal{E})$ is contained in $\mathcal{C}_2$. So if $\mathcal{E}$ has enough injectives then it also has enough right f-acyclics (using the subcategory $\mathcal{C}_2$).

4.1. $rf = Rf$. We explain now why this notion of right f-acyclic coincides with the one defined in [12] and then conclude rf is the right derived functor Rf of f.
Generally, given a triangle functor $F\colon \mathcal{T} \to \mathcal{T}'$ and a fixed Verdier quotient $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{M} =: \mathbb{D}$ (we think of this as a derived category). We will choose $F\colon \mathrm{K}^+(\mathcal{A}) \xrightarrow{\mathrm{K}^+(f)} \mathrm{K}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{F})$ and $Q\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{E})$. Keller constructs (following roughly Deligne) a (possibly zero) triangulated subcategory $\mathcal{U}$ of $\mathcal{T}$ such that $\mathcal{U}/(\mathcal{U} \cap \mathcal{M})$ is a triangulated subcategory of $\mathcal{T}/\mathcal{M}$ such that $F|_{\mathcal{U}}$ factors over a triangle functor $\mathcal{U}/\mathcal{U} \cap \mathcal{M} \to \mathcal{T}'$ (and this triangle functor is isomorphic to the restriction of Deligne's Rf):

Definition 4.8. A triangulated subcategory $\mathcal{U}$ of $\mathcal{T}$ is called **right cofinal** (wrt $\mathcal{M}$) if every morphism $M \to X$ with $M \in \mathcal{M}$, $X \in \mathcal{U}$ factors as $M \to M' \to X$ with $M' \in \mathcal{U} \cap \mathcal{M}$.

In this case, the induced triangle functor $\mathcal{U}/(\mathcal{U}\cap\mathcal{M})\to\mathbb{D}$ is fully faithful (cf. [**12**, 10.3]).
Now let $\ker F$ be the thick subcategory of $\mathcal{T}$ with objects $T\in\mathcal{T}$ such that $F(T)\cong 0$.
This definition differs from [**12**, Lem. 14.1], explanation see below:

Definition 4.9. We say $X\in\mathcal{T}$ is F**-split** if every morphism $M\to X$ in $\mathcal{T}$ with $M\in\mathcal{M}$ factors as $M\to M'\to X$ with $M'\in\mathcal{M}\cap\ker F$.
We call $\mathcal{U}$ be the full subcategory of $\mathcal{T}$ with F-split objects.

Remark 4.10. The characterization of F-split objects in [**12**, 14.1, (iii)] does not seem to give a triangulated subcategory. We use Lipman's stronger definition ([**16**], Def. (2.2.5), Ex. 2.2.8(D)) as in that case the F-split objects are shown to be a triangulated subcategory.

Lemma 4.11. *(cf.* [**16**, Lemma 2.2.5.1]*) $\mathcal{U}$ is a triangulated subcategory of $\mathcal{T}$.*

Observe, that $\mathcal{M}\cap\ker F\subseteq\mathcal{M}\cap\mathcal{U}$ because given $X\in\mathcal{M}\cap\ker F$ and $M\to X$ a morphism with $M\in\mathcal{M}$ we can consider the factorization over $M'=X\xrightarrow{1_X}X$, to see that $X\in\mathcal{U}$. In particular, $\mathcal{U}$ is right cofinal in $\mathcal{T}$.
But also by definition $X\in\mathcal{U}\cap\mathcal{M}\subseteq\ker F$ because now we can take $1_X\colon X\to X$ and it has to factor over $M'\in\ker F\cap\mathcal{M}$ as $X\in\mathcal{U}$. This means $1_{F(X)}=F(1_X)\cong 0$ and therefore $F(X)\cong 0$. By the universal property of the Verdier quotient, F factors over a triangle functor $\mathcal{U}/(\mathcal{U}\cap\mathcal{M})\to\mathcal{T}'$.

Then we just cite the following result as we do not remind the reader of Deligne's definition of the derived functor.

Lemma 4.12. *(*[**12**, section 14]*) The triangle functor $\mathcal{U}/\mathcal{U}\cap\mathcal{M}\to\mathcal{T}'$ coincides with Deligne's $RF|_{\mathcal{U}/\mathcal{M}\cap\mathcal{U}}$.*

We call $\mathcal{C}^{Del}:=\mathcal{U}\cap\mathcal{A}\subseteq\mathrm{K}^+(\mathcal{A})$ the category of **right** F**-acyclics** (i.e. these are the F-split stalk complexes in the homotopy category). Then:

Corollary 4.13. *(of Lemma 4.3)* $\mathcal{C}^{Del}=\mathcal{C}$

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