**Categories of Sheaves**
October 14, 2014

# 1 Chapter 1: Schemes and Varieties

***Summary:*** *We explain affine schemes, projective schemes and how naive varieties can be understood as those.*

Let $R$ be a commutative ring. We define

$$\mathrm{Spec\ R} := \{\mathfrak{p} \subset \mathrm{R} \mid \mathfrak{p} \text{ prime ideal } \}$$

(an ideal $\mathfrak{p}$ is a prime ideal if $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. ). This set has a topology, called **Zariski topology**, defined through

$$A \subset \mathrm{Spec\ R} \text{ closed } \Leftrightarrow \text{there is an ideal } I \subset R \text{ such that}$$
$$A = V(I) := \{\mathfrak{p} \in \mathrm{Spec\ R} \mid \mathrm{I} \subset \mathfrak{p}\}$$

Observe, that we have a natural identification

$$V(I) \to \mathrm{Spec\ R/I}$$
$$\mathfrak{p} \mapsto \mathfrak{p} + I$$

**Lemma 1.** *A basis for the Zariski topology is given by the* ***principal open sets***

$$D(r) := \{\mathfrak{p} \in \mathrm{Spec\ R} \mid \mathrm{r} \notin \mathfrak{p}\}$$

*for all $r \in R$. Observe, that we have a natural identification $D(r) = \mathrm{Spec\ R}[\frac{1}{\mathrm{r}}]$.*

**proof:** $D(r) = \mathrm{Spec\ R} \setminus \mathrm{V((r))}$ is closed. If $I = ((r_t)_{t \in T}) \subset R$ is an ideal, then we can write the open

$$\mathrm{Spec\ R} \setminus \mathrm{V(I)} = \mathrm{Spec\ R} \setminus (\bigcap_{\mathrm{t \in T}} \mathrm{V((r_t))}) = \bigcup_{\mathrm{t \in T}} \mathrm{D(r_t)}.$$

Clearly, it holds $D(r) \cap D(s) = D(rs)$.

**Remark.** (1) A point $\mathfrak{p} \in \mathrm{Spec\ R}$ is closed if and only if $\mathfrak{p}$ is a maximal ideal.

(2) $(0) = \{0\} \in \mathrm{Spec\ R}$ if and only if $R$ has no zero divisors.

The closed sets $V(I)$ have the following properties

(1) $V(I \cap J) = V(I) \cup V(J)$

(2) $V(\sum_{t \in T} I_t) = \bigcap_t V(I_t)$

(3) $V(R) = \emptyset, V((0)) = \mathrm{Spec\ R}$ (the trivial closed sets)

**Example.** (1) If $R$ is a local ring with maximal ideal which is a principal ideal $\mathfrak{m} = (p)$, then $\mathrm{Spec\ R} = \{(0), \mathfrak{m}\}$ consists of a closed and an open point.

(2) $\mathrm{Spec\ \mathbb{Z} = \{(p) \mid p \in \mathbb{N}\ prime\ number\} \cup \{(0)\}}$ and the nontrivial closed sets are of the form

$$\{(p_1), \ldots, (p_r) \mid p_i \text{ prime numbers}\} = V((p_1 \cdots p_r))$$

Except (0) all prime ideals are maximal.

(3) $\mathrm{Spec\ \mathbb{C}[X] = \{(X - c) \mid c \in \mathbb{C}\} \cup \{0\}}$ and the nontrivial closed sets are of the form

$$\{(X - c_1), \ldots, (X - c_r) \mid c_i \in \mathbb{C}\} = V(((X - c_1) \cdots (X - c_r)))$$

(4) If $K$ is a field, we have

$$\mathrm{Spec\ K} = \{(0)\}$$

but we also have $\mathrm{Spec\ K[X]/(X^n) = \{(\overline{X})\}}$ consists only of a point for every $n \in \mathbb{N}$.

**Definition 1.** Let $I \subset R$ be an ideal, we call $\sqrt{I} := \{r \in R \mid \exists n > 0 \colon r^n \in I\}$ the **radical ideal**.

Let us summarize some properties.

(1) $V(I) \subset V(J)$ if and only if $J \subset \sqrt{I}$ and in particular $V(I) = V(\sqrt{I})$.

(2) $R$ has no nontrivial nilpotent elements if and only if $\sqrt{(0)} = \{0\}$, in that case we call $R$ **reduced**. We set $R_{red} := R/\sqrt{(0)}$, then clearly $\mathrm{Spec\ R = V(\{0\}) = V(\sqrt{0}) = Spec\ R_{red}}$. In other words, we can not see reducedness at ring spectra.

But we can see the following property.

**Definition 2.** We call a topological space $X$ **irreducible** if any two open subsets have a nontrivial intersection.

**Remark.** Spec R is irreducible if and only if $R_{red}$ has no nontrivial zero divisors. For example, if $ab = 0$, then

$$\emptyset = D(0) = D(a) \cap D(b)$$

In that case, $\{0\}$ is the unique minimal prime ideal (= the fattest point) in $\mathrm{Spec\ R_{red}}$. In general there are a couple of minimal prime ideals, the closures of these minimal prime ideals are called **irreducible components** and they are the maximal irreducible subsets of Spec R. The minimal prime ideals are called the **generic points**.

If $R$ is noetherian, then Spec R has only finitely many irreducible components.

**Example.** $R = K[X,Y,Z]/(X^2Y^3(Y^2 - Z)^5)$ with $K$ a field. Then $\mathrm{Spec\ R = V((\overline{X})) \cup V((\overline{Y})) \cup V((\overline{Y}^2 - \overline{Z}))}$ is the decomposition into irreducible components. The minimal prime ideals of $R_{red} = K[X,Y,Z]/(XY(Y^2 - Z))$ are $(\overline{X}), (\overline{Y}), (\overline{Y}^2 - \overline{Z})$.

Now given a ring homomorphism $f\colon R \to S$ this induces a map

$$\begin{aligned} \text{Spec f}\colon \text{Spec S} &\to \text{Spec R} \\ \mathfrak{q} &\mapsto f^{-1}(\mathfrak{q}). \end{aligned}$$

This map is continous because

$$(\text{Spec f})^{-1}(\text{V(I)}) = \{\mathfrak{q} \in \text{Spec S} \mid \text{I} \subset \text{f}^{-1}(\mathfrak{q})\} \quad = \{\mathfrak{q} \in \text{Spec S} \mid \text{f(I)} \subset \mathfrak{q}\} = \text{V((f(I))}$$

where $(f(I))$ is the ideal generated by $f(I)$. This means, we have a contravariant functor

$$\text{Spec : (commutative rings)}^{\text{op}} \to \text{(topological spaces)}, \quad \text{R} \mapsto \text{Spec R}.$$

Next: We want *extra data* to make this an equivalence of categories. This will be the **structure sheaf**.

## 1.1 Sheaves

Let $X$ be a topological space. We define a category Top(X) with objects the open subset s of $X$ and with morphisms the inclusion of the subsets, i.e. for $U, V$ open in $X$ we have the homsets have either one point or are empty

$$\text{Hom}_{\text{Top(X)}}(U, V) = \begin{cases} \{i\colon U \subset V\}, & \text{if } U \subset V \\ \emptyset, & \text{else.} \end{cases}$$

**Definition 3.** A (set-valued) **presheaf** is a functor

$$\begin{aligned} \mathcal{O}\colon \text{Top(X)}^{\text{op}} &\to \text{(sets)} \\ U &\mapsto \mathcal{O}(U) \\ U \subset V &\mapsto \rho_U^V\colon \mathcal{O}(U) \to \mathcal{O}(V). \end{aligned}$$

here: You should think of *functions on* $U$ and the $\rho_U^V$ are the maps restricting the function from the bigger set to the smaller.

But *functions* have the extra property: For every open covering, a function on the whole space corresponds to a family of functions on the open sets of the covering which coincide on the intersections. This property is the precisely the **sheaf property**, or is sometimes called **local-to-global property**.
Also since $\emptyset$ is an initial object in Top(X), we want it to be a terminal object in *Sets*, i.e. the set with a single point. Formally, the definition says.

**Definition 4.** A presheaf $\mathcal{O}\colon \text{Top(X)}^{\text{op}} \to \text{(sets)}$ is called a **sheaf** if

(1) $\mathcal{O}(\emptyset) = \{pt\}$ and

(2) for every (not nec. finite) union of opens $U = \bigcup_{\text{ß}\in I} U_i$ we have that

$$\mathcal{O}(U) \xrightarrow{r} \prod_{i\in I} \mathcal{O}(U_i) \overset{r_1}{\underset{r_2}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{O}(U_i \cap U_j)$$

where

$$r(s) := (\rho^U_{U_i}(s))_{i\in I},$$
$$r_1((s_i)_{i\in I}) = (\rho^{U_i}_{U_i\cap U_j}(s_i))_{i,j\in I\times I},$$
$$r_2((s_i)_{i\in I}) = (\rho^{U_j}_{U_i\cap U_j}(s_j))_{i,j\in I\times I},$$

is exact (for set-valued presheaves this says: $r$ is injective and the image of $r$ is precisely the set of elements $a$ where $r_1(a) = r_2(a)$, for sheaves of abelian groups this means that the sequence with $r_1 - r_2$ is exact.)
The elements in $\mathcal{O}(U)$ are called **sections over** $U$ and $\mathcal{O}(X)$ is called the set of **global sections**, also the following notation is common for it $\Gamma(X, \mathcal{O}) := \mathcal{O}(X)$.

**Example.** (1) Let $X$ be a topological space and $\mathcal{O}(U) = \{f : U \to \mathbb{R} : f \text{ continous}\}$.

(2) Let $X$ be a complex manifold and $\mathcal{O}(U) := \{f : U \to \mathbb{C} : f \text{ differentiable}\}$.

Let $X$ be a topological space. We write $Sh(X)$ for the category of (set-valued/abelian group/ring valued) sheaves on $X$, here morphisms of sheaves are just all natural transformations of the functors.

**stalks.** Let $x \in X$ and $\mathcal{O}$ a presheaf on $X$. We consider $Nbh(x)$ = the open subsets containing $x$. This is a partially ordered set with respect to anti-inclusion ($V < U$ iff $U \subset V$). For every $V < U$ in $Nbh(x)$ we have the restriction maps

$$\rho^V_U : \mathcal{O}(V) \to \mathcal{O}(U),$$

then, the stalk of $\mathcal{O}$ at $x$ is defined as

$$\mathcal{O}_x := \mathrm{colim}_{\mathrm{U\in Nbh(x)}} \mathcal{O}(\mathrm{U}).$$

This means that for every $U \in Nbh(x)$ there is a map $\rho^U_x : \mathcal{O}(U) \to \mathcal{O}_x, s \mapsto s_x$ such that for every $V < U$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}(V) & \xrightarrow{\rho^V_U} & \mathcal{O}(U) \\ & {\scriptstyle \rho^V_x}\searrow \quad \swarrow{\scriptstyle \rho^U_x} & \\ & \mathcal{O}_x & \end{array}$$

and $(\mathcal{O}_x, (\rho^U_x)_{U\in Nbh(x)})$ is *universal* with that property, which means whenever you have an object $T$ and maps $t_U : \mathcal{O}(U) \to T$ (in the category in which your presheaf takes

values) such that $\rho_U^V \circ t_U = t_V$ for all $V < U$,
then there is a unique map $c\colon \mathcal{O}_x \to T$ such that $t_U = \rho_x^U \circ c$ for every $U \in Nbh(x)$.
The elements of $\mathcal{O}_x$ can be realized as equivalence classes $[(s,U)]$ of pairs $(s,U)$ with $U \in Nbh(x), s \in \mathcal{O}(U)$ and

$$[(s,U)] = [(t,V)] \Leftrightarrow \exists W \in Nbh(x), W \subset U \cap V\colon \rho_W^U(s) = \rho_W^V(t).$$

One main property of stalks is the following:

**Proposition 1.** *A morphism $\varphi\colon \mathcal{O} \to \mathcal{P}$ of sheaves on a topological space $X$ is an isomorphism if and only if for every point $x \in X$ the maps at the stalks $\varphi_x\colon \mathcal{O}_x \to \mathcal{P}_x$ is an isomorphism.*

### 1.1.1 The structure sheaf

Let $R$ be a commutative ring and $X :=$ Spec R. We define the **structure sheaf $\mathcal{O}_X$** as the following ring-valued sheaf

$$\mathcal{O}_X\colon Top(X)^{op} \to \text{(com. rings including zero)}$$
$$\mathcal{O}(D(r)) := R\left[\frac{1}{r}\right].$$

here $R\left[\frac{1}{r}\right] := R[X]/(rX-1)$ is the zero ring if and only if $r$ is nilpotent.
**Localization.** Let $S \subset R$ be a subset. We say, that $S$ is **multiplicative** if $1 \in S$ and $a, b \in S$ implies $ab \in S$. Let $R[S^{-1}]$ (or $S^{-1}R$) be the ring with elements given by equivalence classes $[\frac{a}{s}], a \in R, s \in S$ with

$$[\frac{a}{s}] = [\frac{b}{t}] \Leftrightarrow \exists u \in S\colon u(at - bs) = 0$$

and $\ell_S\colon R \to R[S^{-1}], r \mapsto [\frac{r}{1}]$, this map is called **the localization with respect to $S$.** It is the universal ringhomomorphism $f\colon R \to T$ with $f(S)$ invertible, i.e. any of those has to factor uniquely over $\ell_S$.
One of its main properties is that it is flat, which means that the functor $-\otimes_R R[S^{-1}]$ from $R$-modules to $R[S^{-1}]$-modules is exact.

We already met the localization $R[\frac{1}{r}]$ which is the localization with respect to $S = \{1, r, r^2, \ldots\}$.

**Lemma 2.** *Let $X =$ Spec R and $x \in X$ a point. Then, the stalk of the structure sheaf at $x$ is the local ring*

$$\mathcal{O}_{X,x} = R_x,$$

*where $R_x$ is the localization with respect to the multiplicative set $S = R\setminus x$. In particular, this is a local ring.*

**proof:** Let $x = \mathfrak{p} \subset R$ a prime ideal.

$$\mathcal{O}_{X,x} = \mathrm{colim}_{\mathrm{r} \notin \mathfrak{p}} \mathrm{R}[\frac{1}{\mathrm{r}}] = \mathrm{R}[\mathrm{S}^{-1}]$$

with $S = R \setminus \mathfrak{p}$ and this is $R_{\mathfrak{p}}$.

□

### 1.1.2 Pushforward of sheaves

Given a continous map $f: X \to Y$, we have a natural transformation

$$f_*: Sh(X) \to Sh(Y), \quad \mathcal{O} \mapsto f_*\mathcal{O}$$

with for $U \subset Y$ open, we define $f_*\mathcal{O}(U) := \mathcal{O}(f^{-1}(U)$. Given a morphism of sheaves $\varphi: \mathcal{O} \to \mathcal{P}$ on $X$. We define $f_*\varphi: f_*\mathcal{O} \to f_*\mathcal{P}$ on an open subset $U \subset Y$ as $f_*\varphi(U) := \varphi(f^{-1}(U))$.

**Definition 5.** We define the category of **locally ringed spaces** *lrTop* as a category with

(1) objects are pairs $(X, \mathcal{O}_X)$ with $X$ a topological space and $\mathcal{O}_X$ a ring-valued sheaf on $X$ with all stalks $\mathcal{O}_{X,x}$ are local rings.

(2) morphisms $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ are given by pairs $(f, f^{\#})$ with $f: X \to Y$ continous and

$$f^{\#}: \mathcal{O}_Y \to f_*\mathcal{O}_X$$

a morphism of sheaves **with the extra condition**: the ring homomorphisms at the stalks $f_x^{\#}: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$, defined as follows, we have a map of partially ordered sets

$$Nbh(f(x)) \to Nbh(x),\ U \mapsto f^{-1}(U)$$

which induces a map of directed systems

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{f^{\#}(V)} & \mathcal{O}_X(f^{-1}(V)) \\ \downarrow \rho_U^V & & \downarrow \rho_{f^{-1}(U)}^{f^{-1}(V)} \\ \mathcal{O}_Y(U) & \xrightarrow{f^{\#}(U)} & \mathcal{O}_X(f^{-1}(U)) \end{array}$$

and that induces a map on the colimits $f_x^{\#}: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. We require that it fulfills

$$(f_x^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$$

(i.e. is a *local* ring homomorphisms).

**Definition 6.** We define the category of **affine schemes** (affine schemes) as the full subcategory of *lrTop* with objects isomorphic to pairs

$$(\mathrm{Spec\ R}, \mathcal{O}_{\mathrm{Spec\ R}}), \quad \mathrm{R\ commutative\ ring}\ .$$

In our interpretation of $\mathcal{O}$ as *functions* on $X$: The map of sheaves $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is given by precomposing functions with the map $f\colon X \to Y$. The stalks are equivalence classes of functions on a neighborhood of $x \in X$ called germs, the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the germs vanishing in the point $x$. The last statement, should be understood as: If you have a germ which is vanishing on $x$ and you precompose with $f$, then you get a function which vanishes in $f(x)$.

But a warning: The condition that the stalks are local rings is for most sheaves not fulfilled. For example sheaves of continous functions do not have always stalks which are local rings.

**Theorem 1.1.** *There is an equivalence of categories*

$$\begin{aligned}\text{Spec} : (\textit{commutative rings})^{\text{op}} &\to (\textit{affine schemes}),\\ R &\mapsto (\text{Spec R}, \mathcal{O}_{\text{Spec R}})\\ h\colon R \to S &\mapsto (f = \text{Spec h}, \text{f}^\#)\colon (\text{Spec S}, \mathcal{O}_{\text{Spec S}}) \to (\text{Spec R}, \mathcal{O}_{\text{Spec R}})\end{aligned}$$

*where* $f^\#\colon \mathcal{O}_{\text{Spec R}} \to f_*\mathcal{O}_{\text{Spec S}}$ *is given on* $D(r) \subset \text{Spec R}$ *as the from* $h$ *induced map*

$$R\left[\frac{1}{r}\right] \to S\left[\frac{1}{h(r)}\right], \quad \frac{a}{r^n} \mapsto \frac{h(a)}{h(r)^n}$$

**proof:** □

The most notorious example of an affine scheme, we give the name **affine ($n$-)space** (over a field $K$), it is defined as

$$\mathbb{A}^n := (\text{Spec K}[\text{x}_1, \ldots, \text{x}_\text{n}], \mathcal{O}_{\text{Spec K}[\text{x}_1,\ldots,\text{x}_\text{n}]})$$

**Definition 7.** (1st definition of schemes) A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that for every point $x \in X$ there is an open $x \in U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

The big plus that looking at schemes gives us, it that we can *glue* affine schemes to not affine schemes. Let us have an examples of a scheme which are not affine.

**Example.** The projective $n$-space. Let $R = K[x_1, \ldots, x_{n+1}]$ be the polynomial ring over a field $K$, we consider this with the standard grading giving the $x_i$ degree 1. An ideal is called homogeneous, if it is generated by homogeneous elements. We call $\mathfrak{m} := (x_1, \ldots, x_n)$, (this will be the ideal corresponding to the zero point in $K^{n+1}$, so we better avoid it). Let

$$\mathbb{P}^n := \text{Proj R} := \{\mathfrak{p} \subset \text{R} \mid \mathfrak{p} \text{ homogeneous prime ideal }\} \setminus \{\mathfrak{m}\}$$

this becomes a topological space by defining the closed subsets as precisely the sets

$$V_+(I) := \{\mathfrak{p} \in \text{Proj R} \mid \text{I} \subset \mathfrak{p}\},$$

for $I \subset R$ a homogeneous ideal. Alternatively on can define it as the topology with basis given by the sets

$$D_+(r) := \{\mathfrak{p} \in \mathrm{Proj\ R} \mid \mathrm{r} \notin \mathfrak{p}\}$$

where $r \in R$ is a homogeneous element. Now, we define the structure sheaf through

$$\mathcal{O}_{\mathbb{P}^n}(D_+(r)) := \left(R\left[\frac{1}{r}\right]\right)^0$$

where $()^0$ refers to taking the subring of elements of degree zero. We claim that, this is in fact a scheme with $D_+(x_i)_{i=1,\dots,n}$ being an open cover of affine schemes, in fact they are all affine $n$-spaces. It holds

$$(D_+(x_i), \mathcal{O}_X|_{D_+(x_i)}) \cong (\mathrm{Spec\ K}\left[\frac{\mathrm{x_1}}{\mathrm{x_i}}, \dots, \frac{\mathrm{x_{n+1}}}{\mathrm{x_i}}\right], \mathcal{O}_{\mathrm{Spec\ K}\left[\frac{\mathrm{x_1}}{\mathrm{x_i}}, \dots, \frac{\mathrm{x_{n+1}}}{\mathrm{x_i}}\right]}).$$

Here, we have to mention the following result about the category of schemes.

**Theorem 1.2.** *The category of schemes has pullbacks. This means given a diagram of morphisms of schemes*

$$Y \xrightarrow{f} X \xleftarrow{g} Z$$

*there is scheme $Y \times_X Z$ together with morphisms $p, q$ making the following diagram commutative*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{p} & Z \\ \downarrow q & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

*such that any commutative diagram*

$$\begin{array}{ccc} T & \xrightarrow{t} & Z \\ \downarrow s & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

*factors over it, i.e. there is a unique $f\colon T \to Y \times_X Z$ with $p \circ f = t, q \circ f = s$.*

For affine schemes Spec (R $\leftarrow$ S $\to$ T) it holds

$$\mathrm{Spec\ R} \times_{\mathrm{Spec\ S}} \mathrm{Spec\ T} = \mathrm{Spec\ (R \otimes_S T)}.$$

It is most useful if we consider $S$-algebras $R$ (as open affines in our scheme) and want to change to $T$-algebras. Then, this is just given by tensoring the rings, i.e. this gives a change of the base ring. We will later look at $K$-algebras (schemes over a field $K$), and just consider passing to a field extensions to assume that our base field is algebraically closed.

### 1.1.3 Open and closed subschemes

**Definition 8.** (1) Let $(X, \mathcal{O}_X)$ be a scheme and $U \subset X$ an open subset, then $(U, \mathcal{O}_U := \mathcal{O}_X|_U)$ is a scheme again and there is a natural morphism of schemes $(U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$, we call $(U, \mathcal{O}_U)$ an **open subscheme** and the morphism an **open embedding**.

(2) Let $(X, \mathcal{O}_X)$ be a scheme. A subsheaf $\mathcal{I} \subset \mathcal{O}_X$ on $X$ is called an ideal sheaf, if for every $U \subset X$ we have that $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal in the ring $\mathcal{O}_X(U)$. Then, we define a topological space (with the induced topology)

$$V(\mathcal{I}) = \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\}$$

and a sheaf of rings on $V(\mathcal{I})$ as follows, for every open affine $U = \mathrm{Spec\ R} \subset \mathrm{X}$ we have $\mathcal{I}(U) =: I \subset R$ is an ideal and $\mathrm{Spec\ R/I} = \mathrm{V(I)} = \mathrm{V}(\mathcal{I}) \cap \mathrm{U}$ is an open affine in $V(\mathcal{I})$, we define

$$\mathcal{O}_{V(\mathcal{I})}(V(I)) := \mathcal{O}_{\mathrm{Spec\ R/I}},$$

then it is easy to see that this defines a scheme $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ and we have a natural scheme morphism

$$(i, i^{\#})\colon (V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})}) \to (X, \mathcal{O}_X)$$

where $i$ is the inclusion and $i^{\#}\colon \mathcal{O}_X \to i_*\mathcal{O}_{V(\mathcal{I})}$ is given on the open $U = \mathrm{Spec\ R} \subset \mathrm{X}$ by the canonical morphism

$$\mathcal{O}(U) = R \to R/I = \mathcal{O}_{V(\mathcal{I})}(i^{-1}(U)).$$

It is easy to see that the closed subschemes of Spec R and the closed immersions are of the form

$$V(I) = \mathrm{Spec\ (R/I)} \xrightarrow{\mathrm{Spec\ (p_I)}} \mathrm{Spec\ R}$$

where $p_I\colon R \to R/I$ is the canonical map. Open subschemes of affine schemes are not so easy to describe. Open subschemes of affine schemes do not need to be affine again. See example (3) below.

**Example.** (1) If $\mathbb{X} = (X, \mathcal{O}_X)$ is a scheme, then there is an ideal sheaf $\mathcal{N} \subset \mathcal{O}_X$ defined by $\mathcal{N}(U) = \sqrt{(0)} \subset \mathcal{O}(U)$ for $U \subset X$ open. The corresponding closed subscheme, is called the **underlying reduced subscheme** $\mathbb{X}_{red}$ and has the same topological space as $\mathbb{X}$, but its affine charts are of the form $\mathrm{Spec\ R}/\sqrt{0}$ for all affine charts Spec R of $\mathbb{X}$.

(2) The $D_+(r)$ give open subschemes of $\mathbb{P}^n$. Every closed subscheme of $\mathbb{P}^n$ is of the form $V_+(I)$ for a homogeneous ideal $I \subset K[x_1, \ldots, x_{n+1}]$.

(3) Look at the open subscheme $X = \mathrm{Spec\ K[x]} \setminus \{\{(\mathrm{x})\} \cup \mathrm{Spec\ K[x]} \setminus \{(\mathrm{x}-1)\} = \mathrm{D(x)} \cup \mathrm{D(x-1)} \subset \mathbb{A}^1$, we claim that this is not an affine scheme...

## 1.2 Projective schemes

**Definition 9.** A **projective scheme** (of finite type over $K$) is a scheme isomorphic to a closed subscheme of a projective space (over $K$), i.e. to a

$$V_+(I) \subset \mathbb{P}^n$$

for an homogeneous ideal $I \subset K[x_1, \ldots, x_{n+1}]$.
A **quasi-projective scheme** is a an open subscheme of a projective scheme.

## 1.3 Naive varieties

Let now $K = \overline{K}$ be an algebraically closed field for the rest of this section. When you are only interested in reduced schemes of finite type over $K$ (i.e. schemes where the coordinate rings are finitely generated $K$-algebras with no nontrivial nilpotent elements), then it is enough to study naive varieties.
To understand the brigde between the two notions, let us start with the functorial point of view. We can idenity a scheme with its *functor of points* (by the lemma of Nakayama)

$$X\colon (\text{schemes})^{op} \to (\text{sets})$$
$$T \mapsto X(T) := \mathrm{Hom}_{(\text{schemes})}(T, X)$$

We call $X(T)$ the $T$-valued points of $X$ and if $T = \text{Spec A}$, we say $A$-valued points of $X$. In fact, since schemes are glued from affine schemes, the functor of points is already uniquely determined by the restriction to affine schemes, i.e. to commutative rings. For example, if $X = \text{Spec R}$ with $R$ is a $K$-algebra, we look at the functor

$$X\colon (\text{K-algebras}) \to (\text{sets})$$
$$A \mapsto X(A) := \mathrm{Hom}_{K-alg}(R, A).$$

Here, the naive question: Given a scheme over $K$ (i.e. the coordinate rings are $K$-algebras), when is $X$ already determined by its $K$-valued points $X(K)$?
For the schemes $V(I) \subset \mathbb{A}^n$ and $V_+(I) \subset \mathbb{P}^n$, we give a positive answers below.

**Remark.** Now, one can define a scheme as a functor

$$X\colon (\text{commutative rings}) \to (\text{sets})$$

which has certain extra properties (being locally of the form $\mathrm{Hom}(R, -)$ ). This approach can be for example found in a book of Demazure.
The usual definition of a group scheme is for example, that it is a scheme with the functor of points is factoring over the forgetful functor ( groups) $\to$ (sets). For example $\mathbf{Gl}_n$ a group scheme with functor of points

$$\mathbf{Gl}_n(R) := \{\phi\colon R^n \to R^n \mid \phi \ \ R-\text{linear and bijective}\}$$

### 1.3.1 Naive affine varieties

We define
$$\mathbb{A}^n(K) := \mathrm{Hom}_{K-alg}(K[x_1,\dots,x_n],K),$$
then, since $K$ is algebraically closed we have that every maximal ideal in $K[x_1,\dots,x_n]$ is of the form $(x_1-a_1,\dots,x_n-a_n)$ with $a_1,\dots,a_n\in K$. Therefore,
$$K^n \cong \mathrm{Hom}_K(K[x_1,\dots,x_n],K) = \{\mathfrak{m}\in \mathrm{Spec\ K[x_1,\dots,x_n]} \mid \mathfrak{m} \text{ maximal}\},$$
i.e. $\mathbb{A}^n(K)=K^n$ consists precisely of the closed points of $\mathrm{Spec\ K[x_1,\dots,x_n]}$. The Zariski topology on $K^n$ can be described as: The closed subsets are precisely the zero sets of polynomials
$$Z(f_1,\dots,f_r) := \{(a_1,\dots,a_n)\in K^n \mid f_i(a_1,\dots,a_n)=0,\ 1\le i\le r\},$$
for $f_1,\dots,f_r\in K[x_1,\dots,x_n]$. Observe, $Z(f_1,\dots,f_r)=Z((f_1,\dots,f_r))$. These sets (with the induced Zariski topology) will be called an algebraic set or a **naive affine varieties**.

Now, given a Zariski-closed subset $X\subset K^n$, we define
$$I(X) := \{f\in K[x_1,\dots,x_n] \mid f(a)=0 \mid a\in X\}$$
this is an ideal. In fact, it is a radical ideal. We have to mention the following old result which might have inspired Grothendieck for his scheme theory.

**Theorem 1.3.** *(Hilbert's Nullstellensatz) Let $K=\overline{K}$ be algebraically closed and $J\subset K[x_1,\dots,x_n]=:R$, then*
$$I(Z(J))=\sqrt{J}.$$
*and we have a bijection*
$$I\colon (\textit{naive affine varieties} \subset \mathbb{A}^n(K)) \leftrightarrow (\textit{radical ideals in } R)\colon Z$$

**Morphisms (defined using regular functions).** A regular function $K^n\to K$ is a function, given by evaluating a polynomial $(a_1,\dots,a_n)\mapsto f(a_1,\dots,a_n)$. So, the ring of regular functions on $\mathbb{A}^n(K)$ is given by definition by $K[x_1,\dots,x_n]=\mathcal{O}_{\mathbb{A}^n}(\mathrm{Spec\ K[x_1,\dots,x_n]})$. Now, let $U$ be an open subset of a naive affine variety in $\mathbb{A}^n(K)$. A function $f\colon U\to K$ is called regular at a point $a\in U$ if there is an open neighborhood (in the naive variety) $a\in V\subset U$ and polynomials $P,Q\in K[x_1,\dots,x_n]$ with $Q(v)\neq 0$ for all $v\in V$ such that $f|_V=(\frac{P}{Q})|_V\colon V\to K$. We say $f$ is regular on $U$, if it is regular in all points (of $U$). Now, we can define morphisms of naive affine varieties as a map $\varphi\colon X\to Y$ such that for every regular function $f$ on a open subset $U$ of $Y$, it holds $\varphi^{-1}(U)\xrightarrow{\varphi} U\xrightarrow{f} K$ is regular again. Let $\mathcal{V}$ be the category of naive affine varieties.

Now, we can formulate this as an equivalence of categories.

**Theorem 1.4.** *Let ( reduced affine schemes) be the full subcategory of the category of schemes with objects reduced affine schemes. Then, there is an equivalence of categories*
$$t\colon \mathcal{V}\to (\textit{ reduced affine schemes})$$

### 1.3.2 Naive projective varieties

Again, $\mathbb{P}^n(K)$ is the set of $K$-valued points which can be identified since $K$ is algebraically closed with the set

$$\{L \subset K^n \mid L \text{ is one dimensional subvector space}\} = (K^{n+1} \setminus \{0\})/(x \equiv \lambda x).$$

Again, this can be identified with the closed points of the underlying topological space of $\mathbb{P}^n$. The induced Zariski topology can be described as: The closed subsets are precisely the sets

$$Z_+(f_1, \dots, f_r) := \{[a_1 \colon \cdots \colon a_{n+1}] \in \mathbb{P}^n(K) \mid f_i(a_1, \dots, a_{n+1}) = 0,\ 1 \le i \le r\},$$

for $f_1, \dots, f_r \in K[x_1, \dots, x_{n+1}]$ homogeneous polynomials. These sets (with the induced Zariski topology) will be called **naive projective varieties**.

Now, given a Zariski-closed subset $X \subset \mathbb{P}^n(K)$, we define

$$I_+(X) := \{f \in K[x_1, \dots, x_{n+1}] \mid f \text{ homogeneous and } f(a) = 0,\ \forall a \in X\}$$

this is a homogeneous ideal. In fact, it is also a radical ideal.

**Theorem 1.5.** *(projective Hilbert's Nullstellensatz) Let $K = \overline{K}$ be algebraically closed and $J \subset K[x_1, \dots, x_{n+1}] =: R$, be a homogeneous ideal $J \neq (x_1, \dots, x_{n+1}) = \mathfrak{m}$, then*

$$I_+(Z_+(J)) = \sqrt{J}.$$

*and we have a bijection*

$$I_+ \colon (\textit{naive projective varieties} \subset \mathbb{P}^n(K)) \leftrightarrow (\textit{radical homogeneous ideals in } R) \setminus \{\mathfrak{m}\} \colon Z_+$$

**Morphisms (defined using regular functions).** Let $U$ be an open subset of projective naive projective variety in $\mathbb{P}^n(K)$. A function $f \colon U \to K$ is regular at $a \in U$ if there is an open neighborhood (in the naive projective variety) $a \in V \subset V$ and homogeneous polynomials $P, Q \in K[x_1, \dots, x_{n+1}]$ of the same degree with $Q(v) \neq 0$ for all $v \in V$ such that

$$f|_V = \frac{P}{Q}|_V \colon V \to K.$$

It is called regular, if it is regular in every point of $U$.
Now, we can define morphisms of naive projective varieties as a map $\varphi \colon X \to Y$ such that for every regular function $f$ on a open subset $U$ of $Y$, it holds $\varphi^{-1}(U) \xrightarrow{\varphi} U \xrightarrow{f} K$ is regular again. Let $\mathcal{P}$ be the category of naive projective varieties.

Now, we can formulate this as an equivalence of categories.

**Theorem 1.6.** *Let ( reduced projective schemes) be the full subcategory of the category of schemes with objects reduced projective schemes. Then, there is an equivalence of categories*

$$t \colon \mathcal{P} \to (\textit{ reduced projective schemes})$$