Realization functors in algebraic triangulated categories

This chapter is joint work with Janina Letz, cf. [22].

1. Synopsis

Let $\mathcal{T}$ be an algebraic triangulated category and $\mathcal{C}$ an extension-closed subcategory with $\operatorname{Hom}(\mathcal{C}, \Sigma^{<0}\mathcal{C}) = 0$. Then $\mathcal{C}$ has an exact structure induced from exact triangles in $\mathcal{T}$. Keller and Vossieck say that there exists a triangle functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion $\mathcal{C} \subseteq \mathcal{T}$.
What is new? We provide the missing details for a complete proof.

2. Introduction

Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a full additive subcategory with an exact structure. A *realization functor* for $\mathcal{C}$ is a triangle functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion. There are various constructions of a realization functor, all requiring an enhancement and restricting to certain subcategories $\mathcal{C}$. The first realization functor was constructed in [2] when $\mathcal{C}$ is the heart of a t-structure in a filtered triangulated category; also see [28, Appendix]. A different construction appears in [24].

In this chapter we work in algebraic triangulated categories; These include all stable module categories and derived categories. Unlike the works mentioned above we consider exact subcategories of $\mathcal{T}$, not hearts of t-structures. There exist exact categories whose bounded derived category does not admit a bounded t-structure; see [25].

The following result appears in [20, 3.2 Théorème]:

Theorem 2.1. *Let $\mathcal{T}$ be an algebraic triangulated category and $\mathcal{C}$ an extension-closed full subcategory with $\operatorname{Hom}_{\mathcal{T}}(\mathcal{C}, \Sigma^{-n}\mathcal{C}) = 0$ for $n \geq 1$. Then $\mathcal{C}$ has an exact structure induced from the triangulated structure on $\mathcal{T}$ and there exists a realization functor.*

The article [20, 3.2 Théorème] provides a sketch of the proof, referring to a construction later appearing in [16]. The main goal of this chapter is to provide the missing details for a complete proof of Theorem 2.1.

The non-negativity condition in Theorem 2.1 for $\mathcal{C}$ is necessary for our construction. It also appears when the realization functor is a triangle equivalence. In fact, whenever the realization functor is fully faithful, then $\mathcal{C}$ has to satisfy the non-negativity condition.

Theorem 2.1 can be considered the standard tool to realize an (algebraic) triangulated as a bounded derived category of an exact category; we provide conditions for when the realization functor is an equivalence in Section 3.4. Therefore, Theorem 2.1 is expected to be used in classifications of exact subcategories of a triangulated category up to (bounded) derived equivalence.

Further, finding a realization functor is an alternative to tilting theory. Tilting subcategories in a triangulated category were defined by Keller; see for example [19]. A subcategory $\mathcal{C}$ of $\mathcal{T}$ is *tilting*, if $\mathcal{C}$ is endowed with the split exact structure, hence $\mathrm{D}^b(\mathcal{C}) = \mathrm{K}^b(\mathcal{C})$, and the realization functor

$\mathrm{K}^b(\mathcal{C}) \to \mathcal{T}$ exists and is a triangle equivalence. There exist realization functors that are equivalences that are not induced by tilting theory; for example the inclusion of a small exact category into its weak idempotent completion induces a triangle equivalence on their bounded derived categories; see [**23**, 1.10].

In general it is not known whether a realization functor of a category $\mathcal{C}$ is unique. However, it is unique with respect to the chosen enhancement. Theorem 2.1 is also central to the search for a universal property defining the bounded derived category of an exact category; cf. [**17**] and for derivators [**26**].

3. Realization functor

The bounded derived category of an exact category $\mathcal{C}$ is the Verdier quotient of the homotopy category of the underlying additive category by the full subcategory of bounded $\mathcal{C}$-acyclic complexes $\mathrm{Ac}(\mathcal{C})$; see [**23**] and also [**6**, Section 10]. We fix a triangulated category $\mathcal{T}$ with suspension functor Σ. A *realization functor* for an additive subcategory $\mathcal{C}$ of $\mathcal{T}$ with an exact structure is a triangle functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion $\mathcal{C} \to \mathcal{T}$.

3.1. Admissible exact subcategories. In this work we focus on subcategories $\mathcal{C}$ of the triangulated category $\mathcal{T}$ that inherit their exact structure from the triangulated structure of $\mathcal{T}$.

Definition 3.1. A full subcategory $\mathcal{C}$ is called *non-negative* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \Sigma^{<0}\mathcal{C}) = 0$; this means $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0$ for any $X, Y \in \mathcal{C}$ and $n < 0$. When $\mathcal{C}$ is additionally closed under extensions and direct summands, we say $\mathcal{C}$ is *admissible exact*.

By [**9**], any extension-closed, non-negative subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ inherits an exact structure from the triangulated structure: The short exact sequences $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathcal{C}$ are precisely those that fit into an exact triangle $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L$.

Remark 3.2. With the notation of 'admissible exact' we follow [**2**, Definition 1.2.5] and [**13**, Section 2]; the former only considers 'admissible abelian', while the latter dropped 'exact'. We use admissible exact to avoid confusion with the notions of left/right admissible in the sense of [**3**, §1].

The crucial condition of admissible exactness is the non-negativity. In fact, when $\mathcal{C}$ is non-negative, then the smallest full subcategory closed under extensions and direct summands containing $\mathcal{C}$ is an admissible exact subcategory.

Example 3.3. We equip an extension-closed subcategory $\mathcal{C}$ of an exact category $\mathcal{E}$ with the induced exact structure; that is $\mathcal{C}$ is a *fully exact* subcategory of $\mathcal{E}$. Then $\mathcal{C}$ is an admissible exact subcategory of $\mathrm{D}^b(\mathcal{E})$.

Example 3.4. The heart of any t-structure on a triangulated category is admissible exact. Any intersection of admissible exact subcategories is admissible exact. Hence the intersection of two hearts is admissible exact; this applies in particular for hearts that are mutations of each other; see [**7**] for HRS tilting and [**4**] for the heart fan of an abelian category.

3.2. Weak realization functor. Next, we consider triangle functors $\mathrm{K}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion for any full subcategory $\mathcal{C}$ of $\mathcal{T}$; such a functor can be considered as a realization functor for $\mathcal{C}$ with the split exact structure. We call such a functor a *weak realization functor*. Under reasonable conditions on the exact structure a weak realization functor induces a realization functor.

Lemma 3.5. *Let $\mathcal{C} \subseteq \mathcal{T}$ be a full subcategory with an exact structure. We assume there exists a weak realization functor $\mathrm{F}\colon \mathrm{K}^b(\mathcal{C}) \to \mathcal{T}$. If any exact sequence $L \to M \to N$ in $\mathcal{C}$ fits into an exact* *triangle $L \to M \to N \to \Sigma L$ in $\mathcal{T}$, then* F *induces a realization functor such that the following diagram commutes*

$$\begin{tikzcd} \mathrm{K}^b(\mathcal{C}) \arrow[r] \arrow[dr] & \mathrm{D}^b(\mathcal{C}) \arrow[d] \\ & \mathcal{T} \end{tikzcd}$$

In particular, this holds when $\mathcal{C}$ is an admissible exact subcategory.

Proof. It is enough to show that F sends acyclic complexes to zero. For this it is enough to show that complexes of the form

$$(\cdots \to 0 \to L \to M \to N \to 0 \to \cdots) = \mathrm{cone}(\mathrm{cone}(L \to M) \to N)$$

are send to zero when $L \to M \to N$ is an exact sequence in $\mathcal{C}$. But this holds by assumption. □

Remark 3.6. The above condition on F, that any exact sequence in $\mathcal{C}$ fits into an exact triangle in $\mathcal{T}$, means that $\mathcal{C} \to \mathcal{T}$ is a δ*-functor* as defined in [**17**].

In the sequel we construct a weak realization functor. However, we do not know of a general criterion for the existence of a weak realization functor. Our construction requires some form of non-negativity. In particular, a weak realization functor may even exist for $\mathcal{C} = \mathcal{T}$.

Example 3.7. Let k be a field and $\mathcal{T} = \mathrm{vect}(k)$, the category of finite-dimensional k-vector spaces with suspension $\Sigma = \mathrm{id}$. We can view $\mathrm{K}^b(\mathcal{T})$ as the category of finite-dimensional $\mathbb{Z}$-graded k-vector spaces $\mathrm{vect}^{\mathbb{Z}}(k)$ with suspension the shift of the grading. The forgetful functor from graded k-vector spaces to ungraded k-vector spaces is a weak realization functor for $\mathcal{C} = \mathcal{T}$. As $\mathrm{D}^b(\mathcal{T}) = \mathrm{K}^b(\mathcal{T})$ we obtain the realization functor

$$\mathrm{D}^b(\mathcal{T}) = \mathrm{vect}^{\mathbb{Z}}(k) \to \mathrm{vect}(k) = \mathcal{T}\,.$$

3.3. Existence. A *Frobenius category* is an exact category with enough projectives and with enough injectives and the projectives and injectives coincide. Let $\mathcal{E}$ be a Frobenius category with $\mathcal{P}$ the full subcategory of projective-injective objects. The ideal quotient $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ with respect to the morphisms factoring through $\mathcal{P}$ has a natural triangulated structure by [**10**, I.2]. A triangulated category is *algebraic*, if it is triangle equivalent to $\underline{\mathcal{E}}$ for some Frobenius category; see [**18**, 3.6].

The key observation for the proof of Theorem 2.1 is the following result, which is stated in [**20**, 3.2].

Proposition 3.8. *Let $\mathcal{E}$ be a Frobenius category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be a non-negative full subcategory and set $\mathcal{B} := q^{-1}(\mathcal{C})$. Then the functor $\mathcal{B} \to \mathcal{C}$ induces an equivalence of triangulated categories*

$$\mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{C})\,.$$

Note, that in the equivalence connects the Verdier quotient of the homotopy category and the homotopy category of an ideal quotient. We postpone the proof to Section 4.

Remark 3.9. In the Proposition we show that the tilting subcategory $\mathcal{B}$ in $\mathrm{K}^b(\mathcal{B})$ is send to the tilting subcategory $\mathcal{C}$ under the Verdier quotient functor $\mathrm{K}^b(\mathcal{B}) \to \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P})$. In general, Verdier quotients need not preserve tilting subcategories.

Without the assumption that the subcategory $\mathcal{C}$ is non-negative the Proposition 3.8 is false in general:

Example 3.10. Let k be a field and $A = k[x]/(x^2)$. Then $\mathcal{E} = \mod A$ is a Frobenius category and $\underline{\mathcal{E}} = \mod k$ is the category of finite-dimensional vector spaces which is a triangulated category with $\Sigma = \mathrm{id}$. We show below that $\mathrm{K}^b(\mod A)/\mathrm{K}^b(\mathrm{proj} A)$ is not equivalent to $\mathrm{K}^b(\mod k)$, that is that the conclusion of Proposition 3.8 does not hold for $\mathcal{C} = \underline{\mathcal{E}}$, which is not non-negative. Observe first that

$\mathrm{K}^b(\operatorname{mod} k) = \mathrm{D}^b(\operatorname{mod} k)$ has no non-trivial thick subcategory. But on the other hand $\mathrm{K}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A)$ admits a non-trivial Verdier quotient

$$\mathrm{K}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A) \to \mathrm{D}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A) \cong \underline{\mathcal{E}}\,;$$

see [**5**, Theorem 4.4.1]. In particular, the kernel of this Verdier quotient is a non-trivial thick subcategory. Therefore they can not be triangle equivalent.

Proposition 3.11. *Let $\mathcal{E}$ be a Frobenius category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be an admissible exact subcategory. Then there exists a weak realization functor $\mathrm{K}^b(\mathcal{C}) \to \underline{\mathcal{E}}$.*

Proof. Set $\mathcal{B} := q^{-1}(\mathcal{C})$. By Proposition 3.8 there exists an equivalence of triangulated categories

$$\mathrm{F}\colon \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{C})\,.$$

There is also an equivalence

$$\mathrm{B}\colon \underline{\mathcal{E}} \to \mathrm{D}^b(\mathcal{E})/\mathrm{K}^b(\mathcal{P})\,;$$

this has been stated in [**20**, Example 2.3] with proofs provided in [**14**, Corollary 2.2] or [**21**, Proposition 4.4.18]. Then the following composition involving the quasi-inverses of the above functors yields the claim

$$\mathrm{K}^b(\mathcal{C}) \xrightarrow{\mathrm{F}^{-1}} \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{E})/\mathrm{K}^b(\mathcal{P}) \xrightarrow{\mathrm{B}^{-1}} \underline{\mathcal{E}}\,. \qquad \square$$

Proof of Theorem 2.1. By Proposition 3.11 there exists a weak realization functor, and it induces a realization functor by Lemma 3.5. $\square$

From Proposition 3.8 we can also deduce the following corollary.

Corollary 3.12. *Let $\mathcal{C}$ be an admissible exact subcategory of $\underline{\mathcal{E}}$. Then $\mathcal{B} = q^{-1}(\mathcal{C})$ is extension-closed in $\mathcal{E}$ and the functor $q\colon \mathcal{B} \to \mathcal{C}$ sends exact sequences to exact triangles. In this case q induces a triangle equivalence*

$$\mathrm{D}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{D}^b(\mathcal{C})\,.$$

Proof. It is straightforward to check that $\mathrm{K}^b(\mathcal{P})$ and $\mathrm{Ac}^b\mathcal{B}$ are Hom-orthogonal in $\mathrm{K}^b(\mathcal{B})$. Then $\mathrm{Ac}^b\mathcal{B}$ is a full subcategory of $\mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P})$ by [**15**, Proposition 1.6.10]. So it is enough to show that the equivalence from Proposition 3.8 restricts to an equivalence of the acyclic complexes $\mathrm{Ac}^b\mathcal{B} \to \mathrm{Ac}^b\mathcal{C}$.

The fully faithfullness of the restriction holds as $\mathrm{Ac}^b\mathcal{B}$ is a full subcategory of $\mathrm{D}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P})$. Essentially surjectivity holds as

$$\operatorname{Ext}^1_{\mathcal{B}}(X,Y) \cong \operatorname{Hom}_{\underline{\mathcal{E}}}(X,\Sigma Y) \cong \operatorname{Ext}^1_{\mathcal{C}}(X,Y)$$

for any $X, Y \in \mathcal{B}$. $\square$

3.4. Fully faithfulness and equivalence. Let $\mathcal{C}$ be an admissible exact subcategory of a triangulated category $\mathcal{T}$. In this section we discuss when a realization functor

$$\mathrm{R}\colon \mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$$

is fully faithful and even an equivalence. The realization functor R induces natural group homomorphisms

$$\Phi_n(X,Y) := (\operatorname{Ext}^n_{\mathcal{C}}(X,Y) \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{D}^b(\mathcal{C})}(X,\Sigma^n Y) \xrightarrow{\mathrm{R}} \operatorname{Hom}_{\mathcal{T}}(X,\Sigma^n Y))$$

for $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$. Here $\operatorname{Ext}^n_{\mathcal{C}}$ are the groups of Yoneda extensions for $n \geq 0$ and we set $\operatorname{Ext}^n_{\mathcal{C}} := 0$ for $n < 0$. For the isomorphism see for example [**21**, Proposition 4.2.11]. These natural morphisms have been considered in [**8**, Lemma 2.11] for hearts of t-structures and in [**27**, A.8] for exact subcategories. The morphism $\Phi_n(X,Y)$ is an isomorphism for $n < 0$ as $\mathcal{C}$ is non-negative, for

$n = 0$ as $\mathcal{C}$ is full, and for $n = 1$ by [**27**, Corollary A.17]. Further, for $n = 2$ it is a monomorphisms by [**27**, Corollary A.17]. The following result appears in [**2**, Remarque 3.1.17] and [**8**, Lemma 2.11] when $\mathcal{C}$ is the heart of a bounded t-structure.

Lemma 3.13. *Let $\mathcal{C}$ be an admissible exact subcategory of $\mathcal{T}$ and let R be a realization functor of $\mathcal{C}$. Then the following are equivalent*

(1) R *is fully faithful;*
(2) $\Phi_n(X, Y)$ *is an isomorphism for all $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$;*
(3) $\Phi_n(X, Y)$ *is surjective for all $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$;*
(4) *For every $X, Y \in \mathcal{C}$, $n \geq 1$ and every morphism $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ there exists a $\mathcal{C}$-deflation $d\colon Z \to X$ with $f \circ d = 0$ in $\mathcal{T}$; and*
(4^{op}) *For every $X, Y \in \mathcal{C}$, $n \geq 1$ and every morphism $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ there exists an $\mathcal{C}$-inflation $i\colon Y \to W$ such that $\Sigma^n i \circ f = 0$ in $\mathcal{T}$.*

Proof. The implication (1) $\implies$ (2) is clear and the converse is an application of *dévissage* using $\mathrm{D}^b(\mathcal{C}) = \mathrm{thick}_{\mathrm{D}^b(\mathcal{C})}(\mathcal{C})$; see for example [**21**, Lemma 3.1.8].

The implication (2) $\implies$ (3) is clear and the converse is shown in [**27**, Corollary A.17].

A standard construction shows that (2) is equivalent to

(5) Every $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ with $X, Y \in \mathcal{C}$ and $n \geq 1$ decomposes as
$$X = X_0 \to \Sigma X_1 \to \Sigma^2 X_2 \to \cdots \to \Sigma^n X_n = \Sigma^n Y$$
for $X_i \in \mathcal{C}$;

see for example [**8**, Lemma 2.1] for the abelian case. Moreover, by induction over n this is also equivalent to

(6) Every $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ with $X, Y \in \mathcal{C}$ and $n \geq 1$ decomposes as $X \to \Sigma U \to \Sigma^n Y$ for some $U \in \mathcal{C}$.

So it is enough to show that (4) and (6) are equivalent. For the backward direction it is enough to observe that any morphism $X \to \Sigma U$ in $\mathcal{T}$ with $X, U \in \mathcal{C}$ induces an exact sequence $U \to Z \xrightarrow{d} X$ in $\mathcal{C}$. For the forward direction let $f\colon X \to \Sigma^n Y$ be a morphism in $\mathcal{T}$ with $X, Y \in \mathcal{C}$ and $n \geq 1$. Then there exists a deflation $d\colon Z \to X$ such that $f \circ d = 0$. We complete d to an exact sequence $U \to Z \xrightarrow{d} X$ in $\mathcal{C}$. Then f factors through the induced morphism $X \to \Sigma U$. This shows (6).

The equivalence of (2) and (4) holds by an analogous argument. □

Remark 3.14. The previous Lemma can be strengthened to yield an explicit description of the image of $\Phi_n(X, Y)$. That is, the subgroup $\mathrm{Im}(\Phi_n(X, Y))$ is the set of all morphisms $f\colon X \to \Sigma^n Y$ with $X, Y \in \mathcal{C}$ such that there exists a $\mathcal{C}$-deflation $d\colon Z \to X$ such that $f \circ d = 0$.

For a subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ we denote by $\mathrm{thick}_{\mathcal{T}}(\mathcal{C})$ the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{C}$.

Corollary 3.15. *Let $\mathcal{C}$ be an admissible exact subcategory of $\mathcal{T}$. A realization functor of $\mathcal{C}$ is an equivalence of triangulated categories if and only if it is fully faithful and $\mathrm{thick}_{\mathcal{T}}(\mathcal{C}) = \mathcal{T}$.* □

Example 3.16. Let $\mathcal{C}$ be a fully exact subcategory of $\mathcal{E}$. Then the induced functor $\mathrm{F}\colon \mathrm{D}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{E})$ is a realization functor for $\mathcal{C} \subseteq \mathrm{D}^b(\mathcal{E})$. The functor F is fully faithful if and only if the inclusion $\mathcal{C} \subseteq \mathcal{E}$ induces isomorphism on the Ext-groups. For example, the latter condition is satisfied by resolving subcategories; see [**1**, Section 2] and also [**11**, Definition 5.1].

The functor F is an equivalence if additionally $\mathcal{E}$ is the smallest additively-closed subcategory closed under the 2-out-of-three property containing $\mathcal{C}$. For example, this is satisfied by finitely resolving subcategories; cf. [**12**, Theorem 3.11(2)].

4. Proof of the main Proposition

For clarity we use different notations for the suspension in the stable category and the homotopy category. We write Σ for the suspension or shift functor in $\underline{\mathcal{E}}$ where $\mathcal{E}$ is a Frobenius exact category. By construction, we have $q(\Omega^n X) = \Sigma^{-n} X$ for any $X \in \mathcal{E}$ where Ω is the syzygy functor. On the other hand, for an additive category $\mathcal{A}$ we write $\mathrm{Ch}(\mathcal{A})$ for the category of chain complexes. In $\mathrm{Ch}(\mathcal{A})$ and the homotopy category $\mathrm{K}(\mathcal{A})$, we denote the degree n shift of a complex X by $X[n]$; this is the complex given by

$$X[n]^\ell = X^{\ell+n} \quad \text{and} \quad d_{X[n]} = (-1)^n d_X\,.$$

For a *map of complexes* $f\colon X \to Y$ we write

$$\partial(f) = d^Y f - f[-1]d^X \colon X \to Y[-1]\,.$$

The map f is a *chain map* if and only if $\partial(f) = 0$. Note, that a map of complexes need not commute with the differential, while a chain map does.

Lemma 4.1. *Let $\mathcal{E}$ be a Frobenius exact category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be a non-negative full subcategory and set $\mathcal{B} := q^{-1}(\mathcal{C})$. For any chain map $f\colon q(X) \to q(Y)$ in $\mathrm{Ch}(\underline{\mathcal{E}})$ with $X \in \mathrm{Ch}^+(\mathcal{B})$ and $Y \in \mathrm{Ch}^-(\mathcal{B})$ there exist chain maps $g\colon \hat{X} \to Y$ and $s\colon \hat{X} \to X$ with $\hat{X} \in \mathrm{Ch}^+(\mathcal{B})$ and $\mathrm{cone}(s) \in \mathrm{Ch}^b(\mathcal{P})$ such that $q(g) = f \circ q(s)$.*

Proof. First we construct an injective resolution I of X in the category of complexes. By [**16**, 4.1, Lemma, b)], there exists a left bounded complex I_0 of projective-injective objects and a chain map $j_0\colon X \to I_0$ that is an inflation in each degree. We denote the cokernel of j_0 by $q_0\colon I_0 \to \Omega^{-1}X$. Continuing this process, we obtain a sequence of chain maps

$$\begin{tikzcd}
 & & \Omega^{-1}X \arrow[dr, tail, "j_1"] & & \Omega^{-2}X \arrow[dr, tail, "j_2"] & & \Omega^{-3}X \arrow[dr, tail] & \\
X \arrow[r, tail, "h_{-1}=j_0"] & I_0 \arrow[ur, two heads, "q_0"] \arrow[rr, "h_0"'] & & I_1 \arrow[ur, two heads, "q_1"] \arrow[rr, "h_1"'] & & I_2 \arrow[ur, two heads, "q_2"] \arrow[rr] & & \cdots
\end{tikzcd}$$

We set $h_{-1} := j_0$ and $h_\ell := j_{\ell+1}q_\ell$. As X is left bounded we may assume that there exists an integer s such that $(I_\ell)^{\leq s} = 0$ for all ℓ; that is s is a universal lower bound. Since the maps j_ℓ are degreewise inflations, every map from $\Omega^{-\ell}X$ to a complex of projective-injective objects factors through j_ℓ.

We take a lift of f to a map of complexes $\hat{f}\colon X \to Y$ in $\mathrm{Ch}(\mathcal{E})$. This map need not commute with the differential. However, as it is the lift of a chain map in $\underline{\mathcal{E}}$ the map $\partial(\hat{f})$ factors through a complex of projective-injective objects. So there exists a map $g_0\colon I_0 \to Y[-1]$ such that $\partial(\hat{f}) = g_0 j_0$. For convenience we set $q_{-1} := \mathrm{id}_X$ and $g_{-1} := \hat{f}$. We now inductively construct maps $g_\ell\colon I_\ell \to Y[-\ell-1]$ with $\partial(g_{\ell-1}) = g_\ell j_\ell q_{\ell-1}$.

We assume that we have constructed the maps for any integer $\leq \ell$ for some $\ell \geq 0$. Then $0 = \partial(g_\ell) j_\ell q_{\ell-1}$ as j_ℓ and $q_{\ell-1}$ are chain maps. As $q_{\ell-1}$ consists of deflations in each degree, we get $0 = \partial(g_\ell) j_\ell$. Hence $\partial(g_\ell)$ factors through q_ℓ and we obtain the commutative diagram

$$\begin{tikzcd}
\Omega^{-\ell}X \arrow[r, tail, "j_\ell"] \arrow[dr, "0"'] & I_\ell \arrow[r, two heads, "q_\ell"] \arrow[d, dashed, "\partial(g_\ell)"] & \Omega^{-\ell-1}X \arrow[dl] \arrow[d, tail, "j_{\ell+1}"] \\
 & Y[-\ell-2] & I_{\ell+1} \arrow[l, dashed, "g_{\ell+1}"]
\end{tikzcd}$$

By the non-negativity of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathrm{Ch}(\underline{\mathcal{E}})}(q(\Omega^{-\ell-1}X), q(Y[-\ell-2])) = \mathrm{Hom}_{\mathrm{Ch}(\underline{\mathcal{E}})}(\Sigma^{\ell+1}q(X), q(Y[-\ell-2])) = 0\,.$$

Hence the map $\Omega^{-\ell-1}X \to Y[-\ell-2]$ factors through $j_{\ell+1}$. Note, that $g_{\ell+1}$ need not be a chain map. We continue this process until the map $g_{\ell+1}$ is a chain map. As Y is right bounded and the I_ℓ's have a universal upper bound, this will happen eventually.

Let t be an integer such that $Y^{\geq t} = 0$. We replace I_ℓ by the truncation $(I_\ell)^{\geqslant t-\ell-1}$. This does not effect the properties of the g_ℓ's, as they are zero in the other degrees. To summarize, we have a sequence of maps

$$\begin{tikzcd} X \arrow[r, tail, "h_{-1}"] \arrow[d, "\hat{f}=g_{-1}"] & I_0 \arrow[r, "h_0"] \arrow[d, "g_0"] & I_1 \arrow[r] \arrow[d, "g_1"] & \cdots \arrow[r] & I_{n-1} \arrow[r, "h_{n-1}"] \arrow[d, "g_{n-1}"] & I_n \arrow[d, "g_n"] \\ Y & Y[-1] & Y[-2] & \cdots & Y[-n] & Y[-n-1] \end{tikzcd}$$

where each I_ℓ is a bounded complex of projective-injective objects, g_n is a chain map and $\partial(g_{\ell-1}) = g_\ell h_{\ell-1}$ and $h_\ell h_{\ell-1} = 0$ for $0 \leq \ell \leq n$.

We take the total complex J of $I_0 \to \cdots \to I_n$. This means as graded module $J = \bigoplus I_i[i]$ with differential

$$d_J|_{I_i[i]} = d_{I_i[i]} + (-1)^i h_i[i] \,.$$

For convenience we use a nonstandard sign convention. We set

$$v := \sum_i g_i[i] \colon J \to Y[-1] \,.$$

This is a chain map, as

$$\begin{aligned} (vd_J)|_{I_i[i]} &= g_i[i-1]d_{I_i[1]} + (-1)^i g_{i+1}[i]h_i[i] \\ &= (-1)^i (g_i[-1]d_{I_i} + g_{i+1}h_i)[i] \\ &= (-1)^i (d_{Y[-i-1]}g_i)[-i] = d_{Y[-1]}g_i[i] = (d_{Y[-1]}v)\big|_{I_i[i]} \,. \end{aligned}$$

One can similarly check that the composition $u := (X \to I_0 \to J)$ is a chain map. By construction we have $\partial(\hat{f}) = vu$. Then $\hat{X} = \Sigma^{-1}\mathrm{cone}(u)$ and $g = (-v, \hat{f})$ and the natural map $s \colon \hat{X} \to X$ satisfy the desired properties. $\square$

Lemma 4.2. *Let $\mathcal{E}$ be a Frobenius exact category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q \colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $X \in \mathrm{K}^b(\mathcal{E})$. If $q(X) = 0$ in $\mathrm{K}^b(\underline{\mathcal{E}})$, then $X \in \mathrm{K}^b(\mathcal{P})$.*

Proof. It is enough to show the claim for a complex of the form

$$X = (\cdots \to 0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \to \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \to 0 \to \cdots)$$

for any $n \geq 0$. We use induction on n.

For $n = 0$, the assumption $q(X) = 0$ implies $q(X^0) = 0$. Hence $X^0 \in \mathcal{P}$.

Let $n \geq 1$. As $q(X) = 0$, the morphism $q(d^0)$ is a split monomorphism in $\underline{\mathcal{E}}$ and there exists a morphism $s \colon X^1 \to X^0$ such that $sd^0 - \mathrm{id}_{X^0} = ba$ for some morphisms $X^0 \xrightarrow{a} P \xrightarrow{b} X^0$ with $P \in \mathcal{P}$. We view P as a complex concentrated in degree zero and set

$$X' := \mathrm{cone}(\Sigma^{-1}a) = (\cdots \to 0 \to X^0 \xrightarrow{\left(\begin{smallmatrix} d^0 \\ a \end{smallmatrix}\right)} X^1 \oplus P \xrightarrow{\left(\begin{smallmatrix} d^1 & 0 \end{smallmatrix}\right)} X^2 \xrightarrow{d^2} X^3 \to \cdots) \,.$$

Since $\left(\begin{smallmatrix} s & -b \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} d^0 \\ a \end{smallmatrix}\right) = \mathrm{id}_{X^0}$, the zero differential of X' is a split monomorphism in $\mathcal{E}$. Therefore, in $\mathrm{K}^b(\mathcal{E})$, the complex X' is isomorphic to a complex Y concentrated between degrees 1 and n. In $\mathrm{K}^b(\underline{\mathcal{E}})$ we have $q(Y) \cong q(X') \cong q(X) = 0$. By induction hypothesis we have $X' \cong Y \in \mathrm{K}^b(\mathcal{P})$. By construction there is an exact triangle $X' \to P \to X \to \Sigma X'$, and as $X', P \in \mathrm{K}^b(\mathcal{P})$, so is X. $\square$

Proof of Proposition 3.8. As $q(\mathrm{K}^b(\mathcal{P})) = 0$ in $\mathrm{K}^b(\underline{\mathcal{E}})$, the functor $q \colon \mathrm{K}^b(\mathcal{B}) \to \mathrm{K}^b(\mathcal{C})$ induces a triangle functor

$$\overline{q} \colon \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{C}) \,.$$

We claim that $\overline{q}$ is an equivalence of triangulated categories. For this we need to show that $\overline{q}$ is full, faithful and essentially surjective.

The functor $\overline{q}$ is full by Lemma 4.1. By Lemma 4.2, whenever $\overline{q}(X) = 0$ then $X = 0$. As we already know that $\overline{q}$ is full, this implies that $\overline{q}$ is faithful by [**29**, p. 446]; also see [**30**, 4.3, 4.4].

It remains to show that $\overline{q}$ is essentially surjective. The essential image of $\overline{q}$ is a thick subcategory containing the complexes concentrated in degree zero. As these complexes generate $\mathrm{K}^b(\mathcal{C})$, the functor $\overline{q}$ is essentially surjective. □

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