An application of tilting theory to infinite type A quiver representations

1. Synopsis

Let Q be an infinite quiver, we look at those whose category of finitely presented quiver representations $\operatorname{rep}^+(Q)$ over a field K is an hereditary abelian category with enough projectives which is Hom- and Ext-finite (i.e. these take values in finite-dimensional K-vector spaces). The easiest class of examples of such infinite quivers are quivers Q which are tree-shaped quivers with finitely many branching points. The abelian categories $\operatorname{rep}^+(Q)$ are always one-sided Auslander-Reiten categories, cf. [**1**]. The purpose of this chapter is to start studying tilting subcategories in $\operatorname{rep}^+(Q)$ for Q an infinite quiver with sufficient finiteness conditions. The obvious first example are quivers of type $\mathbb{A}_\infty$.

We prove the following theorem using tilting theory:

Theorem 1.1. *(cf. Theorem 4.9) Let Q and Q' be two quivers of type $\mathbb{A}_\infty$. Then, there exists a triangle equivalence*

$$\mathrm{D}^b(\operatorname{rep}^+(Q)) \to \mathrm{D}^b(\operatorname{rep}^+(Q'))$$

if and only if one of the following three cases holds

(a) *Q and Q' have a left infinite path*
(b) *Q and Q' have a right infinite path*
(c) *Q and Q' have no infinite path*

In each of the cases the derived equivalence is obtained by composing at most two derived equivalences induced from a tilting subcategory.

The same question can be asked for the other infinite Dynkin types. We give an idea how to answer this in section 5.

2. Preliminaries

We remark that in our situtation tilting subcategories are maximal self-orthogonal:

Remark 2.1. Let $\mathcal{A}$ be an exact category and assume $\operatorname{gldim} \mathcal{A} < \infty$. Let $\mathcal{T}$ be a tilting subcategory and $\mathcal{T} \subseteq \mathcal{S}$ with $\mathcal{S}$ selforthogonal in $\mathcal{A}$. Then we have $\mathcal{T} = \mathcal{S}$, the proof goes as follows:
Since $\operatorname{Ext}^{>0}(\mathcal{T}, \mathcal{S}) = 0$ implies $\mathcal{S} \subset \mathcal{T}^\perp$ and because $\operatorname{gldim} \mathcal{T}^\perp \leq \operatorname{gldim} \mathcal{A} < \infty$ and $\mathcal{T}^\perp$ wep. given by $\mathcal{T}$ it follows that $\operatorname{Ext}^{>0}(\mathcal{S}, \mathcal{T}^\perp) = 0$ and therefore $\mathcal{S} \subset \mathcal{P}(\mathcal{T}^\perp) = \mathcal{T}$.

Lemma 2.2. *Assume that $\mathcal{T}$ is a 1-tilting subcategory in an abelian category $\mathcal{A}$. Then the category of finitely presented functors $\operatorname{mod}_1 \mathcal{T}$ has enough projectives, i.e. equals $\operatorname{mod}_\infty \mathcal{T}$ and is abelian.*

Proof. We show first the inclusion $\operatorname{mod}_1 \mathcal{T} \subseteq \operatorname{mod}_\infty \mathcal{T}$: Let $f\colon T_1 \to T_0$ be a morphism in $\mathcal{T}$, we denote by $f'\colon T_1 \to \operatorname{Im} f$ the induced morphism to the image. We have $\operatorname{Im} f \in \operatorname{gen}_\infty \mathcal{T}$ (using $\operatorname{pres} \mathcal{T} = \operatorname{gen} \mathcal{T} = \operatorname{gen}_\infty \mathcal{T}$, using [**7**], Lem 4.4). So we find an exact sequence $T_2' \xrightarrow{g} T_1' \xrightarrow{h} \operatorname{Im} f \to 0$

with $T_i' \in \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{A}}(T, T_2') \to \mathrm{Hom}_{\mathcal{A}}(T, T_1') \to \mathrm{Hom}_{\mathcal{A}}(T, \mathrm{Im}\, f) \to 0$ for all T in $\mathcal{T}$. Let us form the pullback of h along f' in the abelian category $\mathcal{A}$

$$\begin{tikzcd}
 & \ker f \arrow[r, equal] \arrow[d, tail] & \ker f \arrow[d, tail, "i"] \\
\ker h \arrow[r, tail, "j"] \arrow[d, equal] & H \arrow[r, two heads, "\tilde{h}"] \arrow[d, two heads] & T_1 \arrow[d, two heads, "f'"] \\
\ker h \arrow[r, tail] & T_1' \arrow[r, two heads, "h"] & \mathrm{Im}\, f
\end{tikzcd}$$

One can observe that the second row is even split exact (but we are not going to use this). Now, we look at the induced morphism $g' \colon T_2' \to \mathrm{Im}\, g = \ker h$. Using the bicartesian commutative diagram we conclude $f'\tilde{h}jg' = 0$ and so by the universal property of the kernel there is a unique morphism $t \colon T_2 \to \ker f$ such that $it = \tilde{h}jg'$. Now we check that $it \colon T_2' \to T_1$ is a weak kernel of f. For that it suffices to see that for every T in $\mathcal{T}$ the map $\mathrm{Hom}(T, T_2') \twoheadrightarrow \mathrm{Hom}(T, \ker f)$ is surjective. Let T be an object in $\mathcal{T}$ and $s \colon T \to \ker f$ a morphism. Using the bicartesian commuting square, we see that $T \to \ker f \to H \to T_1' \xrightarrow{h} \mathrm{Im}\, f$ is zero, so it has to factor uniquely over a morphism $T \to \ker h$. But as g' is a right $\mathcal{T}$-approximation of $\ker h$, it follows that there exists a morphism $s' \colon T \to T_2'$ with $ts' = s$.
As every morphism has a weak kernel, the claim follows, cf. e.g. [4, Lemma 2.1.6].

□

There is also the following other case when we can conclude that $\mathrm{mod}_\infty \mathcal{T}$ is abelian.

Lemma 2.3. *If $\mathcal{T}$ is a contravariantly finite subcategory of an abelian category $\mathcal{A}$, then $\mathcal{T}$ has weak kernels. In particular $\mathrm{mod}_\infty \mathcal{T} = \mathrm{mod}_1 \mathcal{T}$ is abelian with enough projectives.*

Proof. Let $f \colon T \to T'$ be a morphism in $\mathcal{T}$. Take $T_f \to \ker f$ a right $\mathcal{T}$-approximation. Then, consider the composition $g \colon T_f \to \ker f \to T$. It is straightforward to see that g is a weak kernel of f. □

That a tilting subcategory is contravariantly finite is an extra property. It is equivalent to that we have a torsion class associated to it:

Definition 2.4. A pair $(\mathcal{R}, \mathcal{F})$ of full subcategories in an abelian category $\mathcal{A}$ is a torsion pair if:

(TP1) $\mathrm{Hom}(R, F) = 0$ for all $R \in \mathcal{R}, F \in \mathcal{F}$,
(TP2) For each $Z \in \mathcal{A}$ exists a short exact sequence $0 \to X \to Z \to Y \to 0$ with $X \in \mathcal{R}, Y \in \mathcal{F}$.

We recall the following

Corollary 2.5. *(of [2, Prop.1.2]) If $\mathcal{R}$ is a torsion class in an abelian category $\mathcal{A}$, then the $\mathcal{R}$ is contravariantly finite in $\mathcal{A}$.*

Proof. By [2, Prop. 1.2], the inclusion $i \colon \mathcal{R} \to \mathcal{A}$ has a right adjoint $R \colon \mathcal{A} \to \mathcal{R}$. Then, for $Z \in \mathcal{A}$, the counit $\epsilon_Z \colon iR(Z) \to Z$ of the adjunction is a right $\mathcal{R}$-approximation. □

Lemma 2.6. *Let $\mathcal{T}$ be a 1-tilting subcategory in an abelian category. Then the following are equivalent:*

(1) *$\mathcal{T}^\perp$ is a torsion-class.*
(2) *$\mathcal{T}^\perp$ is contravariantly finite in $\mathcal{A}$.*
(3) *$\mathcal{T}$ is contravariantly finite in $\mathcal{A}$.*

Proof. Assume $\mathcal{T}^\perp$ is a torsion class, then $\mathcal{T}^\perp$ is contravariantly finite in $\mathcal{A}$ by the previous lemma. Since $\mathcal{T}$ is contravariantly finite in $\mathcal{T}^\perp$ (since $\mathcal{T}^\perp$ has enough projectives given by $\mathcal{T}$) it follows that $\mathcal{T}$ is contravariantly finite in $\mathcal{A}$. So we have (1) implies (2) implies (3)
Conversely, assume (3), i.e. that $\mathcal{T}$ is contravariantly finite in $\mathcal{A}$. Define $\mathcal{R} = \mathcal{T}^\perp$ and $\mathcal{F} = \{F \in \mathcal{A} \mid \mathrm{Hom}(T, F) = 0 \,\forall\, T \in \mathcal{T}\}$. Let $R \in \mathcal{R}, F \in \mathcal{F}$ and $f\colon R \to F$ be a morphism. By definition, there exists an epimorphism $p\colon T \to R$ with $T \in \mathcal{T}$ and $f \circ p = 0$. This implies $f = 0$ and (TP1).
Now let $Z \in \mathcal{A}$ be arbitrary. By assumption, there exists a right $\mathcal{T}$-approximation $f_Z\colon T_Z \to Z$, in particular $X = \mathrm{Im}(f_Z) \in \mathcal{R}$. Let $Y := \mathrm{coker}(f_Z)$. We consider the short exact sequence $0 \to X \xrightarrow{j} Z \to Y \to 0$ and apply $\mathrm{Hom}(T, -)$ with $T \in \mathcal{T}$. We look at $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Z)$ and want to see that this map is surjective. So, given $g\colon T \to Z$, we use that there is an $h\colon T \to T_Z$ such that $f_Z \circ h = g$. Since $f_Z\colon T_Z \xrightarrow{q} X \xrightarrow{j} Z$ factors over its image as $f_Z = j \circ q$, it follows $g = j(qh)$ and therefore $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Z)$ is surjective. This implies that $Y \in \mathcal{F}$ and therefore (TP2). □

Definition 2.7. We say that an object X in $\mathcal{A}$ is **noetherian** if it satisfies the ascending chain (acc) condition, i.e. whenever there is a chain of subobjects of X

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

then it eventually stabilizes.
Let $\mathcal{C}$ be a small category, then one says $\mathrm{Mod}\,\mathcal{C}$ (also denoted by $\mathrm{Rep}\,\mathcal{C}$) is **noetherian** if every finitely generated $\mathcal{C}$-module X fulfills the (acc) for chains of finitely generated submodules.

Lemma 2.8. *Let k be a field. Let $\mathcal{A}$ be an abelian Hom-finite k-category and $\mathcal{T}$ a 1-tilting subcategory with countably many indecomposables in it. If every object in $\mathcal{A}$ is noetherian, then $\mathcal{T}$ is contravariantly finite in $\mathcal{A}$.*

Proof. Let X be in $\mathcal{A}$. We choose a numbering of the indecomposables of $\mathcal{T}$, e.g. $T_n, n \in \mathbb{N}$. We define $X_n \subseteq X$ to be $\sum_{f \in \mathrm{Hom}_{\mathcal{A}}(\bigoplus_{i=1}^n T_i, X)} \mathrm{Im}\, f$. By assumption there is an $N \in \mathbb{N}$ such that $X_N = X_n$ for all $n \geq N$. This means that every morphism $T \to X$, $T \in \mathcal{T}$ must factor over X_N. We observe $X_N \in \mathrm{pres}(\mathcal{T}) = \mathrm{gen}(\mathcal{T}) = \mathcal{T}^\perp$. Since $\mathcal{T}^\perp$ is an exact category with enough projectives given by $\mathcal{T}$ there is a *projective cover* $T \to X_N$, with $T \in \mathcal{T}$ and this is clearly a right $\mathcal{T}$-approximation. □

3. Representations of strongly locally finite infinite quivers

Here we follow [1]. We fix a **strongly locally finite quiver** Q, this means every vertex has finitely many arrows arriving and starting at it and for every two (possibly equal) vertices there are only finitely many paths from one to the other.
A representation (over an always fixed field K) of Q is an assignment of a K-vector space V_i to every vertex $i \in Q_0$ and a linear map $V_i \to V_j$ to every arrow $a\colon i \to j$ in Q_1. This defines the objects in an abelian category $\mathrm{Rep}(Q)$. For every vertex $x \in Q_0$ one defines a Q-representation P_x with top S_x (the one-dimensional representation supported at x) and let $\mathcal{P} := \mathrm{proj}Q = \mathrm{add}\{P_x \colon x \in Q_0\}$ be the category of finitely generated projectives in $\mathrm{Rep}(Q)$. A quiver is called **noetherian** if every object in $\mathcal{P}$ defined as above satisfies the ascending chain property (definition of [3]). In [5] this is called left noetherian and it is shown that this implies that $\mathrm{Rep}(Q)$ is a locally noetherian abelian category in loc. cit. Theorem 1.1.

Lemma 3.1. *Let Q be an infinite quiver. If the underlying graph is a (possibly infinite) tree with only finitely many branching points, then the quiver is noetherian.*

Proof. Since Q is a strongly locally finite quiver and its underlying graph is a tree with only finitely many branch points, it follows that the graph of $P(Q)_v$ has finitely many branch points for any $v \in Q$. We get that $P(Q)_v$ is barren (in the sense of [3]) and thus Q is notherian by [3, Theorem 3.6]. □

We define

$$\mathrm{rep}^+(Q) := \mathrm{mod}_1 \mathcal{P}$$

as the finitely presented Q-representations, this is an extension-closed subcategory of $\mathrm{Rep}(Q)$.

Lemma 3.2. *([1, Lem 1.14]) Let K be a field. Then $\mathrm{rep}^+(Q)$ is hereditary abelian and is a Hom-finite K-category with finite-dimensional Ext^1-groups.*

Corollary 3.3. *In particular $\mathcal{P}$ has weak kernels and $\mathrm{rep}^+(Q) = \mathrm{mod}_\infty \mathcal{P}$ has enough projectives which are given by $\mathcal{P}$.*

Following [1, Cor 2.2] an abelian Krull-Schmidt category $\mathcal{C}$ is a **right Auslander-Reiten category** if every indecomposable non-projective is ending term of an almost split sequence and all indecomposable projectives have a simple top. It is called a **left Auslander-Reiten category** if its opposite is a right Auslander-Reiten category.
We call a quiver of the form $\bullet \to \bullet \to \cdots$ a **right infinite path** and its opposite a **left infinite path**. We call a quiver $\cdots \to \bullet \to \bullet \to \cdots$ a **double infinite path**.

Theorem 3.4. *([1], Thm 3.7, Cor. 3.8) Let Q be a strongly locally finite quiver. Then*

(1) $\mathrm{rep}^+(Q)$ *is left Auslander-Reiten if and only if Q has no right infinite path.*
(2) $\mathrm{rep}^+(Q)$ *is right Auslander-Reiten if and only if Q has no left infinite path or else Q is a left infinite path or double infinite path.*
(3) $\mathrm{rep}^+(Q)$ *is Auslander-Reiten if and only if Q has no infinite path or Q is a left infinite path.*

Observe that: Q having no right infinite path is equivalent to $\mathrm{rep}^+(Q)$ coinciding with the category of finite dimensional Q-representation (i.e. Q-representations V such that $\dim_K \bigoplus_{i \in Q_0} V_i < \infty$).

We will also use the following definition.

Definition 3.5. Let $\mathcal{A}$ be a left (and/or right) abelian Auslander-Reiten category. A **weak slice** in $\mathcal{A}$ is an additively closed subcategory $\mathcal{X}$ such that the indecomposables in $\mathcal{X}$ fulfill the following:

(1) The indecomposables in $\mathcal{X}$ are all in the same component of the Auslander-Reiten quiver and they are a full representing system of the $\tau^{\pm}$-orbits.
(2) The full subquiver of the Auslander-Reiten quiver defined by the indecomposables of $\mathcal{X}$ is path-closed (i.e. if there is a path given by a sequence of arrows $X_1 \to X_2 \to \cdots \to X_n$ in the Auslander-Reiten quiver with X_1, X_n in $\mathcal{X}$, then all X_i are in $\mathcal{X}$).
(3) Given an almost split sequence $M \rightarrowtail L \twoheadrightarrow N$ with one summand of L in $\mathcal{X}$, then either M or N are in $\mathcal{X}$.

We say a weak slice is a **slice** if it defines a 1-tilting subcategory of $\mathcal{A}$.

Ringel showed in [6], section 4.2: If $\mathcal{A}$ is also hereditary exact and $\mathcal{X}$ is a slice then the subcategory $\mathcal{X}$ is selforthogonal (i.e. there are no non-split n-extensions between any two objects in $\mathcal{X}$ for all $n \geq 1$).

3.1. Reflection at a set of sinks. Let Q be a strongly locally finite quiver and $\mathbf{a} \subset Q_0$ a subset consisting of sinks (this can be a single vertex or it may also be an infinite set). We write $\mu_{\mathbf{a}}(Q)$ for the quiver (Q_0', Q_1') with $Q_0' = Q_0$ and
$Q_1' = \{\alpha \colon i \to j \mid j \notin \mathbf{a}\} \cup \{\alpha^* \colon a \to i \mid \alpha \colon i \to a, a \in \mathbf{a}\}$.
To distinguish the finitely generated projective $\mu_{\mathbf{a}}(Q)$-representations from those for Q, we denote them by $\overline{P}_x$, $x \in (\mu_{\mathbf{a}}(Q))_0 = Q_0$.
We define the reflection functor

$$\mathbb{S}_{\mathbf{a}} \colon \mathrm{Rep}(Q) \to \mathrm{Rep}(\mu_{\mathbf{a}}(Q))$$

as follows, for a Q representation M and every $a \in \mathbf{a}$ we have a linear map $M_\alpha\colon M_i \to M_a$ for every $\alpha\colon i \to a$. This induces a linear map from the direct sum

$$0 \to N_a \to \bigoplus_{\alpha \in Q_1\colon \alpha\colon i \to a} M_i \to M_a$$

where we call N_a the kernel of this map. We define $(\mathbb{S}_{\mathbf{a}}(M))_x = M_x$ for $x \notin \mathbf{a}$ and $(\mathbb{S}_{\mathbf{a}}(M))_a = N_a$ for every $a \in \mathbf{a}$. On all arrows α in Q_1' not ending at an $a \in \mathbf{a}$, we define $(\mathbb{S}_{\mathbf{a}}(M))_\alpha = M_\alpha$. An arrow $\alpha^*\colon a \to i$ in Q_1' with $a \in \mathbf{a}$, corresponds by definition to an arrow $\alpha\colon i \to a$ in Q_1, therefore we we can define $\mathbb{S}_{\mathbf{a}}(M)_{\alpha^*}\colon N_a \to \bigoplus_{\beta\colon j \to a} M_j \xrightarrow{pr_\alpha} M_i$.
It is clear that this functor restricts to finite-dimensional quiver representations. It is not immediately clear that $\mathbb{S}_{\mathbf{a}}$ would restrict to the subcategory of finitely represented quiver representations.
We look at the special tilting subcategory in $\operatorname{rep}^+(Q)$ with respect to $\mathcal{M} = \operatorname{add}\{\mathcal{P}_x \mid x \notin \mathbf{a}\}$. For $a \in \mathbf{a}$, the following is the $\mathcal{M}$-approximation of P_a

$$0 \to P_a \to \bigoplus_{\alpha \in Q_1\colon \alpha\colon i \to a} P_i.$$

Let R_a be the cokernel, $\mathcal{T}_{\mathbf{a}} = \mathcal{M} \vee \operatorname{add}\{R_a \mid a \in \mathbf{a}\}$. Observe that:

Lemma 3.6. *The reflection functor restricts to an equivalence of categories*

$$\mathcal{T}_{\boldsymbol{a}} \to \mathcal{P}(\operatorname{rep}^+(\mu_{\boldsymbol{a}}(Q)))$$

mapping $P_x \mapsto \overline{P}_x$ for $x \notin \boldsymbol{a}$ and $R_a \mapsto \overline{P}_a$ for $a \in \boldsymbol{a}$.
We have the following commutative diagram

$$\begin{tikzcd} \operatorname{Rep}(Q) \arrow[r, "\mathbb{S}_{\boldsymbol{a}}"] \arrow[dr, "\Phi"] & \operatorname{Rep}(\mu_{\boldsymbol{a}}(Q)) \\ & \operatorname{Mod}\mathcal{T}_{\boldsymbol{a}} \arrow[u, "\cong"] \end{tikzcd}$$

with $\Phi(M) = \operatorname{Hom}_{\operatorname{Rep}(Q)}(-, M)|_{\mathcal{T}_a}$.

In particular, we can restrict $\mathbb{S}_{\mathbf{a}}$ to $\mathcal{T}_{\mathbf{a}}^{\perp} = \operatorname{gen}\mathcal{T}_{\mathbf{a}} \subseteq \operatorname{rep}^+(Q)$, i.e. to a functor

$$\mathbb{S}_{\mathbf{a}}\colon \mathcal{T}_{\mathbf{a}}^{\perp} \to \operatorname{mod}_\infty - \mathcal{T}_a \cong \operatorname{rep}^+(\mu_a(Q))$$

But every indecomposable not in $\operatorname{gen}\mathcal{T}_{\mathbf{a}}$ is a simple S_a, $a \in \mathbf{a}$ and $\mathbb{S}_{\mathbf{a}}(S_a) = 0$ which is finitely presented. Therefore, we have a well-defined reflection functor

$$\mathbb{S}_{\mathbf{a}}\colon \operatorname{rep}^+(Q) \to \operatorname{rep}^+(\mu_{\mathbf{a}}(Q))$$

which can be identified with the *tilting functor* of the tilting category $\mathcal{T}_{\mathbf{a}}$.

Proof. By definition we have that $\mathbb{S}_{\mathbf{a}}(P_x) = \overline{P}_x$ for $x \notin \mathbf{a}$, and $\mathbb{S}_a(R_a) = \overline{P}_a$ for $a \in \mathbf{a}$.
As both categories are Krull-Schmidt categories, it is enough to show that $\mathbb{S}_{\mathbf{a}}$ induces isomorphisms on Hom-spaces of indecomposables. For M in $\mathcal{T}_a$, $x \notin \mathbf{a}$ we have natural isomorphisms

$$\operatorname{Hom}_{\operatorname{rep}^+(Q)}(P_x, M) \cong M_x \cong \operatorname{Hom}_{\operatorname{rep}^+(\mu_{\mathbf{a}}(Q))}(\mathbb{S}_{\mathbf{a}}(P_x), \mathbb{S}_{\mathbf{a}}(M))$$

For $x = a \in \mathbf{a}$, we apply $\operatorname{Hom}_{\operatorname{rep}^+(Q)}(-, M)$ to the short exact sequence
$0 \to P_a \to \bigoplus_{\alpha\colon i \to a} P_i \to R_a \to 0$

$$0 \to \operatorname{Hom}(R_a, M) \to \bigoplus_{\alpha\colon i \to a} \operatorname{Hom}(P_i, M) \to \operatorname{Hom}(P_a, M)$$

Now, by the first natural isomorphism this left exact sequence identifies naturally with

$$0 \to N_a \to \bigoplus_{\alpha\colon i \to a} M_i \to M_a$$

In particular $\operatorname{Hom}_Q(R_a, M) \cong N_a \cong \operatorname{Hom}_{\mu_a(Q)}(\overline{P}_a, \mathbb{S}_{\mathbf{a}}(M))$. $\square$

Corollary 3.7. *In particular, the reflection functor induces a triangle equivalence on the bounded derived categories*

$$\boldsymbol{S}_{\boldsymbol{a}}^{+} \colon \mathrm{D}^b(\mathrm{rep}^+(Q)) \to \mathrm{D}^b(\mathrm{rep}^+(\mu_{\boldsymbol{a}}(Q))).$$

We write $\boldsymbol{S}_{\boldsymbol{a}}^{-}$ for the quasi-inverse of $\boldsymbol{S}_{\boldsymbol{a}}^{+}$.

Here $\mathbf{S}_{\mathbf{a}}^{-}$ can be constructed dually using the special cotilting subcategory associated to the set of sources.

4. Representations of infinite quivers of type A-infinity

We call orientations of the following graph

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } \cdots$$

quivers of type $\mathbb{A}_\infty$. For every such quiver, $a \leq b$ in $\mathbb{N}$, we define the interval module $E_{a,b}$ as the indecomposable module with dimension vector $(\underline{\dim}\, E_{a,b})_i = 1$ if $a \leq i \leq b$ and zero else.

4.1. A-infinity.

4.1.1. *Left infinite path.* We first look at the following infinite quiver Q

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$$

Let $\mathcal{A} = \mathrm{rep}^+(Q)$ be the category of finite-dimensional Q-representations. All indecomposables are finite-dimensional interval modules, with projectives

$$P_n = E_{1,n}, \quad n \in \mathbb{N}, \quad \text{set } \mathcal{P} := \mathcal{P}(\mathcal{A}) = \mathrm{add}\{P_n \mid n \in \mathbb{N}\}.$$

In this case, we have an Auslander-Reiten category with Auslander-Reiten quiver

$$\begin{tikzcd}
 & & & & E_{1,5} & \cdots \\
 & & & E_{1,4} \arrow[ur] \arrow[dr] & & \\
 & & E_{1,3} \arrow[ur] \arrow[dr] & & E_{2,4} \arrow[ll, dotted, "\tau"'] & \cdots \\
 & E_{1,2} \arrow[ur] \arrow[dr] & & E_{2,3} \arrow[ur] \arrow[dr] \arrow[ll, dotted, "\tau"'] & & \\
E_{1,1} \arrow[ur] & & E_{2,2} \arrow[ur] \arrow[ll, dotted, "\tau"'] & & E_{3,3} \arrow[ll, dotted, "\tau"'] & \cdots
\end{tikzcd}$$

Let $I = [a, b]$ with $a \leq b$ be an intervall in $\mathbb{N}$. We define $\mathcal{C}_I$ to be the full additive subcategory given by objects whose composition factors are simples $S_i, a \leq i \leq b$. Alternatively,
$\mathcal{C}_I = \mathrm{add}\{E_{ij} \mid a \leq i \leq j \leq b\}$. This is a fully exact subcategory of $\mathcal{A}$ which is deflation-closed and inflation-closed and even a Serre subcategory. Furthermore, it is an abelian subcategory with enough injectives (indeomposable injectives: $E_{i,b}, a \leq i \leq b$), with enough projectives (indecomposable projectives: $E_{a,j}, a \leq j \leq b$) and unique indecomposble projective injective $E_{a,b}$. It is obvious that $\mathcal{C}_I$ is equivalent to the quiver representations of the full subquiver (of Q) with vertices I. This is a linear oriented quiver of type A.
Furthermore, every tilting subcategory fulfills $\mathcal{T}^\perp = \mathcal{P}^{<\infty}(\mathcal{T}^\perp)$ since $\mathcal{A} = \mathcal{P}^{<\infty}(\mathcal{A})$ by [7, Lemma 5.7].

Proposition 4.1. *Let $\mathcal{A}$ be the exact category described before. Let $\mathcal{T}$ be full additive subcategory in $\mathcal{A}$ closed under summands. The following are equivalent:*

(1) *$\mathcal{T}$ is a 1-tilting subcategory.*
(2) *$|\mathcal{T} \cap \mathcal{P}| = \infty$ and for every indecomposable $E_{1,n} \in \mathcal{T}$ we have that $\mathcal{T} \cap \mathcal{C}_{[1,n]}$ is a tilting subcategory of $\mathcal{C}_{[1,n]}$.*

(2') $|\mathcal{T} \cap \mathcal{P}| = \infty$ *and for every indecomposable* $E_{a,b} \in \mathcal{T}$ *we have that* $\mathcal{T} \cap \mathcal{C}_{[a,b]}$ *is a tilting subcategory of* $\mathcal{C}_{[a,b]}$.

Proof. (1) implies (2): Assume that $\mathcal{T}$ is 1-tilting (i.e. selforthogonal and $\mathcal{P} \subset \mathrm{Cores}_1(\mathcal{T})$). Since $\mathcal{P} \subset \mathrm{copres}(\mathcal{T})$, it follows that $\mathcal{T}$ contains infinitely many indecomposable projectives. First we look at projectives $E_{1,n} \in \mathcal{T}$. Clearly $\mathcal{T} \cap \mathcal{C}_{[1,n]}$ is selforthogonal in $\mathcal{C}_{[1,n]}$. The inclusion $\mathcal{P} \subset \mathrm{Cores}_1(\mathcal{T})$ implies for $P = E_{1,m}$ with $m \leq n$ that there is an exact sequence $P \rightarrowtail T_0 \twoheadrightarrow T_1$ in $\mathcal{A}$ with $T_i \in \mathcal{T}$. It is easy to see that $\mathrm{Hom}_{\mathcal{A}}(-, T)$ is exact on it for every $T \in \mathcal{T}$. This means, we can choose a minimal left $\mathcal{T}$-approximation $f\colon P \to T_0'$. Let $m < m' \leq n$ be minimal with $E_{1,m'} \in \mathcal{T} \cap \mathcal{C}_{[1,n]}$, then clearly $E_{1,m'} \in \mathrm{add}(T_0')$ and every other morphism $P \to T$ with $T \in \mathcal{T}$ indecomposable, $T \notin \mathcal{C}_{[1,n]}$, must factor over $P \to E_{1,m'}$. This implies that $T_0' \in \mathcal{C}_{[1,n]}$. Let $L := \mathrm{coker}\, f$, then we have an induced monomorphism $L \to T_1$, this means $L \in \mathrm{pres}(\mathcal{T}) \cap \mathrm{copres}(\mathcal{T}) = \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{T} = \mathcal{T}$ since $\mathcal{T}^{\perp} = \mathcal{P}^{<\infty}$ (so: being left perpendicular on the projectives in $\mathcal{T}^{\perp}$ implies being projective). Since $\mathcal{C}_{[1,n]}$ is closed under quotients, $L \in \mathcal{T} \cap \mathcal{C}_{[1,n]}$.
(2) implies (2'): For general $E_{a,b} \in \mathcal{T}$ the claim follows since one can find a projective $E_{1,n} \in \mathcal{T}$ with $\mathcal{C}_{a,b} \subset \mathcal{C}_{1,n}$ and since this is well-known for tilting modules over linear oriented A_n quivers (since restrictions of binary trees to a full subtree starting at branching point are binary trees).
(2') implies (1): For any two indecomposable summands X, Y in $\mathcal{T}$ there exists a projective $E_{a,n} \in \mathcal{T}$ such that $X, Y \in \mathcal{C}_{[1,n]}$. By assumption and since $\mathcal{C}_{[1,n]}$ is extension-closed, it follows that $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$. Furthermore, for every projective $P = E_{1,m} \notin \mathcal{T}$ there is a projective $E_{1,n} \in \mathcal{T}$ with $E_{1,j} \notin \mathcal{T}$ for $m < j < n$. Since $\mathcal{T} \cap \mathcal{C}_{[1,n]}$ is tilting in $\mathcal{C}_{[1,n]}$ we have an exact sequence $P \rightarrowtail T_0 \twoheadrightarrow T_1$ for a $T_i \in \mathcal{C}_{[1,n]} \cap \mathcal{T}$. $\square$

Remark 4.2. From the previous result it follows that weak slices only give tilting subcategories if they contain infinitely many indecomposable projectives. (As in the representation finite case, they are precisely the tilting subcategories with $\mathrm{gldim}\,\mathrm{mod}_{\infty}\, \mathcal{T} = 1$, the remaining ones fulfill $\mathrm{gldim}\,\mathrm{mod}_{\infty}\, \mathcal{T} = 2$.)

4.1.2. *Right infinite path.* Let Q be the following quiver

$$1 \to 2 \to 3 \to \cdots$$

Its Auslander-Reiten quiver has two components: the preprojective consisting only of the (infinite dimensional) projectives

$$\cdots \to P_3 \to P_2 \to P_1$$

and another component consisiting of all finite-dimensional modules.

$$\begin{tikzcd}
\cdots & E_{1,5} \arrow[dr] & & & & \\
& & E_{1,4} \arrow[dr] & & & \\
\cdots & E_{2,4} \arrow[ur] \arrow[dr] & & E_{1,3} \arrow[ll, dashed, "\tau"'] \arrow[dr] & & \\
& & E_{2,3} \arrow[ur] \arrow[dr] & & E_{1,2} \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
\cdots & E_{3,3} \arrow[ur] & & E_{2,2} \arrow[ll, dashed, "\tau"'] \arrow[ur] & & E_{1,1} \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

Lemma 4.3. *Let* $\mathcal{T}$ *be a tilting subcategory. Then:*

(0) $P_1 \in \mathcal{T}$
(1) *For every* $E_{ij} \in \mathcal{T}$ *we have that* $\mathcal{T} \cap \mathcal{C}_{[i,j]}$ *is tilting in* $\mathcal{C}_{[i,j]}$.
(2) *For* $P_n \in \mathcal{T}$ *we have* $\mathcal{T} \cap \mathcal{C}_{\geq n}$ *is tilting in* $\mathcal{C}_{\geq n}$.
(3) *If* $\mathrm{gen}(P_i) \cap \mathcal{T}$ *contains infinitely many indecomposables, then* $P_i \in \mathcal{T}$ *and* $P_\ell \notin \mathcal{T}$ *for all* $\ell > i$.

(4) *Let $E_{ij} \in \mathcal{T}$. If $P_i \notin \mathcal{T}$ then there is an E_{aj}, $a < i$ or an E_{ib}, $b > j$ in $\mathcal{T}$.*

Proof. (0) As $\mathcal{T}$ is tilting there has to exist an exact sequence $P_1 \rightarrowtail T_0 \twoheadrightarrow T_1$ with $T_i \in \mathcal{T}$. Then we find an inflation $P_1 \rightarrowtail T_1'$ where $T_1' \in \operatorname{add}(T_1)$ is the summand with top supported at $S_1 = E_{1,1}$. As P_1 is infinite-dimensional, at least one summand of T_1' has to be P_1 itself.

(1) Clearly $\operatorname{add}(T) := \mathcal{T} \cap \mathcal{C}_{[i,j]}$ is still selforthogonal. The indecomposable projectives in $\mathcal{C}_{[i,j]}$ are E_{it}, $t \leq j$ and by assumption, the projective-injective $E_{ij} \in \operatorname{add}(T)$. We only show that $E_{it} \in \operatorname{Cores}_1(T)$ for every $t < j$. There exist only finitely many indecomposables modules $E_{a,b}$ with $\operatorname{Hom}(E_{it}, E_{ab}) \neq 0$. Let $\mathcal{Z} := \operatorname{add}\{E_{ab} \mid \operatorname{Hom}(E_{it}, E_{ab}) \neq 0\}$, then $\mathcal{T}_{\mathcal{Z}} := \mathcal{T} \cap \mathcal{Z}$ contains only finitely many indecomposables, so there is a minimal left $\mathcal{T}_{\mathcal{Z}}$-approximation $f\colon E_{it} \to T_0$ with $T_0 \in \mathcal{T}_{\mathcal{Z}}$. Since we have a monomorphism $E_{it} \to E_{ij}$ and $E_{ij} \in \mathcal{T}_{\mathcal{Z}}$, it follows that $T_0 \in \mathcal{C}_{[i,j]}$ and f is a monomorphism. Let $R := \operatorname{coker} f$, since $\mathcal{C}_{[ij]}$ is a wide subcategory it follows that $R \in \mathcal{C}_{[ij]}$. Then we have for every $T \in \mathcal{T}_Z$ an exact sequence

$$\operatorname{Hom}(R, T) \rightarrowtail \operatorname{Hom}(T_0, T) \twoheadrightarrow \operatorname{Hom}(E_{it}, T)$$

If we have an indecomposable $T \in \mathcal{T}, T \notin \mathcal{T}_Z$, then it holds $\operatorname{Hom}(E_{it}, T) = 0$. It follows that $R \in {}^{\perp}\mathcal{T}$ and also that $R \in \mathcal{T}^{\perp}$ (the last inclusion can be seen by applying $\operatorname{Hom}(T, -)$ to the short exact sequence $E_{it} \rightarrowtail T_0 \twoheadrightarrow R$). Now, $\mathcal{T}^{\perp}$ is a full exact subcategory with enough projectives given by $\mathcal{T}$ itself. So, for every $Y \in \mathcal{T}^{\perp}$ there is an exact sequence

$$T^1 \rightarrowtail T^0 \twoheadrightarrow Y$$

with $T^i \in \mathcal{T}$, by applying $\operatorname{Hom}(R, -)$ one concludes $\operatorname{Ext}^1(R, Y) = 0$ and therefore $R \in \mathcal{P}(\mathcal{T}^{\perp}) = \mathcal{T}$.

(2) For $m > n$ we have an exact sequence $0 \to P_m \to T_0 \to T_1 \to 0$ with $T_i \in \mathcal{T}$.

Every non-zero homomorphism $P_m \to E_{ij}$ or $P_m \to P_i$ with $i < n$ factors through $P_m \to P_n$. By leaving out the summands we find a left $\mathcal{T}$-approximation $0 \to P_m \to T_0' \to R \to 0$. As in (1) we conclude that $R \in \mathcal{T}$.

(3) Assume that $P_i \notin \mathcal{T}$, then $i > 1$. There exists a short exact sequence $P_i \rightarrowtail T_0 \twoheadrightarrow T_1$ with $T_i \in \mathcal{T}$. As P_i is infinite dimensional there exists an $j < i$ such that $P_j \in \operatorname{add}(T_0)$ and we also assume that there is a summand $E_{jk} \in \operatorname{add}(T_1)$. By assumption there exists $E_{is} \in \mathcal{T}$ with $s > k$. Then $E_{is} \rightarrowtail E_{ik} \oplus E_{js} \twoheadrightarrow E_{jk}$ is a non-split short exact sequence contradicting $\mathcal{T}$ being selforthogonal. This shows $P_i \in \mathcal{T}$. Assume $P_\ell \in \mathcal{T}$ with $\ell > i$, there exists $E_{is} \in \mathcal{T}$ with $s \geq \ell$ but then $\operatorname{Ext}^1(E_{is}, P_\ell) \neq 0$ contradicting $\mathcal{T}$ being selforthogonal.

(4) Observe that for $E_{ab}, E_{cd} \in \mathcal{T}$ we have two binary trees, one in $\mathcal{C}_{[a,b]}$ and one in $\mathcal{C}_{[c,d]}$ then these intervals $[a, b]$ and $[c, d]$ have to be either one contained in the other or they have to be disjoint and if $a \leq b < c \leq d$, then $c - b > 1$. In all other cases we find a non-split extension between E_{ab} and E_{cd}.

Take $E_{ij} \in \mathcal{T}$ and assume that $E_{ab} \notin \mathcal{T}$ for all $[i, j] \subseteq [a, b]$ (this means that $\mathcal{C}_{[i,j]} \cap \mathcal{T}$ is a not a proper subtree of a $\mathcal{C}_{[a,b]} \cap \mathcal{T}$). Then we want to see that $P_i \in \mathcal{T}$. As $\mathcal{T}$ is maximal selforthogonal, it is enough to see that $\operatorname{Ext}^1(\mathcal{T}, P_i) = 0$. For that we look at the indecomposables E_{ts} with $t < i \leq s$ and we need to see that $E_{ts} \notin \mathcal{T}$. But as we remarked before, the next of these binary trees on the right has at least one distance from this one, this implies the claim.

□

Definition 4.4. Let $\Gamma_0 = \Gamma_0^p \cup \Gamma_0^f$ be the set of vertices of the Auslander-Reiten quiver where Γ_0^p denotes the vertices corresponding to the projectives and Γ_0^f the vertices corresponding to the finite-dimensional modules.
A **binary tree** on Γ_0 consists of $\mathbb{T} \cup \mathbb{P}$ with $\mathbb{T} \subset \Gamma_0^f, \mathbb{P} \subset \Gamma_0^p$ such that:

(i) $P_1 \in \mathbb{P}$.
(ii) If there are infinitely many $E_{ij_n} \in \mathbb{T}$, $n \in \mathbb{N}$ then $P_i \in \mathbb{P}$ and $P_\ell \notin \mathbb{P}$ for all $\ell > i$.
(iii) For every $E_{ij} \in \mathbb{T}$ is $\mathcal{C}_{ij} \cap \mathbb{T}$ a binary tree on the Auslander-Reiten quiver of $\mathcal{C}_{ij}$ in the sense of Hille and

either $P_i \in \mathbb{P}$, or
there is an $E_{aj}, a < i$ or an $E_{ib}, b > j$ in $\mathbb{T}$.
If also $E_{ts} \in \mathbb{T}$, then
either $E_{ab} \in \mathbb{T}$ with $a = \min(t, i), b = \max(s, j)$, or
$[t, s] \cap [i, j] = \emptyset$ and $|t - j| \geq 2$.

(iv) Given $\ell \in \mathbb{N}$ assume there is no $t < \ell \leq s$ with $E_{ts} \in \mathbb{T}$ then $P_\ell \in \mathbb{P}$.

Remark 4.5. Given the set $\mathbb{T}$ fulfilling (iii), there is always a unique set $\mathbb{P}$ defined by the properties (i)-(iv), such that the union is a binary tree.

Remark 4.6. As an indexing set we take always $N \subseteq \mathbb{N}$ with $N = \emptyset$ or $N = [1, n]$ or $N = \mathbb{N}$. We have a two types of binary trees:

(a) If $\mathbb{P}$ is infinite, then we have an indexing set N and sequence of pairwise disjoint intervals $i_n \leq j_n$, $n \in N$ with $j_n < i_{n+1} - 1$ such that $\mathcal{C}_{i_n j_n} \cap \mathbb{T}$ is a binary tree and every $\mathbb{T}$ is the union of these.
(b) If $\mathbb{P}$ is finite with $i = \max\{a \mid P_a \in \mathbb{P}\}$, then there is a finite indexing set N and a sequence $i_n \leq j_n < i - 1$, $j_n < i_{n+1} - 1$, $n \in N$ such that $\mathcal{C}_{i_n \cdot j_n} \cap \mathbb{T}$ are binary trees and there is an infinite nested sequence $\mathcal{C}_{i,t_s} \subset \mathcal{C}_{i,t_{s+1}}$ such that $\mathcal{C}_{i,t_s} \cap \mathbb{T}$ is a binary tree. Again $\mathbb{T}$ has to be the union of this finite sequence and the nested sequence of binary trees.

THEOREM 4.7. *Let $\mathcal{T}$ be a subcategory, then it is tilting if and only if the vertices give a binary tree on the vertices of the Auslander-Reiten quiver as defined before.*

We first remark the following

PROOF. Let $\mathcal{T}$ be a tilting subcategory. By lemma 4.3, we have (i),(ii) and (iii) are fulfilled. Properties (iv) follows since a tilting subcategory is maximal selforthogonal (see remark 2.1). Conversely, consider $\mathcal{T} = \operatorname{add}\{X \mid X \in \mathbb{T} \cup \mathbb{P}\}$ with $\mathbb{T}, \mathbb{P}$ fulfilling the properties (i)-(iv). Then we have $\mathcal{T}$ is selforthogonal- this follows from the easy observation:

$$\operatorname{Ext}^1(E_{ij}, P_\ell) \neq 0 \Leftrightarrow \quad i < \ell \leq j$$

We need to see that $\mathcal{P} \subset \operatorname{copres}_1(\mathcal{T})$. If $P_\ell \notin \mathcal{T}$, then $\ell \geq 2$ and there exists a $E_{ts} \in \mathcal{T}$ with $t < \ell \leq s$ such that E_{ts} is the root of one of the binary trees in $\mathbb{T}$ with also $P_t \in \mathbb{P}$. In case that $E_{\ell s} \in \mathcal{T}$, we have $P_\ell \rightarrowtail E_{\ell s} \oplus P_t \twoheadrightarrow E_{ts}$ shows that claim. Else, as $\mathcal{C}_{[t,s]} \cap \mathcal{T}$ is tilting in $\mathcal{C}_{[t,s]}$, we have an exact sequence $E_{\ell s} \rightarrowtail E_{st} \oplus T_0 \twoheadrightarrow T_1$ with $T_i \in \mathcal{C}_{[t,s]} \cap \mathcal{T}$. Then, we look at the inflation $P_\ell \rightarrowtail P_t \oplus T_0$, the push out along $P_\ell \to E_{\ell s}$ is just the inflation $E_{\ell s} \rightarrowtail E_{st} \oplus T_0$, and both inflation have the same cokernel in $\mathcal{T}$. □

Remark 4.8. In this case, the notion of a slice is empty as all non-projective tilting subcategories have indecomposables from both connected components of the Auslander-Reiten quiver. Assume that we have Q' another orientation differing in an interval $[1, n]$. Now for the interval $[1, n]$ we can realize any orientation of a type $\mathbb{A}_n$ quiver as a binary tree $\mathbb{T}$ in $\mathcal{C}_{[1,n]}$, then define $\mathbb{P} = \{P_1\} \cup \{P_{n+1}, P_{n+2}, \cdots\}$. Then take the corresponding tilting subcategory $\mathcal{T}$ and $\operatorname{mod}_\infty \mathcal{T}$ is $\operatorname{rep}^+(Q')$.

4.1.3. *Derived equivalences between different orientations.*

THEOREM 4.9. *Let Q and Q' be two quivers of type $\mathbb{A}_\infty$. Then, there exists a triangle equivalence*

$$\mathrm{D}^b(\operatorname{rep}^+(Q)) \to \mathrm{D}^b(\operatorname{rep}^+(Q'))$$

if and only if one of the following three cases holds:

(a) *Q and Q' have a left infinite path.*
(b) *Q and Q' have a right infinite path.*
(c) *Q and Q' have no infinite path.*

One implication is an immediate corollary of the following result

Theorem 4.10. *([1], Thm 7.11) Let Q be a strongly locally finite infinite quiver, then $\mathrm{D}^b(\mathrm{rep}^+(Q))$ has (left/right) almost split triangles if and only if $\mathrm{rep}^+(Q)$ has no (right/left) infinite path.*

Therefore, it is enough to prove that if Q and Q' in the previous conjecture both fulfill (a) (resp. (b) , resp. (c)), then there exists a triangle equivalence as stated.

Proof. We show that in each of the cases (a), (b) and (c) a derived equivalence between categories of finitely represented quiver representations of two different orientations can be obtained by two tilting derived equivalences.

(a) Let Q'' be the orientation given by one left infinite path and $\mathcal{A} = \mathrm{rep}^+(Q'')$. Then we find two slices for both orientations Q, Q' and the corresponding tilting categories induce then derived equivalence.
$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \leftarrow \mathrm{D}^b(\mathrm{rep}^+(Q'')) \rightarrow \mathrm{D}^b(\mathrm{rep}^+(Q'))$$
(b) Let Q'' be the orientation given by one left infinite path and $\mathcal{A} = \mathrm{rep}^+(Q'')$. We do not find slices in this case but we can find tilting subcategories which fulfill the same task, cf. remark 4.8. Take the two tilting subcategories corresponding to the two different orientations, their tilting functors give derived equivalences
$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \leftarrow \mathrm{D}^b(\mathrm{rep}^+(Q'')) \rightarrow \mathrm{D}^b(\mathrm{rep}^+(Q'))$$
(c) Here we take $\mathcal{A}$ to be the one described below. We will look at another abelian category $\mathcal{A}$ (see below) and find two tilting subcategories inducing derived equivalences
$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \leftarrow \mathrm{D}^b(\mathcal{A}) \rightarrow \mathrm{D}^b(\mathrm{rep}^+(Q'))$$

□

Let from now on $\mathcal{A}$ denote the category $\mathrm{rep}^b(\Delta)$ where Δ is the quiver (of type $\mathbb{A}_\infty^\infty$)
$$\cdots \leftarrow (-3) \leftarrow (-2) \leftarrow (-1) \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \cdots$$
and rep^b denotes the subcategory of all quiver representations of total finite dimension.
Observe that $\mathcal{A}$ is a hereditary abelian Auslander-Reiten category without non-zero projectives or non-zero injectives. Its Auslander-Reiten quiver can be pictured as follows...

$$\begin{tikzcd}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & E_{-3,0} \arrow[dr] & & E_{-2,1} \arrow[ll, dashed] \arrow[dr] & & E_{-1,2} \arrow[ll, dashed] \arrow[dr] & & E_{0,3} \arrow[ll, dashed] \arrow[dr] & & \\
\cdots & E_{-3,-1} \arrow[ur] \arrow[dr] & & E_{-2,0} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & E_{-1,1} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & E_{0,2} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & E_{1,3} \arrow[ll, dashed] & \cdots \\
 & & E_{-2,-1} \arrow[ur] \arrow[dr] & & E_{-1,0} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & E_{0,1} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & E_{1,2} \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & \\
\cdots & E_{-2,-2} \arrow[ur] & & E_{-1,-1} \arrow[ll, dashed] \arrow[ur] & & E_{0,0} \arrow[ll, dashed] \arrow[ur] & & E_{1,1} \arrow[ll, dashed] \arrow[ur] & & E_{2,2} \arrow[ll, dashed] & \cdots
\end{tikzcd}$$

A slice has an associated quiver by just taking the full subquiver of the Auslander-Reiten quiver with the vertices given by the slice. We say a slice does not contain a left (resp. right) infinite path if the associated quiver does not contain a right (resp. left) infinite path.

Lemma 4.11. *Weak slices in the Auslander-Reiten quiver of $\mathcal{A}$ give tilting subcategories if and only if the slice does not contain a left or right infinite path.*

PROOF. Let $\mathcal{T}$ be a the full additively closed subcategory of $\mathcal{A}$ with indecomposables given by the vertices of a slice in the Auslander-Reiten category. Let us first see that a slice with an infinite path can not be a tilting subcategory. If the slice does contain a left infinite path, we look at an indecomposable in this path and apply τ^{-1} to it, call this X. Then X is not in $\mathrm{pres}(\mathcal{T})$ and it is not a subobject of an object in $\mathrm{pres}(\mathcal{T})$, therefore (T2) is not fulfilled.
If the slice does not contain a left infinite path, we see $\mathrm{pres}(\mathcal{T})$ as the additive closure of all indecomposables of the slice and of all indecomposables on the right (i.e. after applying τ^{-n}, $n \geq 1$) of it (as going-down arrows in the Auslander-Reiten quiver are all epimorphisms). Assume that the slice contains a right infinite path, and let Y be τ of an indecomposable corresponding to a vertex of the right infinite path. Then Y is not a subobject of an object in $\mathrm{pres}(\mathcal{T})$.
From now on assume that the slice does not contain an infinite path. The description of $\mathrm{pres}(\mathcal{T})$ as above implies $\mathrm{pres}(\mathcal{T}) = \mathcal{T}^{\perp}$. As $\mathcal{T}$ is contravariantly finite, we want to see that the kernel of a right $\mathcal{T}$ approximation of an object in $\mathrm{pres}(\mathcal{T})$ is in $\mathcal{T}^{\perp}$ (this implies (T1)). But this follows directly from applying $\mathrm{Hom}(T, -)$ with $T \in \mathcal{T}$ to the short exact sequence.
Now, take any indecomposable object A in $\mathcal{A}$, we want to see that $A \in \mathrm{Cores}_1(\mathcal{T}^{\perp})$. Wlog. A is in a τ-orbit of an indecomposable in $\mathcal{T}$. We look at the right infinite arrow going-up (i.e. of monomorphisms) starting at A in the Auslander-Reiten quiver. As the slice does not contain a right infinite path, the right infinite path going-up starting at A will eventually meet a vertex in the slice. This gives a monomorphism $A \to T_0$ with $T_0 \in \mathcal{T}$. We look at the short exact sequence $A \rightarrowtail T_0 \twoheadrightarrow B$ and as $B \in \mathrm{pres}(\mathcal{T})$, (T2) follows. $\square$

5. Other Dynkin types?

The same question as in Theorem 4.9 can be probably answered for the other infinite Dynkin types only using sink/source reflections and results from [**1**]. Let us first pose the following question: An infinite quiver is called *strongly locally bounded* (cf. [**1**]) if at every vertex there are only finitely many arrows ending and starting and between any given two vertices there are only finitely many paths.

Open question 5.1. Let Q, Q' be two orientations of a graph, both strongly locally bounded infinite quiver without an infinite path. Is there a finite sequence of sink/source mutations passing from one to the other?

For the rest **assume** that the question has a positive answer for quivers of Dynkin type.

THEOREM 5.2. *(assuming: Yes in 5.1) Let Q and Q' be two quivers of type $\mathbb{D}_\infty$. Then we find the same three triangle equivalence classes as in Theorem 4.9.*

PROOF. We sketch the argument as follows: Here, again that we have at least these three equivalence classes follows from [**1**], combine Thm 5.22, Prop. 7.9, Thm 7.10. Inside (a) and (b) we have that (single) sink/source reflection operate transitively, so they are all triangle equivalent. Inside (c), we would need potentially infinite sequences of single sink/source reflections to pass between two orientations (and that is not a valid argument). But by introducing mutation of possibly infinite sets of sources/sinks (cf. subsection 2.1) we can overcome this problem and see that all orientations without an infinite path will induce a triangle equivalence as in the theorem. $\square$

Now, for an infinite quiver Q of type $\mathbb{A}_\infty^\infty$ we define some numbers:
$\ell :=$ number of maximal left infinite paths in Q,
$r :=$ number of maximal right infinite paths in Q
So $0 \leq r, \ell \leq 2$, $r + \ell \leq 2$ and $r = \ell = 0$ means either Q is a double infinite quiver or has no infinite paths.
In case $r = \ell = 1$, there is a finite number c of arrows in one direction and infinitely in the other.

Theorem 5.3. *(assuming: Yes in 5.1) Let Q and Q' be two quiver of type $\mathbb{A}_\infty^\infty$. Then, there exists a triangle equivalence*

$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \to \mathrm{D}^b(\mathrm{rep}^+(Q'))$$

if and only if one of the following three cases holds

(a) *Q and Q' are both double infinite paths.*
(b) *Q and Q' have the same numbers ℓ, r and $(\ell, r) \neq (1,1)$ and are not a double infinite path.*
(c) *Q and Q' have the same numbers $(\ell, r) = (1,1)$ and c.*

As before, we sketch the proof. To see that these orientations are pairwise non-derived equivalent: Look at the description of the Auslander Reiten quiver components of $\mathrm{D}^b(\mathrm{rep}^+(Q))$ for all quivers of type $\mathbb{A}_\infty^\infty$ in [**1**, Thm. 5.17, Thm 7.9]. Here, in case (c), the number c appears as the number of τ-orbits in the finite wing (cf, Thm 5.17, (4)) and therefore for different c they are pairwise non-derived equivalent.
To see that in each case (a), (b), (c) we have the claimed triangle equivalences: First, observe that the underlying graph automorphism σ which maps the vertices as $\sigma(x) = (-x)$ induces an isomorphism of categories $\mathrm{rep}^+(Q) \cong \mathrm{rep}(\sigma Q)$ and this induces a derived equivalence. Observe that the numbers ℓ, r and also in case (c) the number c are preserved. This shows e.g. that the two double infinite paths in (a) are derived equivalent.
Once we take these isomorphims into account, one can see that in case (c) reflection functors at sinks and sources operate transitively. In case (b) also, but we need reflection functors at infinitely many sinks and sources.

Bibliography

[1] R. Bautista, S. Liu, and C. Paquette, *Representation theory of strongly locally finite quivers*, Proc. Lond. Math. Soc. (3) **106** (2013), no. 1, 97–162. MR3020740

[2] A. Beligiannis and I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207. MR2327478

[3] E. Enochs, J. R. García Rozas, L. Oyonarte, and S. Park, *Noetherian quivers*, Quaest. Math. **25** (2002), no. 4, 531–538. MR1949331

[4] H. Krause, *Homological theory of representations*, Cambridge Studies in Advanced Mathematics, vol. 195, Cambridge University Press, Cambridge, 2022. MR4327095

[5] ———, *Locally noetherian quiver representations*, 2024.

[6] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer-Verlag, Berlin, 1984. MR774589

[7] J. Sauter, *Tilting theory in exact categories*, 2022.