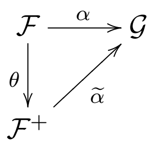
Categories of Sheaves
October 28, 2014

1 Chapter 2: Categories of Sheaves

Summary: *Sheaves with values in an abelian groups. Coherent sheaves. The equivalence of coherent sheaves over an affine scheme to the finitely generated modules over the coordinate ring. Vector bundles. The equivalence of vector bundles to locally free sheaves. The equivalence of coherent sheaves on a noetherian projective schemes to the quotient category given by finitely generated graded modules over the coordinate ring modulo the finite dimensional graded modules. Gabriel's theorem - how to recover a noetherian scheme from the category of quasi-coherent sheaves.*

1.1 The category of sheaves with values in abelian groups $Sh(X)$

Sheaves on topological spaces have studied in full generality for example in the not easy to read book of Godement, [God73].
Let (Ab) be the category of abelian groups (but it could be any abelian category) and denote $Sh(X)$ the category of sheaves on a topological space X with a values in (Ab). If we write $PSh(X)$ for the category of all presheaves on X with values in (Ab). Associated to a presheaf, we can get a sheaf $\mathcal{F}^+$ together with a morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ called **sheafification** such that for every morphism $\alpha\colon \mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, there is a unique morphism $\widetilde{\alpha}\colon \mathcal{F}^+ \to \mathcal{G}$ such that



commutes. This defines a functor called **sheafification**

$$()^+\colon PSh(X) \to Sh(X), \quad \mathcal{F} \mapsto \mathcal{F}^+,$$

explicitly

$$\mathcal{F}^+(U) := \{f\colon U \mapsto \bigsqcup_{x\in U} \mathcal{F}_x \mid \forall x \in U \exists x \in V \subset U, s \in \mathcal{F}(V)\colon f(x) = s_x\}$$

it holds $\theta\colon \mathcal{F} \to \mathcal{F}^+$ induces isomorphisms $\theta_x\colon \mathcal{F}_x \to \mathcal{F}^+_x$ for every $x \in X$.

Remark. Sheafification is left adjoint to the natural inclusion $i\colon Sh(X) \to PSh(X)$, this means

$$\mathrm{Hom}_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) = \mathrm{Hom}_{PSh(X)}(\mathcal{F}, i(\mathcal{G})).$$

Right adjoint functors preserve limits (are left exact) and left adjoint functors preserve colimits (are right exact).
In particular, the category $Sh(X)$ is **bicomplete** (has all limits and colimits): By the above, we know that a limit of sheaves in the category of presheaves gives a sheaf again, this defines the limit in the category of sheaves.
Given $D\colon I \to Sh(X)$ a functor from a small category, say a diagramm of sheaves, then $\mathrm{colim(i \circ D)}$ is in general only a presheaf, but $(\mathrm{colim(i \circ D)})^+$ defines the colimit in the category of sheaves.

Then, the category $Sh(X)$ is an abelian category.
To see that:

(1) It is an additive category by $\mathcal{F} \oplus \mathcal{G}$ is defined by direct sums at the evaluation $U \mapsto \mathcal{F}(U) \oplus \mathcal{G}(U)$.

(2) Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ a morphism of sheaves on X
(i.e. a natural transformation, given by $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for each open $U \subset X$ such that $\alpha(U)\rho_U^V = \widetilde{\rho}_U^V\alpha(V)$ where $\rho_U^V, \widetilde{\rho}_U^V$ are the restriction maps of $\mathcal{F}$ and $\mathcal{G}$for $U \subset V$ open in X).
The kernel of α is defined for an open $U \subset X$ as

$$(\ker\alpha)(U) := \ker(\alpha(U)),$$

this defines a subsheaf $\ker\alpha \subset \mathcal{F}$ in $Sh(X)$.
In general the presheaf $U \mapsto \mathrm{Im}(\alpha(U))$ is not a sheaf.
The image of α is defined as the sheafification of this presheaf, we call it $\mathrm{Im}\,\alpha$.

(3) Given $\mathcal{F} \subset \mathcal{G}$ in $Sh(X)$, the quotient sheaf $\mathcal{G}/\mathcal{F}$ is given by the sheafification of the presheaf

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U).$$

For the isomorphism theorems: Prove that the induced maps on stalks are isomorphisms.

A sequence $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ is exact in the category $Sh(X)$ if $\mathrm{Im}\,\alpha = \ker\beta$.

Lemma 1. *A sequence $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$ in $Sh(X)$ is exact if and only if for all $x \in X$ the induces sequences*

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$$

are exact.

proof: By definition $(\ker(\beta))_x = \mathrm{colim_{U \in Nbh(x)}}\ker\beta(\mathrm{U}) = \ker(\beta_\mathrm{x})$. Since a presheaf has the same stalks as the sheafification, we get $(\mathrm{Im}\,\alpha)_x = \mathrm{colim_{U \in Nbh(x)}}\,\mathrm{Im}(\alpha(\mathrm{U})) = \mathrm{Im}(\alpha_\mathrm{x})$. So clearly $\ker(\beta) = \mathrm{Im}\,\alpha$ implies all their stalks are equal. Vice versa, assume that $\ker(\beta_x) = \mathrm{Im}\,\alpha_x$ for all $x \in X$, then it holds that $\mathrm{Im}(\alpha) \subset \ker(\beta)$ and since this morphism is equality on stalks it is equality. □

Lemma 2. *Let $\mathcal{R}$ be a sheaf of rings on a topological space X. The category full subcategory $\mathcal{R}-mod$ of $Sh(X)$ consisting of $\mathcal{R}$-modules is an abelian subcategory. It has* ***enough injective objects****. It is a closed symmetric monoidal category and it is bicomplete.*

proof: Explanation... □

There are several notions replacing injective resolutions, like flasque, mu,... Godement resolutions...

1.2 The category of coherent sheaves $Coh(X)$ and quasi-coherent sheaves $QCoh(X)$

Literature.: [Har77],[Ser55], [Gab62]

Let $X =$ Spec R. For an R-module M through we define an associated $\mathcal{O}_X$-module $\widetilde{M}$ through

$$\widetilde{M}(D(r)) = M\left[\frac{1}{r}\right] = S^{-1}M,$$

where $S = \{1, r, r^2, \ldots\}$.

Definition 1. Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is called **quasi coherent** if there is an open affine covering $\{U_i\}_{i\in I}$ of X such that $\mathcal{F}|_{U_i} \cong M_i$ for an $\mathcal{O}_X(U_i)$-module M_i.
If additionally all M_i are finitely generated $\mathcal{O}_X(U_i)$-modules, then $\mathcal{F}$ is called **coherent**.

Lemma 3. *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then*

(1) The $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open affine $U \subset X$ there is an $\mathcal{O}_X(U)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$.

(2) Assume X is noetherian (i.e. it has an open affine covering by spectra of noetherian rings). The $\mathcal{O}_X$-module $\mathcal{F}$ is coherent if and only if for every open affine $U \subset X$ there is a finitely generated $\mathcal{O}_X(U)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$.

proof: Let $U =$ Spec B be open affine. Let $\{U_i = \text{Spec B}_\text{i}\}_{\text{i}\in\text{I}}$ be the open affine covering such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. For every refinement of the open affine covering using principal open subsets the property still holds true. Therefore without loss of generality U is a union of U_i's and the intersection $U_i \cap U_j$ is of the form $D(b_i) \cap D(b_j) = D(b_i b_j)$ for certain $b_i, b_j \in B$, in particular $\mathcal{O}_X(U) = B \to B_{b_i} = B\left[\frac{1}{b_i}\right] = \mathcal{O}_X(U_i)$ is the localization map. By the sheaf property we have a short exact sequence

$$0 \to \mathcal{F}|_U \to \prod \mathcal{F}|_{U_i} \xrightarrow{\eta} \prod_{i,j} \mathcal{F}|_{U_i \cap U_j}$$

Set $M = \mathcal{F}(U)$. Evaluation at U (is left exact and) gives

$$0 \to M \xrightarrow{\phi} \prod_i M_i \to \prod_{i,j} M_i \otimes_{B_{b_i}} B_{b_i b_j}.$$

Now, consider M_i as B-modules via the localization maps $B \to B_{b_i}$, then this is B-linear. Then, there is a natural isomorphism $\alpha\colon \widetilde{M} \to \mathcal{F}|_U$ given on $D(b)$ by seeing that the image of the injective map $\phi_b\colon M[\frac{1}{b}] \to \prod_i M_i[\frac{1}{b}]$ is precisely the kernel of $\eta(D(b))$. If additionally all $M_i = M_{b_i}$ are finitely generated B_{b_i}-modules, then using that B is noetherian one can conclude that M is a finitely generated B-module, see [Har77], II, Prop. 5.4, page 113. □

Theorem 1.1. *([Har77], Chapter II, Corollary 5.5) If $X = \mathrm{Spec\ R}$ and $R - Mod$ is the category of all R-modules and $R - mod$ the subcategory of finitely generated R-modules. There is an equivalence of (abelian) categories*

$$\widetilde{()}\colon R - Mod \leftrightarrow QCoh(X)\colon \Gamma$$

where the quasi-inverse Γ is the global section functor. The equivalence is strict symmetric monoidal.
If R is noetherian, this restricts to an equivalence of (abelian) categories

$$\widetilde{()}\colon R - mod \leftrightarrow Coh(X)\colon \Gamma.$$

proof: Given $\mathcal{F}$ a quasi-coherent sheaf on $X = \mathrm{Spec\ R}$. Then, $M := \Gamma(X, \mathcal{F})$ is an R-module.
We claim that $\mathcal{F}$ is (naturally) isomorphic to $\widetilde{M}$. We find the natural isomorphism $\alpha\colon \widetilde{M} \to \mathcal{F}$ as in the previous proof.
Given an R-module M, then $\widetilde{M}$ is a quasi-coherent sheaf. By definition $\Gamma(X, \widetilde{M}) = \widetilde{M}(D(1)) = M$. □

Since sheaves are determined locally, we get often can assume that the scheme is affine and the functors $\widetilde{()}, \Gamma$ are exact.

Corollary 1.1.1. *([Har77], II, Prop.5.7, p. 114) If X is a scheme. The kernel, cokernel, and image of any morphism of quasi-coherent sheaves are quasi-coherent. Any extension of quasi-coherent sheaves is quasi-coherent. If X is noetherian, the same is true for coherent sheaves.*

Example. (1) If $(X, \mathcal{O}_X)$ is a scheme, then $\mathcal{O}_X$ is coherent.

(2) Let $X = \mathrm{Spec\ K[t]_{(t)}} = \{(0), (\mathrm{t})\}$, and (0) is an open point (the generic point). Define an $\mathcal{O}_X$-module $\mathcal{F}$ by

$$\mathcal{F}(\{(0)\}) = K(t), \quad \mathcal{F}(X) = \{0\},$$

then it can not be quasi-coherent because assume $\mathcal{F} = \widetilde{M}$, then $M = \mathcal{F}(X) = 0$ and it follows $\mathcal{F} = 0$ which is a contradiction.

(3) Let $f\colon X \to Y$ be a morphism of schemes. Then $f_*\colon \mathcal{O}_X - mod \to \mathcal{O}_Y - mod$ defines a functor. If X is noetherian, then it maps quasi-coherent sheaves to quasi-coherent ones. This is no longer true for coherent shaeves, for example: consider $f = \mathrm{Spec\ (K[t]_{(t)} \to K(t))\colon X = Spec\ K(t) \to Spec\ K[t]_{(t)} = Y}$ the inclusion of the generic point from the previous example. Assume $\mathcal{G} := f_*\mathcal{O}_X$ is coherent, let us say $\widetilde{M}$ for a finitely generated $K[t]_{(t)}$-module. Then,

$$M = \mathcal{G}(Y) = \mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(X) = K(t)$$

but this is not a finitely generated $K[t]_{(t)}$-module.

(4) Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is **locally free** if X can be covered by open sets U such that $\mathcal{F}|_U$ is isomorphic to $((\mathcal{O}_X)|_U)^{\oplus I}$ for some (possibly infinite) set I. We call the cardinality of I the rank of $\mathcal{F}$ on U. If X is connected, the rank is constant.
Any locally free sheaf is quasi-coherent. If it is of finite rank, it is coherent.
For $X = \mathrm{Spec\ R}$, the above equivalence restricts to an equivalence between projective R-modules and locally free sheaves. For a noetherian ring R, it restricts to finitely generated projective R-modules and locally free sheaves.

Then, we also want to mention.

Example. Let X be a scheme. The full subcategory (of $Coh(X)$) of locally free sheaves of finite rank is equivalent to the category of vector bundles $Vect_X$, we describe how to construct a vector bundle from a locally free sheaf (in general see [Har77], p.128). A vector bundle (of rank n) over X is a morphism of schemes $f\colon Y \to X$ together with an open (wlog affine) covering $\{U_i\}$ of X and isomorphisms ψ_i making the following diagram commutative

$$\begin{array}{ccc} f^{-1}(U_i) & \xrightarrow{\psi_i} & \mathbb{A}^n_{U_i} \\ \downarrow f & \swarrow can & \\ U_i & & \end{array}$$

with for $U_i = \mathrm{Spec\ A}$ we have $can = \mathrm{Spec\ (A \subset A[t_1, \ldots, t_n])}$, such that for every i, j and $V = \mathrm{Spec\ B \subset U_i \cap U_j}$ it holds $\psi_j \circ \psi_i^{-1}|_V\colon \mathbb{A}^n_V \to \mathbb{A}^n_V$ is given by a B-algebra homomorphism

$$B[t_1, \ldots, t_n] \to B[t_1, \ldots, t_n], t_i \mapsto \sum_j b_{ij} t_j$$

for $b_{ij} \in B$ (giving an invertible matrix).
Now, giving a locally free sheaf $\mathcal{E}$ of rank n on X, we define a morphism $f\colon Y \to X$ as follows.
Let $S(\mathcal{E}) = \bigoplus_{n \geq 0} \mathcal{E}^{\otimes n}/(x \otimes y - y \otimes x)$ be the sheaf of $\mathcal{O}_X$-algebras, called symmetric algebra over $\mathcal{E}$. For every open $U \subset X$ we get an affine scheme $\mathrm{Spec\ (S(\mathcal{E})(U))}$, these glue together to a scheme Y. We take $f = \mathrm{Spec\ (\mathcal{O}_X \subset S(\mathcal{E}))}$ be the morphism which is given over an open affine $U = \mathrm{Spec\ A \subset X}$ with $\mathcal{E}|_U$ is free of rank n as $can = \mathrm{Spec\ (A = \mathcal{O}_X(U) \subset S(\mathcal{E}(U)) = A[t_1, \ldots, t_n])}$.

Reminder on quotient categories, see [Gab62], section III Let A be an abelian category. A full subcategory $C \subset A$ is called **Serre subcategory** if it is closed under taking subobjects, quotients and extensions. Given such a category, there is a quotient category A/C and a functor $\ell\colon A \to A/C$ defined as follows:

(ob) the objects of A/C are the objects of A, the functor is the identity on objects.

(mor) let M, N be objects of A/C. We see them as objects of A and look at the canonical morphisms
$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N/N')$$
given for $M' \subset M, N' \subset N$ by pre- and postcomposition, but only those with $M', N/N'$ in C. Then, take the limit over all those
$$\mathrm{Hom}_{A/C}(M, N) := \lim_{M', N'} \mathrm{Hom}_A(M', N/N')$$
and the functor ℓ on morphisms is given by the canonical map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{A/C}(M, N)$ into the limit.

Then, one shows

(1) A/C defines a category, it is abelian and the functor ℓ is an exact functor, called **localization functor** and C is called the kernel, sometimes we write this as $0 \to C \to A \to A/C \to 0$.

(2) $\ell\colon A \to A/C$ answers the universal property of taking quotients for abelian categories [...].

We will need the following two examples.

Example. (1) Let R be a positively graded ring and $R - Grmod$ the category of $\mathbb{Z}$-graded R-modules with degree zero homomorphisms. This is an abelian category with a shift automorphism. We consider $(R - Grmod)_0$ the full subcategory of modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ with all but finitely many $M_n = 0$, then this is a Serre subcategory.

(2) Let X be a variety[1] over a field K. Let $i\colon Z \subset X$ be a closed subset and $j\colon U = X \backslash Z \to X$ be the open complement.For a coherent sheaf $\mathcal{F}$ on X, the set $supp(\mathcal{F}) = \{x \in X\colon \mathcal{F}_x \neq 0\}$ is a closed set. We consider $Coh_Z(X)$ the full subcategory of coherent sheaves on X with support in Z. Then, this defines a Serre subcategory and we have
$$0 \to Coh_Z(X) \to Coh(X) \xrightarrow{j^* := ()|_U} Coh(U) \to 0.$$
To see that we use the following lemma.

[1] here: Any scheme of finite type over K which is separated

Lemma 4. *([?], Lemma 3.2 and 3.3)*

(1) Let $F\colon A \to B$ be an exact functor between abelian categories. Assume F has a right adjoint G which is fully faithful (i.e. $FG \xrightarrow{can} 1_B$ is an isomorphism). Then $KerF$ is a Serre subcategory and we have

$$0 \to \ker F \to A \xrightarrow{F} B \to 0.$$

(2) Let $A' \subset A$ a full abelian subcategory of an abelian subcategory and $C \subset A$ a Serre subcategory. Assume for any $M \in A'$ and $N \in C$ a subobject or quotient of M, it holds $N \in A'$.
Then, the canonical functor $A'/(I \cap A') \to A/I$ is fully faithful.

Now, given a graded R-module. We can define a sheaf $\underset{\sim}{M}$ on $X = \text{Proj R}$ as follows, for $D_+(r), r \in R_d, d > 0$ we set

$$\underset{\sim}{M}(D_+(r)) := (M\left[\frac{1}{r}\right])^0$$

i.e. it is the grade zero part of the $\mathbb{Z}$-graded module $M\left[\frac{1}{r}\right]$. Then, one can show that for $\mathfrak{p} \in X$ it holds

$$\underset{\sim}{M}_{\mathfrak{p}} = (M\left[S^{-1}\right])^0 = (M_{\mathfrak{p}})^0$$

where $S = R \setminus \mathfrak{p}$. Observe, that we defined $\underset{\sim}{R} = \mathcal{O}_X$.

Let $R = \bigoplus_{d \geq 0} R_d$ be a graded ring, set $R_d = 0$ for $d < o$. For $n \in \mathbb{Z}$ define a graded R-module $R(n) := \bigoplus_{d \in \mathbb{Z}} R(n)_d$ by $R(n)_d := R_{d+n}$. As R-module $R(n) \cong R$ but not as graded R-module. More generally, if M is a $\mathbb{Z}$-graded R-module we define another graded R-module called the **shift or twist** $M(n)$ by $M(n)_d := M_{d+n}$ and $R(n) \otimes_R M(m) \cong M(n+m)$ as graded R-modules.

Now, for the structure sheaf $\mathcal{O}_X$ we of $X = \text{Proj R}$, we define a coherent sheaf $\mathcal{O}_X(n)$, $n \in \mathbb{Z}$ as follows

$$\mathcal{O}_X(n) := \underset{\sim}{R(n)}$$

and similar for any quasi-coherent $\mathcal{F}$ on X we define the n-th twist as

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

for example, by definition

$$\mathcal{O}_X(n)(X) = R(n)^0 = R_n.$$

So, what about getting back the whole module by twisting taking direct sums of global sections of all twists? So we define for a quasi-coherent $\mathcal{F}$

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \mathcal{F}(n)(X)$$

this defines a graded R-module via, let $r \in R_d = \mathcal{O}_X(d)(X)$ and $s \in \mathcal{F}(n)(X)$ we set rs to be the image of $r \otimes s$ under the canonical map

$$\mathcal{F}(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) = \mathcal{F} \otimes \mathcal{O}_X(n+d) \to \mathcal{F}(n+d).$$

For example $\Gamma_*(\mathcal{O}_X) = R$ as graded R-module.

Lemma 5. *Assume that R is generated as R_0-algebra by finitely many elements in R_1, $X = \text{Proj R}$ and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, then there is a natural isomorphism $\alpha\colon \mathcal{F} \to \Gamma_*\underset{\sim}{(\mathcal{F})}$.*

Theorem 1.2. *(Serre, 1955 see [Ser55]) Let $X = \text{Proj R}$ be a closed subscheme of $\mathbb{P}^n_K$ for a field K. Then, the previously defined functors give an equivalence of abelian categories*

$$\underset{\sim}{(\,)}\colon R - Grmod/(R - Grmod)_0 \leftrightarrow QCoh(X)\colon \Gamma_*$$

The equivalence is strict symmetric monoidal.
If R is noetherian, this restricts to an equivalence of (abelian) categories

$$\underset{\sim}{(\,)}\colon R - grmod/(R - grmod)_0 \leftrightarrow Coh(X)\colon \Gamma_*$$

proof: [Ser55].

Definition 2. We say a Serre subcategory C of an abelian category is of **finite type** if it has an object such that it is the smallest Serre subcategory of A containing this object. We say that a Serre subcategory C is **irredicible** if it is not equal to 0 and it is not generated by two proper Serre subcategories.

Theorem 1.3. *(Gabriel's thesis,[Gab62]) Let X be a variety over a field K. The map $Z \mapsto Coh_Z(X)$ from the set of closed subsets of X to the set of Serre subcategories of finite type of $Coh(X)$ is a bijection. The closed irreducible subsets correspond to the irreducible Serre subcategories.*

Corollary 1.3.1. *Let X be a variety over a field K. Then the abelian category $Coh(X)$ determines the variety in the category of schemes.*

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