Homological Theory of Exact Categories

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Contents

Forword

This thesis is aiming at a reader already familiar with homological algebra for abelian categories who is open-minded to widen his perspective to exact categories.
As I have been studying representations of finite-dimensional algebras, they are overrepresented in examples. I also have to apologize for the (somewhat) unfinished state, in the end the submission deadline (given by the Wissenschaftszeitvertragsgesetz[1]) came earlier than I thought it would. Also I want to thank my husband Bill for his support and our babysitter Nina, without her nothing would work.

[1] German law which restricts the time of temporary employment of scientists

Why exact categories?

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The heart of homological algebra. Exact categories were introduced by Quillen (1972) in [**157**] to define algebraic K-theory of an exact category. He axiomatized properties of extension-closed subcategories in an abelian category. So, an exact category consists of an additive category together with a class of kernel-cokernel pairs called short exact sequences (or conflations). These have to satisfy axioms which ensure that equivalence classes of short exact sequences define an additive bifunctor $\mathrm{Ext}^1_{\mathcal{E}}$ called the extension functor. Using longer exact sequences one can define higher Ext-functors and show that every short exact sequence (observe the different arrows marking the short exact sequence)

$$X \rightarrowtail Y \twoheadrightarrow Z$$

gives rise to a long exact sequence in abelian groups

$$0 \to \mathrm{Hom}(V, X) \to \mathrm{Hom}(V, Y) \to \mathrm{Hom}(V, Z) \to \mathrm{Ext}^1(V, X) \to \mathrm{Ext}^1(V, Y) \to \cdots$$

For the author, this is at the heart of everything named **homological algebra**. Homological invariants are conditions on Ext-groups (e.g. certain ext-vanishing). Exact categories provide precisely the minimal set of axioms such that all classical concepts of homological algebra are defined. Many cohomology groups of interest are instances of Ext-groups in exact categories (sheaf cohomology, group cohomology, Hochschild cohomology, singular cohomology). Bühler [**49**] showed that every well-known diagram lemma generalizes from abelian to exact categories.

Ubiquity. Exact categories are studied in many *algebraic* contexts, for example

- (*) in **algebra** as subcategories of (graded) module categories over a (graded) ring: as filtered modules over a filtered ring [**145**], as torsion modules [**63**], flat modules, Gorenstein projective modules and relative exact structures [**70**], [**98**], as almost modules [**78**], for more specific rings: as Cohen Macaulay modules and lattices over orders [**166**], [**165**], as modules filtered by standard modules over a quasi-hereditary algebra [**65**], as perpendicular categories for categories of quiver representations [**80**], as monomorphism categories, as semistable modules [**125**], as modules of finite projective dimension, as Auslander-Solberg exact structures on finitely generated modules over an artin algebra [**15**], [**13**], [**14**], as selforthogonal subcategories in finitely generated module categories of artin algebra (in homological conjectures [**74**] or in tilting theory [**7**]),
- (*) also as representations of groups, representations of posets, representations of bocses and differential biquivers [**32**],
- (*) in **algebraic geometry** as subcategories of quasi-coherent categories of sheaves on a scheme: as coherent sheaves, vector bundles, torsion sheaves, supported on a closed subscheme [**89**], also other categories of sheaves: Sheaves on sites, coherent sheaves on complex analytic spaces, flabby sheaves on a topological space [**113**], [**115**],
- (*) in **functional analysis** as subcategories of locally convex spaces [**64**]: Banach spaces, barrelled spaces, Schwartz spaces, Frechet spaces, Montel spaces [**149**], [**164**].

Flexibility. Constructive methods for exact categories are very flexible (which admittedly makes it difficult to be systematic about it), for example:
You can filter with respect to objects, take perpendicular categories (wrt. to Ext- or Hom-functors) or intersections of exact subcategories. You can pass to exact substructures, e.g. look at an exact

substructure making a left exact functor exact. You can look at the category of all short exact sequences in an exact category, or the category of complexes in an exact category. You can look at (many) functor categories with extra properties. You can Ind-complete your exact category (completion with respect to arbitrary small filtered colimits). You can look at recollements of exact categories. And much more.

The main open problem... We know many ways of finding exact categories but then we know hardly anything about their homological algebra. When you pass to an exact substructure or an exact subcategory, everything can happen to your homological invariants. One way to prevent this is to study if the inclusion functor is *homologically exact* (i.e. induces isomorphisms on Ext-groups) - this only applies to exact subcategories. For exact substructures new ideas are needed. Or another open question: when is the global dimension preserved under Ind-completion (cf. [**152**])?

Derived/stable and singular equivalences. Despite of this ubiquity of exact categories, they are surprisingly rarely studied generally from a homological algebra point of view. The primary invariant here is of course the bounded derived category of an exact category. There are very few authors considering them, mostly because of the (sometimes) missing t-structures. Secondly the singularity category is a homological invariant, it 'annihilates' all objects of finite projective dimension in the derived category. Also Auslander's effaceable functor categories reflect some homological properties of an exact category (see next paragraph).
Therefore we ask to study **derived / singular** and **stable equivalence** for exact categories. Of course, this means first that we investigate the three associated categories. When it comes to the desired equivalences, of course, even for finite-dimensional modules over finite-dimensional algebras we only know in some special situations answers to these questions.

Auslander's ideas work for exact categories. As a vague approximation to his ideas, Auslander promotes to study module categories of artin algebras through their categories of finitely presented functors. The most famous is the category of functors represented by deflations, called effaceable functors (or Auslander defect category). Together with Idun Reiten he developed the theory of almost split exact sequences (which correspond to simple effaceable functors). Enomoto carried these ideas into the generality of exact categories [**72**].
Auslander correspondence (and Auslander formula) are telling us that the category of all finitely presented functors (on a small abelian category) can recover the abelian category. The same ideas work for small exact categories, cf. [**90**], [**68**], [**83**].
Also, in a series of papers [**20**], [**21**], [**22**], [**23**], [**24**] Auslander-Reiten started to study stable equivalence of artin algebras (and more general dualizing varieties). They investigated homological properties of effaceable functors. The interesting observation is that homological properties of an exact category are reflected in its effaceable functor category.
We think all of Auslander's work should be generalized to exact categories because it is useful for an understanding of homological properties of exact categories.

Tilting and support τ-tilting. Ideas from representation theory of finite dimensional algebras which are purely based on homological algebra generalize trivially to exact categories. This includes tilting theory and support tau tilting theory. But to understand induced derived equivalence we first need to see that we have to *replace* the 'endomorphism of a tilting module' with a certain functor category (functors represented by admissible morphisms - cf. previous paragraph).
Support τ-tilting subcategories were advertised as a mutation-completion of tilting subcategories [**2**]. For exact categories this is no longer true. The challenge is here: Find a new construction to mutation-complete support τ-tilting subcategories.

Homological conjectures, tame-wild dichotomy. Once we are restricting the study of exact categories to Krull-Schmidt categories (and possibly assuming more properties), it becomes also reasonable to ask if *Homological conjectures* for modules over artin algebras are true in these classes of exact categories (for example: Auslander and Solberg [**14**] showed that the finitistic dimension conjecture for artin algebras is equivalent to the same conjecture for Auslander-Solberg exact substructures).

By the Krull-Schmidt assumption, *finite type* means only finitely many indecomposable objects. But how *wild* can the infinite types look like? In ongoing work, Schlegel defines *finitely definable* subcategories in modules categories over an artin algebra and then conjectures a dichotomy of 'finite' or 'strongly unbounded' type (generalizing the second Brauer Thrall conjecture).
Enomoto investigated properties of exact categories of finite representation-type in [**72**].

Why not more general? So, why not directly study any of the following generalizations of exact categories

(1) one-sided or weak exact categories, e.g. [**92**], [**91**] or [**29**] (leave out some axioms)
Then we have no ext-bifunctor. Proto-exact categories - drop the assumption that the underlying category is additive. Again the ext-functor is not valued in abelian groups.
(2) extriangulated categories [**140**] (axiomatizing extension-closed in triangulated)
Then we do not know a sensible definition of a derived category. The axioms are long. We inherit all known problems with the axioms for triangulated categories (which have lead to a big number of different enhancements of triangulated categories). Often, the only known examples are just exact or triangulated categories.
(3) dg exact categories, model structures or other enhanced situations
(such as A_∞, or ∞-categories). This becomes quickly homotopical category theory. Even though these generalizations may encompass much of the theory of exact categories, we are loosing the simplicity and flexibility that make exact categories so appealing.
(4) n-exact categories [**111**] (higher homological algebra) - these make sense to be studied together with exact categories - they are exclusively occurring as cluster tilting subcategories ([**132**]) and then we are back in the realm of representation theoretic ideas.

But why should we leave an easy algebraic theory behind when it is so little studied and so central in homological algebra? Exact categories have the very tempting balance of impressive potential for creating abstract theory but still being simplistic enough to provide explicit examples. Even for seemingly well-studied categories (e.g. take the category of finitely generated abelian groups, do you know all its exact substructures?).

Quick overview of the project. Every chapter starts with a synopsis explaining its content and my contribution.

(I) Subcategories and functor categories
This part is on constructive methods. First, we look at the lattices of exact substructures (Chapter 2) and the much bigger lattice of exact subcategories (Chapter 3).
We study categories of functors represented by (certain) morphisms (Chapter 4) and faithfully balancedness for (the usually considered) functor categories (Chapter 5).
(II) Derived methods
We start with the definition and existence of the derived category (Chapter 6), then we look at tilting subcategories in an exact category (Chapter 7) and have a closer look at tilting subcategories for infinite quivers (Chapter 8). Then we discuss how one can find derived equivalences more generally (Chapter 9,10).
(III) Singular and stable equivalence
This part is not complete (so far we have not really addressed the title): We introduce the singularity category and the concept of a non-commutative resolution with exact substructures in Chapter 11. Then we only start the study of effaceable functors in Chapter 12.

We do not cover the following topics (but may do so in future):
Support τ-tilting, homological conjectures, tame-wild dichotomy, recollements of exact categories (a more thorough treatment of derived functors of additive functors between exact categories is required for this).

Notation and conventions for exact categories

$\mathcal{E} = (\mathcal{A}, \mathcal{S})$ being an exact category means $\mathcal{A}$ is an additive category and $\mathcal{S}$ is a collection of kernel-cokernel pairs in $\mathcal{A}$ satisfying the axioms below. We call elements in $\mathcal{S}$ **short exact sequences** (in the literature these are usually called **conflations**)[2] Given a short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$, we call $i\colon X \rightarrowtail Y$ an **inflation** and $p\colon Y \twoheadrightarrow Z$ a **deflation**. The axioms of an exact category are:

(E0) all split exact sequences are in $\mathcal{S}$,
(E1) deflations are closed under composition and inflations too,
(E2) Pull backs of deflations along arbitrary morphisms exist and are again deflations. Push outs of inflations along arbitrary morphisms exist and are again inflations.

An **admissible morphism** f is one that factors as $f = j \circ p$ with p a deflation and j an inflation. Given a(n integer interval indexed) sequence of composable morphisms f_n, $n \in I$ (for an interval $I \subseteq \mathbb{Z}$ with at least two elements)

$$\cdots \to X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to \cdots$$

we call it **exact** (or **acyclic**) at X_{n+1} if the morphisms factor as $f_n = j_n p_n$ with j_n and inflation and p_n a deflation (i.e. are admissible) and (j_n, p_{n+1}) is a short exact sequence. If we call such a sequence exact, it means exact at every inner object (here: 'inner' means not at the boundary of the interval).

$\mathrm{Ext}^1_{\mathcal{E}}(X, Y)$ is the class of all short exact sequences $Y \rightarrowtail Z \twoheadrightarrow X$ up to isomorphism of short exact sequences fixing the end terms.
For $n > 1$: $\mathrm{Ext}^n_{\mathcal{E}}(X, Y)$ is the class of all exact sequences $Y \rightarrowtail Z_1 \to \cdots \to Z_n \twoheadrightarrow X$ up to the equivalence relation generated by morphisms of n-exact sequences fixing the end terms.
$\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{I}(\mathcal{E})$) are the full subcategories of projectives (resp. injectives)
$\mathrm{pd}_{\mathcal{E}}\, X \leq n$ means $\mathrm{Ext}^{n+1}_{\mathcal{E}}(X, -) = 0$
$\mathcal{P}^{\leq n}(\mathcal{E})$ (resp. $\mathcal{I}^{\leq n}(\mathcal{E})$) denotes the subcategory of objects of projective (resp. injective) dimension at most n
$\mathcal{P}^n(\mathcal{A}) := \mathcal{P}^{\leq n}(\mathrm{mod}_\infty \mathcal{A})$ is a special case of the former, see below for the functor category $\mathrm{mod}_\infty \mathcal{A}$
$\mathcal{P}^{<\infty}(\mathcal{E}) = \bigcup_n \mathcal{P}^{\leq n}(\mathcal{E})$ (resp. $\mathcal{I}^{<\infty}(\mathcal{E}) = \bigcup_n \mathcal{I}^{\leq n}(\mathcal{E})$)
Most common properties of subcategories: Given a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in an exact category

(*) extension-closed: if X, Z are inside the subcategory then Y too
(*) inflation-closed: if X, Y are inside the subcategory then Z too
(*) deflation-closed: if Y, Z are inside the subcategory then X too

thick subcategory[3] means all 2-out-of-3-properties (see above) and closed under summands.
Serre subcategory means extension-closed and if a middle term is contained then both outer terms are as well.
A subcategory $\mathcal{G}$ is a **generator** in an exact category if for every object X, there exists a deflation $d\colon G \twoheadrightarrow X$ with G in $\mathcal{G}$.
Resolving means extension-closed, deflation-closed, summand-closed and a generator.

[2] Be aware, e.g. in [167], short exact sequence is a synonym for kernel cokernel pair in an additive category.
[3] we often add: 'in the exact category'- do not confuse this with thick in the triangulated sense.

Functor categories. Let $\mathcal{A}$ be a small additive category, (Ab) the category of abelian groups. For A in $\mathcal{A}$, we define $P_A\colon \mathcal{A}^{ob} \to (Ab), P_A(X) := \mathrm{Hom}_{\mathcal{A}}(X, A)$ and

$$\begin{aligned}
\mathrm{Mod}\,\mathcal{A} &= \text{ category of all additive functors } \mathcal{A}^{op} \to (Ab) \\
\mathrm{mod}\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists\ P_A \twoheadrightarrow A\} \\
\mathrm{mod}_1\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists \text{ exact seq. } P_{A_1} \to P_{A_0} \twoheadrightarrow F\} \\
\mathrm{mod}_n\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists \text{ exact seq. } P_{A_n} \to \cdots \to P_{A_0} \twoheadrightarrow F\} \\
\mathrm{mod}_\infty\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists \text{ exact seq. } \cdots \to P_{A_n} \to \cdots \to P_{A_0} \twoheadrightarrow F\}
\end{aligned}$$

Part 1

Subcategories and functor categories

CHAPTER 1

Homologically exact functors

1. Synopsis

Homologically exact functors are exact functors between (Quillen-) exact categories which induce isomorphisms on higher extension groups. To understand and find these functors is of course of core interest for a homological study of exact categories. There is no good characterization of this property known. We also look at exact structures making an additive functor exact. These constructions are applied to some adjoint functors appearing in recollements and we study when these functors become homologically exact.
What is new? Some Lemmata and the study of exact structures making an additive functor exact are not often considered. It does not seem to be known that a result of Rump shows that there is always a unique maximal one.

This chapter should be seen as a gentle introduction to homologically exact functors - but we aim at an audience which has seen homological algebra for abelian categories. We give the following warning: For an understanding of this notion we already mention the derived category which we thoroughly study in a much later chapter - if you are unfamiliar with this, ignore it for the moment and come back to it later.

2. Homologically exact functors

Definition 2.1. Given a subcategory $\mathcal{X}$ in an exact subcategory, we say it is **fully exact** if $\mathcal{X}$ is extension-closed *and* we see it as already equipped with the exact structure restricted from the ambient one.

Definition 2.2. Given an exact functor $f\colon \mathcal{E} \to \mathcal{F}$ between exact categories, we consider $\mathrm{Hom} = \mathrm{Ext}^0$ as an extension group. Let $n \in \mathbb{N}_0$. We call f n**-homologically exact** (resp. n**-homologically faithful**) if the natural maps on all n-th extension groups are isomorphisms (resp. injective). We call f **homologically exact/homologically faithful** if it is n-homologically exact/n-homologically faithful for all $n \geq 0$. It is called **fully homologically exact** if it induces a triangle equivalence on the bounded derived categories.
If $\mathcal{E}$ is a fully exact category of $\mathcal{F}$, we call $\mathcal{E}$ homologically exact (resp. n-homologcally exact, resp. homologically faithful etc.) in $\mathcal{F}$ if the inclusion $\mathcal{E} \subseteq \mathcal{F}$ fulfills this property.

Lemma 2.3. *Every exact functor which is n-homologically exact for some $n \geq 0$ is also $(n+1)$-homologically faithful.*

Proof. For $n = 0$, we consider an exact fully faithful functor f. Observe that $f(p)$ split epimorphism implies p split epimorphism. This translates into a monomorphism on Ext^1-groups.
Let $f\colon \mathcal{E} \to \mathcal{F}$ be n-homologically exact, $n \geq 1$.
We take an $(n+1)$-extension σ such that $[f(\sigma)] = 0 \in \mathrm{Ext}^{n+1}_{\mathcal{F}}(f(X), f(Y))$ and we want to see $[\sigma] = 0 \in \mathrm{Ext}^{n+1}_{\mathcal{E}}(X, Y)$. As $(n+1) \geq 2$ we can find σ is a concatenation of a short exact sequence $\sigma_1\colon Y \rightarrowtail A \twoheadrightarrow Z$ with another exact sequence σ_2. Now, we look at the long exact sequence obtained from applying $\mathrm{Hom}(X, -)$. The connecting morphism $\mathrm{Ext}^n_{\mathcal{E}}(X, Z) \to \mathrm{Ext}^{n+1}_{\mathcal{E}}(X, Y)$ is "concatenation with" σ_1 ([**126**, Cor. 4.2.12]), so we have $[\sigma_2] \mapsto [\sigma]$ under it. Now, we look at the induced commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^n_{\mathcal{E}}(X,A) & \longrightarrow & \mathrm{Ext}^n_{\mathcal{E}}(X,Z) & \longrightarrow & \mathrm{Ext}^{n+1}_{\mathcal{E}}(X,Y) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^n_{\mathcal{F}}(f(X),f(A)) & \longrightarrow & \mathrm{Ext}^n_{\mathcal{F}}(f(X),f(Z)) & \longrightarrow & \mathrm{Ext}^{n+1}_{\mathcal{F}}(f(X),f(Y))
\end{array}$$

By assumption the first two maps are isomorphisms. We just discussed $[\sigma]$ lies in the image of the connecting morphism. Now, an easy diagram chase shows that $[\sigma] = 0$. □

Corollary 2.4. *Every inclusion of an exact substructure is* 0*-homologically exact and therefore* 1*-homologically faithful.*

In particular, every hereditary exact substructure (i.e. gldim ≤ 1) is homologically faithful.

Definition 2.5. Let $f\colon \mathcal{E} \to \mathcal{F}$ be an functor between exact categories. Then, f is called an **exact equivalence** if it is an exact functor, an equivalence of categories such that its quasi-inverse is an exact functor as well.

Remark 2.6. Here a warning to readers used to abelian categories: An additive equivalence (even if it is an exact functor) between exact categories is in general not homologically exact nor does it have an exact quasi-inverse.

Lemma 2.7. *Let $i\colon \mathcal{E} \to \mathcal{F}$ be the inclusion of an exact substructure. Then the following are equivalent*

(1) *i is an exact equivalence (i.e. $\mathcal{E} = \mathcal{F}$))*
(2) *i is homologically exact*
(3) *i is* 1*-homologically exact*

Proof. To see (3) $\Rightarrow$ (1): Consider an $\mathcal{F}$-exact sequence σ and since $\mathrm{Ext}^1_{\mathcal{E}}(Y,X) \to \mathrm{Ext}^1_{\mathcal{F}}(Y,X)$ is surjective, there exists an $\mathcal{E}$-exact sequence equivalent to σ. But this means it is even isomorphic to σ and therefore σ is $\mathcal{F}$-exact. □

Proposition 2.8. *Let $f\colon \mathcal{E} \to \mathcal{F}$ be an exact functor. Assume f is an equivalence of categories with quasi-inverse g. The following are equivalent:*

(1) *f is* 1*-homologically exact.*
(2) *f is homologically exact.*
(3) *g is homologically exact.*
(4) *g is exact.*

Proof. By passing to the essential image of f with the from $\mathcal{E}$ transferred exact structure, we can assume without loss of generality that $\mathcal{E}$ is the inclusion of an exact substructure, in particular $\mathrm{Ext}^1_{\mathcal{E}} \subseteq \mathrm{Ext}^1_{\mathcal{F}}$ is a subfunctor and $f = \mathrm{id}, g = \mathrm{id}$. Then the statement of the Proposition is a consequence of Lemma 2.7. □

The next corollary says that homological exact functors always compose as an exact equivalence followed by an inclusion of an homologically exact subcategory.

Corollary 2.9. *Let $f\colon \mathcal{E} \to \mathcal{F}$ be an exact functor between exact categories. Then we factor the functor over its essential image*

$$f\colon \mathcal{E} \xrightarrow{f'} \mathrm{Im}\, f \xrightarrow{i} \mathcal{F}$$

The following are equivalent

(1) *f is homologically exact*
(2) *$\mathrm{Im}\, f$ is an homologically exact subcategory of $\mathcal{F}$ and f' is an exact equivalence.*

PROOF. Obviously, (2) implies (1). So assume (1). First observe that $\operatorname{Im} f$ has to be extension-closed and f' an equivalence of categories (because homologically exact functors are fully faithful). The extension-closedness is straight-forward using the isomorphism induced by f on Ext^1's. Now we look at this factorization $f = if'$. As i is inclusion of a a fully exact subcategory, it is 1-homologically exact. But as f is also 1-homologically exact, it follows that f' is 1-homologically exact. By Prop. 2.8 it follows that f' is homologically exact. Again f and f' homologically exact imply i is homologically exact and therefore (2) holds. □

Considering the same factorization and same reasons as in the previous Corollary one sees:

Corollary 2.10. *Every fully faithful* 1*-homologically exact functor factors as* $i \circ \varphi$ *with* φ *an exact equivalence and* i *an inclusion of a fully exact subcategory.*

Conversely, every inclusion of a fully exact subcategory is 1-homologically exact and 2-homologically faithful.

In particular, every fully exact subcategory of gldim ≤ 2 is homologically faithful.

Let $\mathcal{E}$ be exact category. We say an object X has *projective dimension* $\leq n$ if $\operatorname{Ext}^{n+1}_{\mathcal{E}}(X, -) = 0$, then we write $\operatorname{pd}_{\mathcal{E}} X \leq n$. We define $\mathcal{P}^{<\infty}(\mathcal{E})$ to be the full subcategory of all objects of finite projective dimension. We say $\mathcal{E}$ is **regular** if $\mathcal{E} = \mathcal{P}^{<\infty}(\mathcal{E})$.

Remark 2.11. Assume $f \colon \mathcal{E} \to \mathcal{F}$ is homologically faithful exact functor between exact categories. Then it is faithful and $f^{-1}(\mathcal{P}^{<\infty}(\mathcal{F})) \subseteq \mathcal{P}^{<\infty}(\mathcal{E})$. If $\mathcal{F}$ is regular, then $\mathcal{E}$ is also regular and $\operatorname{gldim}(\mathcal{E}) \leq \operatorname{gldim}(\mathcal{F})$.
If f is homologically exact then it restricts to an exact functor $\mathcal{P}^{<\infty}(\mathcal{E}) \to \mathcal{P}^{<\infty}(\mathcal{F})$, i.e. it induces a triangle functor on the so-called singularity categories, cp. later chapter.

We also know an example of a 1-homologically exact functor which is not faithful:

Example 2.12. The following gives an example from [**133**] of an exact functor that is not 0-homologically exact (i.e. not fully faithful) but nevertheless it is 1-homologically exact. Let $\mathcal{E}$ be a Frobenius exact category, we denote by $\underline{\mathcal{E}}$ the stable category. This is the ideal quotient with respect to morphisms factoring through a projective. This is a triangulated category with suspension given by the cosyzygy functor $\Sigma = \Omega^-$, [**87**]. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be a full additively closed subcategory which is extension-closed and $\operatorname{Hom}(C, \Sigma^{-n} C') = 0$ for all C, C' in $\mathcal{C}$, $n > 0$. Then $\mathcal{C}$ inherits an exact structure from $\underline{\mathcal{E}}$ by taking as short exact sequences those pairs of morphisms which belong to a distinguished triangle. Now let $\mathcal{B} \subseteq \mathcal{E}$ be the full subcategory of objects which map under the ideal quotient $\pi \colon \mathcal{E} \to \underline{\mathcal{E}}$ to $\mathcal{C}$. This is a fully exact subcategory. The ideal quotient restricts to an exact functor $\mathcal{B} \to \mathcal{C}$. This functor is not faithful und it is easy to see that it is 1-homologically exact.

2.1. Criteria for homologically exactness.

This is the best understood example of a homologically exact functor.

Example 2.13. (*homological ring epimorphisms*) Let $f \colon A \to B$ be a ring homomorphism, then we have an exact restriction of scalars functor $f^* \colon B \operatorname{Mod} \to A \operatorname{Mod}$. The morphism f is an epimorphism (in the category of rings) if and only if f^* is fully faithful. Now, f^* is homologically exact if and only if f is a ring epimorphism and ${\operatorname{Tor}_i}^A(B, B) = 0$ for all $i > 0$, [**80**, Thm 4.4].

Here is another example from reductive group theory.

Example 2.14. If G is a reductive group over a field k and P is a parabolic subgroup, then the restriction functor $\operatorname{res}^{\mathrm{P}}_{\mathrm{G}} \colon \operatorname{rep}_{\mathrm{k}} \mathrm{G} \to \operatorname{rep}_{\mathrm{k}} \mathrm{P}$ is homologically exact, cf. [**110**, Cor. 4.7], p. 233.

We recall characterizations of homological exactness from the literature.

Lemma 2.15. *([**126**, Lemma 4.2.13]). Let $f\colon \mathcal{E} \to \mathcal{F}$ is an exact functor between exact categories and consider the induced triangle functor $D^b(f)\colon D^b(\mathcal{E}) \to D^b(\mathcal{F})$. The following are equivalent*

(1) *f is homologically exact*
(2) *$D^b(f)$ is fully faithful*

Lemma 2.16. *("BBD-criterium", [**40**, Rem. 3.1.17]) Let $f\colon \mathcal{E} \to \mathcal{F}$ be the inclusion of a fully exact subcategory. The following are equivalent:*

(1) *f is homologically exact.*
(1') *The induced morphisms on all Ext-groups are surjective.*
(2) *For every $n \geq 2$ and every $\mathcal{F}$-exact sequence $\eta\colon X \rightarrowtail Z_0 \to \cdots \to Z_{n-1} \twoheadrightarrow Y$ with X, Y in $\mathcal{E}$ there exists an $\mathcal{E}$-deflation $p\colon P \to Y$ such that the pullback of η along p is a split exact sequence (i.e. zero in $\operatorname{Ext}^n_{\mathcal{F}}(P, X)$).*
(3) *For every $n \geq 2$ and every $\mathcal{F}$-exact sequence $\eta\colon X \rightarrowtail Z_0 \to \cdots \to Z_{n-1} \twoheadrightarrow Y$ with X, Y in $\mathcal{E}$ there exists an $\mathcal{E}$-inflation $i\colon X \to I$ such that the push-out of η along i is a split exact sequence (i.e. zero in $\operatorname{Ext}^n_{\mathcal{F}}(Y, I)$).*

Proof. We wrote a detailed proof in [**133**, Lemma 2.15]. □

This is cirtierium can be found in Keller's [**119**], the naming is from Krause [**126**, in section 4.2].

Definition 2.17. Let $f\colon \mathcal{E} \to \mathcal{F}$ be the inclusion of a fully exact subcategory. Then f is called **left cofinal** if for every $\mathcal{F}$-inflation $a\colon E \to F$ with $E \in \mathcal{E}$ there exists an $\mathcal{E}$-inflation $E \to E'$ which factor over a.
It is called **right cofinal** if for every $\mathcal{F}$-deflation $b\colon F \to E$ with $E \in \mathcal{E}$ there exists an $\mathcal{E}$-deflation $E' \to E$ which factors over b.

Then, one can see directly that e.g. right cofinal implies (2) in the BBD-criterium (look at the Appendix, case $n = 1$).

Lemma 2.18. *If the inclusion of a fully exact subcategory $\mathcal{E} \subseteq \mathcal{F}$ is left cofinal, then it is homologically exact and even $D^+(\mathcal{E}) \to D^+(\mathcal{F})$ is fully faithful.*
If it is right cofinal then it is homologically exact and even $D^-(\mathcal{E}) \to D^-(\mathcal{F})$ is fully faithful.

Proof. [**126**, Prop. 4.2.15, Rem. 4.2.16] □

Observe that one can see left/right cofinal as an easy corollory from Lemma 2.16. In the same spirit one can generalize this to another criterium (just academically, this has no applications as far as I know):

Definition 2.19. Let $f\colon \mathcal{E} \to \mathcal{F}$ be the inclusion of a fully exact subcategory. Then f is called **right 2-cofinal** if for every $\mathcal{F}$-admissible morphism $G \to F$ with cokernel E in $\mathcal{E}$ there exists a short exact sequence $G \rightarrowtail H \twoheadrightarrow P$ with P in $\mathcal{E}$, an $\mathcal{E}$-deflation $P \twoheadrightarrow E$ and a commutative diagram

$$\begin{array}{ccccc} G & \longrightarrow & F & \twoheadrightarrow & E \\ \| & & \uparrow & & \uparrow\!\!\uparrow \\ G & \rightarrowtail & H & \twoheadrightarrow & P \end{array}$$

Then, a fully exact subcategory that is right 2-cofinal is always homologically exact. To see this, use the BBD-criterium and the diagram fill-in from the Appendix.

This is the best known subclass of left and right cofinal subcategories.

Definition 2.20. A fully exact subcategory $\mathcal{E} \subseteq \mathcal{F}$ is called **resolving** if it is deflation-closed (i.e. closed under kernels of deflations) and generates (i.e. for every object X in $\mathcal{F}$ there is a deflation $E \to X$ with $E \in \mathcal{E}$) and is closed under taking summands.
A fully exact subcategory is **coresolving** if it is inflation-closed and cogenerating.

We generalize this slightly to the following class.

Definition 2.21. A fully exact subcategory $\mathcal{E} \subseteq \mathcal{F}$ is called **partially resolving** if has enough projectives with $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{P}(\mathcal{F})$ and is closed under summands.
It is **partially coresolving** if its opposite category is partially resolving.

Lemma 2.22. *(Same proof as in* [**71**, Lemma 2.5]*) Every partially resolving subcategory is deflation-closed.*

Remark 2.23. By definition, every resolving and also every partially resolving subcategory is right cofinal.
If $\mathcal{E}$ is closed under summands right cofinal in $\mathcal{F}$ then it is deflation-closed in $\mathcal{F}$ (cp. [**126**, Rem. 4.2.14].

An easy non-example.

Example 2.24. Inside the module category of an artin algebra:
Faithful functorially finite torsion classes are coresolving (cp. [**2**]). If they are not faithful they may also be not homologically exact, such as $\mathcal{T} = \mathrm{add}(S_3 \oplus P_1 \oplus S_1)$ in $\mathrm{mod}\,\Lambda$ with
$\Lambda = K(1 \xrightarrow{a} 2 \xrightarrow{b} 3)/(ba)$. The fully exact structure on $\mathcal{T}$ is semi-simple but $\mathrm{Ext}^2_\Lambda(S_1, S_3) \neq 0$

2.2. Criteria for homological faithfulness. Examples of homologically faithful functors are not often studied. One reason is that checking this is harder than testing for homological exactness. Another reason is that homological faithfulness is not implying that the the induced triangle functor on the bounded derived category is faithful (see next example). But proper exact substructures are never homologically exact (cp. Cor. 2.7) but sometimes homologically faithful.

Example 2.25. The inclusion of the split substructure is always homologically faithful but induces on the derived categories a Verdier localization which is not faithful (if it is not the identity).

2.2.1. *2-homologically faithfulness.*

Definition 2.26. Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category. We call a morphism $\alpha \in \mathrm{Mor}(\mathcal{A})$ **hereditary** (or $\mathcal{E}$-hereditary) if it is $\mathcal{E}$-admissible and it is the diagonal in a cartesion square with all sides are $\mathcal{E}$-inflations or $\mathcal{E}$-deflations.

For example, this means all inflations and all deflations are $\mathcal{E}$-hereditary.

Now, observe that every admissible morphism α gives a 2-extension via
$[\alpha] := [(\ker \alpha, \alpha, \mathrm{coker}\, \alpha)] \in \mathrm{Ext}^2_{\mathcal{E}}(\mathrm{coker}(\alpha), \ker(\alpha))$.

Lemma 2.27. *An admissible morphism α is hereditary if and only if $[\alpha] = 0$ in* $\mathrm{Ext}^2_{\mathcal{E}}(\mathrm{coker}(\alpha), \ker(\alpha))$.
In particular, $\mathcal{E}$ is hereditary exact ($\mathrm{gldim}\, \mathcal{E} \leq 1$) *if and only if all admissible morphisms are hereditary.*

Proof. The claim follows from the diagram fill-in in the case $n = 2$, Appendix. ☐

Lemma 2.28. *Let $f \colon \mathcal{E} \to \mathcal{F}$ be an exact functor. Then the following are equivalent*

(1) *f reflects hereditary morphisms (i.e. if α is $\mathcal{E}$-admissible and $f(\alpha)$ is $\mathcal{F}$-hereditary, then α is also $\mathcal{E}$-hereditary.*

(2) *f is 2-homologically faithful.*

Proof. Let α be an $\mathcal{E}$-admissible morphism. We look at $[\alpha]_{\mathcal{E}} \in \operatorname{Ext}^2_{\mathcal{E}}(Y,X) \to \operatorname{Ext}^2_{\mathcal{F}}(f(Y), f(X))$ such that its image $[f(\alpha)]_{\mathcal{F}} = 0$. This means $f(\alpha)$ is $\mathcal{F}$-hereditary. Injectivity of this map means f reflects hereditary morphisms. $\square$

2.2.2. *More generally.* The following provides non-trivial examples of homologically faithful functors.

Lemma 2.29. *Let $f\colon \mathcal{E} \to \mathcal{F}$ be an exact functor which is n-homologically faithful (resp. n-homologically exact) for some $n \geq 1$. Assume that $\mathcal{E}$ has enough projectives $\mathcal{P}(\mathcal{E})$ and that $\operatorname{pd}_{\mathcal{F}} f(\mathcal{P}(\mathcal{E})) \leq n-1$. Then f is m-homologically faithful (resp. m-homologically exact) for all $m \geq n$.*

In particular, the inclusion of a fully exact subcategory $\mathcal{E} \subseteq \mathcal{F}$ is always 2-homologically faithful, so if $\mathcal{E}$ has enough projectives with $\operatorname{pd}_{\mathcal{F}} \mathcal{P}(\mathcal{E}) \leq 1$ then the inclusion is homologically faithful.

Proof. Let E, X be in $\mathcal{E}$, we pick an $\mathcal{E}$-short exact sequence $\Omega \rightarrowtail P \twoheadrightarrow E$, then we apply $\operatorname{Hom}_{\mathcal{E}}(-,X)$ and we apply $\operatorname{Hom}_{\mathcal{F}}(f(-), f(X))$, this induces a commuting diagram for $m \geq n$

$$\begin{array}{ccc} \operatorname{Ext}^m_{\mathcal{E}}(\Omega, X) & \xrightarrow{\sim} & \operatorname{Ext}^{m+1}_{\mathcal{E}}(E,X) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^m_{\mathcal{F}}(f(\Omega), f(X)) & \xrightarrow{\sim} & \operatorname{Ext}^{m+1}_{\mathcal{F}}(f(E), f(X)) \end{array}$$

where $\sim$ indicates that this is an isomorphism of groups. Now, the claim is an easy inductive argument (starting with $m = n$). $\square$

We are not aware of a good criterium for homologically faithfulness (e.g. of an exact substructure). Hereditary exact substructure are always homologically faithful. Fully exact subcategories with gldim ≤ 2 are also always homologically faithful.

Example 2.30. Let $i\colon \mathcal{E} \to \mathcal{F}$ be the inclusion of a 2-homologically faithful exact substructure. If $\mathcal{E}$ has enough projectives and $\operatorname{pd}_{\mathcal{F}} \mathcal{P}(\mathcal{E}) \leq 1$ then i is homologically faithful.

Let us look at two exact substructures which even have a very similar Auslander-Reiten quiver.

Example 2.31. First we look at Λ of type A_n-equioriented modulo radical square zero. Every exact substructure has enough projectives and enough injectives (because they are of finite type). For every exact substructure and every two indecomposables, in the exact substructure we either have the same Ext-group as for Λ or zero. Therefore all are homologically faithful (and have therefore gldim $\leq n-1$).
Now we look at Λ of type A_n-equioriented and we take the generator $G = \Lambda \oplus \bigoplus_{X \in \mathcal{L}} X$ where $\mathcal{L}$ contains all indecomposable non-projectives of vector space dimension larger or equal 2. Then the exact substructure $\mathcal{E}$ with projectives add(G) has global dimension $n-1$ and is therefore not homologically faithful in $\Lambda\,\mathrm{mod}$.

3. Exact structures making additive functors exact

We would like to introduce a construction of an exact structure following [66].
Let $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ be idempotent complete exact categories and $f\colon \mathcal{A} \to \mathcal{B}$ an additive functor. We denote by $\mathcal{S}_f = \mathcal{S}_{f,\mathcal{T}} \subseteq \mathcal{S}$ the class of exact sequences η such that $f(\eta)$ is in $\mathcal{T}$. Observe, that this depends also on the exact structures and not just on the additive functor.

Lemma 3.1. *(*[66, Lem.1.9, Prop.1.10]*) Then the following are equivalent*

(1) $\mathcal{S}_f$ *is an exact structure*
(2) *Given a short exact sequence in* $\mathcal{S}_f$ *and a morphism to the third object of the sequence (resp. starting at the first object in the sequence), then the pullback (resp. pushout) of the short exact sequence is in* $\mathcal{S}_f$.

It is also straight-forward to prove that (2) implies (1) by checking that compositions of $\mathcal{S}_f$-deflations (resp. -inflations) are $\mathcal{S}_f$-deflations (resp. -inflations).
We will now look at particular situations ensuring that $\mathcal{S}_f$ becomes an exact structure.

Corollary 3.2. *Assume that* $f\colon (\mathcal{A},\mathcal{S}) \to (\mathcal{B},\mathcal{T}_0)$ *is an exact functor between exact categories and let* $\mathcal{T} \subseteq \mathcal{T}_0$ *be an exact substructure then* $\mathcal{S}_f$ *considered with respect to* $\mathcal{T}$ *is an exact structure.*

Given an exact category $\mathcal{E}$, we write $\mathrm{ex}(\mathcal{E})$ the poset of exact substructures of $\mathcal{E}$. The previous corollary translates to: Every exact functor $f\colon \mathcal{E} \to \mathcal{F}$ induces a morphism of posets

$$f^*\colon \mathrm{ex}(\mathcal{F}) \to \mathrm{ex}(\mathcal{E}), \quad \mathcal{T} \mapsto \mathcal{S}_{\mathrm{f},\mathcal{T}}$$

Proof. We take $\eta\colon 0 \to A \to B \to C \to 0$ in $\mathcal{S}$ such that $f(\eta) \in \mathcal{T}$ and a morphism $c\colon C' \to C$. The pullback η' of η along c exists and maps under f to the pullback of $f(\eta) \in \mathcal{T}_0$ along $f(c)$. But $\mathcal{T}$ is an exact substructure of $\mathcal{T}_0$ and as $f(\eta) \in \mathcal{T}$ it follows that $f(\eta') \in \mathcal{T}$. The rest of the proof is the dual statement. □

Lemma 3.3. *Let* $f\colon (\mathcal{A},\mathcal{S}) \to (\mathcal{B},\mathcal{T})$ *be an additive functor between exact categories and assume that* $\mathcal{B}$ *is weakly idempotent complete. Assume either*

(1) f *is right exact (i.e. if* $0 \to A \to B \to C \to 0$ *is in* $\mathcal{S}$*, then* $f(A) \to f(B)$ *has an image in* $\mathcal{B}$ *and* $\mathrm{Im}(f(A) \to f(B)) \rightarrowtail f(B) \twoheadrightarrow f(C)$ *is in* $\mathcal{T}$*), or*
(2) f *is left exact (i.e. the opposite functor between the opposite exact categories is right exact)*

Then $\mathcal{S}_f$ *is an exact structure.*

Proof. We prove only (1) as (2) is analogous. Assume f is right exact. Take $\eta\colon A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{S}_f$. Let $\gamma\colon C' \to C$ be any morphism. Now, we pull-back $\eta \in \mathcal{S}$ along γ to an $\eta'\colon A \rightarrowtail B' \twoheadrightarrow C'$ in $\mathcal{S}$. Applying f gives a commutative diagram with exact rows

$$\begin{array}{ccccc}
f(A) & \xrightarrow{a} & f(B') & \twoheadrightarrow & f(C') \\
\| & & \downarrow{\scriptstyle b} & & \downarrow \\
f(A) & \rightarrowtail & f(B) & \twoheadrightarrow & f(C)
\end{array}$$

Since $b \circ a$ is an inflation it follows that a is an inflation (using that $\mathcal{B}$ is idempotent complete) and therefore $\eta' \in \mathcal{S}_f$. Now, we pushout η along a morphism $\alpha\colon A \to A''$ to an $\eta''\colon A'' \rightarrowtail B'' \twoheadrightarrow C$ in $\mathcal{S}$. Recall from [**49**, Prop. 2.12], that we also get an an induced short exact sequence $\tilde{\eta}\colon A \rightarrowtail A'' \oplus B \twoheadrightarrow B''$.
Applying f gives a commutative diagram with exact rows

$$\begin{array}{ccccc}
f(A) & \overset{d}{\rightarrowtail} & f(B) & \twoheadrightarrow & f(C) \\
\downarrow{\scriptstyle c} & & \downarrow & & \| \\
f(A'') & \longrightarrow & f(B'' & \twoheadrightarrow & f(C)
\end{array}$$

Let D be the push-out of d, c, since $\mathcal{T}$ is an exact structure, we get an exact sequence $f(A) \rightarrowtail f(A'') \oplus f(B) \twoheadrightarrow D$ (in $\mathcal{T}$). In particular, the first map is an inflation. Now apply f to $\tilde{\eta}$ to get a right exact sequence $f(A) \to f(A'') \oplus f(B'') \twoheadrightarrow f(B'')$, since the first map is an inflation, we conclude that $\tilde{\eta} \in \mathcal{S}_f$ and $D \cong f(B'')$. This implies that that the lower row coincides with the push-out short exact sequence in $\mathcal{T}$, in particular $\eta'' \in \mathcal{S}_f$.

□

Remark 3.4. In the situation before, if $\mathcal{S}_f$ is an exact structure, then the composition $(\mathcal{A}, \mathcal{S}_f) \xrightarrow{id} (\mathcal{A}, \mathcal{S}) \xrightarrow{f} (\mathcal{B}, \mathcal{T})$ is an exact functor, we still denote it as f.

Example 3.5. Now given an object $P \in \mathcal{A}$ and consider $f = \mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to (Ab)$. As this functor is left exact, the previous Lemma applies and $\mathcal{S}_P := \mathcal{S}_f$ is an exact substructure. For a small subcategory $\mathcal{P}$ we define $\mathcal{S}_{\mathcal{P}} := \bigcap_{P \in \mathcal{P}} \mathcal{S}_P$. This exact substructure was found by Auslander-Solberg [**15**]. All exact substructure with enough projectives are of this form.

Example 3.6. Let R be a ring and I be a 2-sided ideal. Then, we have an exact substructure of $R\,\mathrm{Mod}$ given by all short exact sequences $0 \to M \to N \to L \to 0$ such that multiplication with I induces a split exact sequence. (Consider $f\colon \mathrm{Hom}_R(R/I, -)\colon R\,\mathrm{Mod} \to R\,\mathrm{Mod}$. This is left exact, so $\mathcal{S}_f$ is an exact substructure. Then apply Cor. 3.2 to the exact functor $f\colon (R\,\mathrm{Mod}, \mathcal{S}_f) \to R\,\mathrm{Mod}$ and the exact substructure $\mathcal{T}$ given by the split exact structure).

We also may often encounter the following situation:
Given a full additive subcategory $i\colon \mathcal{B}' \subseteq \mathcal{B}$ we say that $\mathcal{T}$ *restricts to* $\mathcal{B}'$ if $\mathcal{S}_i$ is an exact structure on $\mathcal{B}'$.

Remark 3.7. We will study this in our next chapter. Mainly, the following will be used:
If $\mathcal{B}'$ is 1) extension-closed or 2) inflation- and deflation-closed (i.e. for every short exact sequence with the middle and one of the outer in $\mathcal{B}'$, the third is in $\mathcal{B}'$) then $\mathcal{T}$ restricts to $\mathcal{B}'$.

Lemma 3.8. *Assume f is fully faithful. If $\mathcal{T}$ restricts to an exact structure on the essential image then $\mathcal{S}_f$ is an exact structure.*

Proof. By assumption, we can replace $(\mathcal{B}, \mathcal{T})$ with the essential image of f with the restricted exact structure, so we can assume wlog that f is additive equivalence. We can pull-back exact structures along equivalences, i.e. $f^{-1}(\mathcal{T})$ gives an exact structure in $\mathcal{A}$. Now, $\mathcal{S}_f = \mathcal{S} \cap f^{-1}(\mathcal{T})$ is again an exact structure. □

Example 3.9. Let us consider a noetherian scheme X and the global section functor $\Gamma\colon \mathrm{coh(X)} \to \mathrm{Ab}$. This is a left exact functor therefore $\mathcal{S}_\Gamma$ is an exact structure on coh(X). As $\Gamma = \mathrm{Hom}_{\mathrm{coh(X)}}(O_X, -)$, this is just the exact making $\mathcal{O}_X$ a projective. For example, let $X = \mathbb{P}^1_k$ with k a field. Then $(\mathrm{coh}(\mathbb{P}^1_{\mathrm{K}}, \mathcal{S}_\Gamma)$ is an Auslander-Reiten exact category with AR-sequence all AR-sequences such that they are not ending in $\mathcal{O}_X$ (i.e. we have lost only one AR-sequence).

Open question 3.10. Let $f\colon Y \to X$ be a proper morphism of locally noetherian schemes then we have adjoint functors $f^*\colon \mathrm{coh(X)} \to \mathrm{coh(Y)}\colon \mathrm{f}_*$. Therefore, we have an exact substructure $\mathcal{E}_{f^*}$ of coh(X) and an exact substructure $\mathcal{E}_{f_*}$ on coh(Y) making these functors exact. If f is a resolution of singularities, we ask if $f^*\colon \mathcal{E}_{f^*} \to \mathrm{coh(Y)}$ has a homologically faithful restriction to a fully homologically exact subcategory of coh(X).

Example 3.11. Given an abelian category $\mathcal{B}$, Dickson in [**63**] defined a torsion pair to be a pair of two full subcategories $(\mathcal{T}, \mathcal{F})$ satisfying $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and for every B in $\mathcal{B}$ there exists an exact sequence $B_t \rightarrowtail B \twoheadrightarrow B_f$ with B_t in $\mathcal{T}$, B_f in $\mathcal{F}$. Then $\mathcal{T}$ and $\mathcal{F}$ are fully exact subcategories in $\mathcal{B}$ but they may or may not be homologically exact. The assignment $B \mapsto B_t$ extends to a functor $t\colon \mathcal{B} \to \mathcal{T}$ (cf. loc. cit. Cor. 2.5), this functor preserves inflations but is in general not left exact. But it is left exact for a so-called *hereditary torsion pair* (cf.[**176**, Ch. VI, Prop. 1.7]) - characterized by $\mathcal{T}$ being closed under subobjects. So assume we have an hereditary torsion pair. By Lemma 3.3, we have an exact substructure $\mathcal{B}_t = (\mathcal{B}, \mathcal{S}_t)$. In $\mathcal{B}$ every exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ gives rise to a left exact sequence $t(X) \rightarrowtail t(Y) \to t(Z)$ and a right exact sequence $X/t(X) \to Y/t(Y) \twoheadrightarrow Z/t(Z)$. Short exact sequences in $\mathcal{B}_t$ are those for which both these sequences are short exact sequences (using the snake Lemma), i.e. they can be seen as an extension between an exact sequence in $\mathcal{T}$ and one in $\mathcal{F}$. By definition, both inclusion functor $\mathcal{T} \to \mathcal{B}_t$ and $\mathcal{F} \to \mathcal{B}_t$ are exact and the pair of subcategories $(\mathcal{T}, \mathcal{F})$ in $\mathcal{B}_t$ is a torsion pair in an exact category (cf. e.g. [**1**, Ex. 3.5]). Now, we show:

$$\mathrm{gldim}\, \mathcal{B}_t = \max(\mathrm{gldim}\, \mathcal{T}, \mathrm{gldim}\, \mathcal{F})$$

As $t\colon \mathcal{B}_t \to \mathcal{T}$ is an exact functor with $t \circ i \cong \mathrm{id}_{\mathcal{T}}$ it follows that $\mathcal{T}$ is homologically faithful in $\mathcal{B}_t$ (and similarly $\mathcal{F}$ is homologically faithful). But they are also homologically exact in $\mathcal{B}_t$ as the map on Ext^n is also obviously surjective (given $[\eta] \in \mathrm{Ext}^n_{\mathcal{B}_t}(t(X), t(Y))$, then $[t(\eta)]$ is in the image $\mathrm{Ext}^n_{\mathcal{T}}(t(X), t(Y)) \to \mathrm{Ext}^n_{\mathcal{B}_t}(t(X), t(Y))$ but by definition $t(\eta)$ and η are equivalent because we have a morphism $t(\eta) \to \eta$ of exact sequences with fixed end terms). This implies $\operatorname{gldim} \mathcal{B}_t \geq \max(\operatorname{gldim} \mathcal{T}, \operatorname{gldim} \mathcal{F})$. We claim that we have equality. To this, we define $n - 1 = \max(\operatorname{gldim} \mathcal{T}, \operatorname{gldim} \mathcal{F}) < \infty$. So, given an n-exact sequence in $\mathcal{B}_t$, say $\eta\colon X \rightarrowtail X_1 \to \cdots \to X_n \twoheadrightarrow Y$, we want to see that $[\eta] = 0$ in $\mathrm{Ext}^n_{\mathcal{B}_t}(Y, X)$. We have $t(\eta)$ and $\eta/t(\eta)$ are both exact sequences which are composed of short exact sequences in $\mathcal{T}$ (resp. $\mathcal{F}$), so they are in the image of $\mathrm{Ext}^n_{\mathcal{T}}(t(Y), t(X)) \to \mathrm{Ext}^n_{\mathcal{B}_t}(t(Y), t(X))$, but as $\mathrm{Ext}^n_{\mathcal{T}} = 0 = \mathrm{Ext}^n_{\mathcal{F}}$ it follows that both $[t(\eta)] = 0$ and $[\eta/t(\eta)] = 0$, let us call this property n-split. But n-split exact sequences are closed under extensions (this is an easy exercise using the derived category $D^b(\mathcal{B}_t)$). It follows that $[\eta] = 0$. We apply this in the following situation: Given R a principal ideal domain $\mathcal{B} = R\,\mathrm{mod}$ and $\mathcal{T}$ be the full subcategory of torsion modules (i.e. they have a non-zero annihilator) and $\mathcal{F}$ the subcategory of free modules. This gives a hereditary torsion pair in $\mathcal{B}$. We conclude that in this case $\operatorname{gldim} \mathcal{B}_t = 1$, i.e. $\mathcal{B}_t$ is still an hereditary exact category.

4. The maximal exact structure making a functor exact

We recall from [**167**] the definition of a left exact strucure on an additive category $\mathcal{A}$. It consists of a class of cokernels $\mathcal{D}$ (called deflations) such that

(C) $\mathcal{D}$ is closed under composition, and contains the identity id_A for every object A in $\mathcal{A}$
(P) pullbacks of morphisms in $\mathcal{D}$ along morphisms in $\mathcal{A}$ exist and are in $\mathcal{D}$,
(Q) If $b \circ a$ is in $\mathcal{D}$ and b has a kernel then b is in $\mathcal{D}$

A right exact structure is a class of kernels $\mathcal{I}$ (called inflations) which satisfies the same axioms in $\mathcal{A}^{op}$.

THEOREM 4.1. *(Rump, [**167**, Thm 1]) If $\mathcal{A}$ is an additive category with a left exact structure $\mathcal{D}$ and a right exact structure $\mathcal{I}$ there is an exact structure $\mathcal{S}$ given by all kernel-cokernels pairs (i, d) with i in $\mathcal{I}$, $d \in \mathcal{D}$.*

THEOREM 4.2. *(Rump, [**167**, Thm 2, Prop. 3]) For every additive category $\mathcal{A}$ and every class of morphsims $\mathcal{D}$ which is closed under composition and contains all split epimorphisms there exists a unique maximal subclass $\mathcal{D}' \subseteq \mathcal{D}$ which is a left exact structure.*

Then as a corollary one can find unique maximal exact structure in many situations.

Corollary 4.3. *([**167**, Cor 2]) Every additive category has a unique maximal exact structure.*

PROOF. Take $\mathcal{D} = \mathrm{Mor}\mathcal{A} = \mathcal{I}$. □

Corollary 4.4. *Let $f\colon \mathcal{E} \to \mathcal{F}$ be an additive functor between exact categories. Then there exists a maximal exact substructure $\mathcal{E}_{f,max}$ of $\mathcal{E}$ such that $f\colon \mathcal{E}_{f,max} \to \mathcal{F}$ is exact.*

PROOF. Take $\mathcal{D} = \{d\ \mathcal{E}-\text{deflation}\colon \text{f(d) } \mathcal{F}\text{-deflation}\}$,
$\mathcal{I} = \{i\ \mathcal{E}-\text{inflation}\colon \text{f(i) } \mathcal{F}\text{-inflation}\}$. □

Remark 4.5. (1) Observe that the short exact sequences for this maximal exact structure $\mathcal{S}_{f,max} \subseteq \mathcal{S}_f$ with equality if and only if $\mathcal{S}_f$ already gives an exact structure.
(2) Alternatively, given an additive functor $f\colon \mathcal{A} \to \mathcal{F}$ to an exact category there exists a unique maximal exact structure on $\mathcal{A}$ such that f becomes exact.

5. Application to recollements of abelian categories

We try to find additive functors $f\colon \mathcal{E} \to \mathcal{F}$ between exact categories such that $f\colon \mathcal{E}_f \to \mathcal{F}$ is homologically exact. Observe that an homologically exact functor is equivalent to a fully faithful functor whose essential image is a homological exact subcategory (in particular extension-closed).
We look now for fully faithful functors with extension-closed essential image, by Lemma 3.8, then we can always make the functor exact by passing to a substructure. A good place to find them are recollements (we restrict to the abelian case, see remark 5.1):
We look at three abelian categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ together with the two pairs of adjoint triples (ℓ, e, r) (i.e. (ℓ, e) and (e, r) are adjoint pairs) and (q, i, p) as indicated below. Then, $(\mathcal{A}, \mathcal{B}, \mathcal{C}), (\ell, e, r), (q, i, p)$ is a **recollement of abelian categories** if

(1) ℓ, r, i are fully faithful
(2) $\operatorname{Im} i = \ker e$

This definition is from [**75**], the notation follows [**130**]. Recollements of abelian categories were invented originally for triangulated categories in [**40**]. Homological properties of these functors have been more systematically studied in [**155**].

Remark 5.1. This has been generalized to exact (and extriangulated categories), cf. [**181**], [**90**, Def. 4.29]. But the definition has more axioms, so we stick to the abelian case here. A recollement of abelian categories is always a recollement of exact categories, cf. [**155**].

Since $\mathcal{B}$ is abelian, we can also define a seventh functor $c\colon \mathcal{C} \to \mathcal{B}$ given by $C \mapsto \operatorname{Im}(\ell(C) \to r(C))$.

The seven functors in a recollement of abelian category:

$$\mathcal{A} \;\underset{\xleftarrow[p]{}}{\overset{\xleftarrow{q}}{\xrightarrow{\;i\;}}}\; \mathcal{B} \;\underset{\xleftarrow[r]{}}{\overset{\xleftarrow{\ell}}{\xrightarrow{\;e\;}}}\; \mathcal{C}, \qquad c\colon \mathcal{C} \to \mathcal{B}$$

Four of these are candidates to be made *homologically exact*, namely ℓ, r, c and i
Observe that i is fully faithful (by axiom (1)) and has an extension-closed essential image (by axiom (2)).

Example 5.2. Let A be a ring and $e \in A$ an idempotent element, then there is a recollement of (left) module categories

$$A/(e)\,\mathrm{Mod} \;\underset{\xleftarrow[p]{}}{\overset{\xleftarrow{q}}{\xrightarrow{\;i\;}}}\; A\,\mathrm{Mod} \;\underset{\xleftarrow[r]{}}{\overset{\xleftarrow{\ell}}{\xrightarrow{\;e\;}}}\; eAe\,\mathrm{Mod}$$

where $e\colon M \mapsto eM$ is the multiplication with e and i is the natural inclusion. The left adjoint functors ℓ, q can be expressed as tensor functors and the right adjoint functors as Hom-functors induced by the two bimodules ${}_{A/(e)}A/(e)_A$ and ${}_A Ae_{eAe}$.
In general this will only restrict partially to recollements on smaller abelian categories. But there is one easy special case: If A is artinian, then also $A/(e)$ and eAe then the category of finitely generated modules is abelian and in this case all six functors restrict to subcategories of finitely presented modules.

Lemma 5.3. *The functors ℓ, r, c are fully faithful with extension-closed essential images.*

Proof. We have $\operatorname{Im} c = \ker q \cap \ker q$ (cf. [**75**, Prop. 4.11]) it is enough to see that $\ker q$ and $\ker p$ are extension closed. But since q (resp. p) is left (resp. right) adjoint to i, it is right (resp. left) exact and therefore $\ker q$ (resp. $\ker p$) extension-closed.
For $t = \ell$ or r. We have $t \cong tet$, so apply to a $\mathcal{B}$-exact sequence $tX \rightarrowtail E \twoheadrightarrow tY$, the exact functor e and then t again, to obtain a complex $tX \to teE \to tY$. We want to see that E and teE are

isomorphic. For $t=\ell$, it is enough to see that $\ell X \to \ell e E$ is an inflation. But we have the canonical morphism $\ell e E \to E$ and postcomposition gives the inflation $\ell X \rightarrowtail E$. Therefore the claim follows from the obscure axiom. For $t=r$ use the dual argument. □

The functors $t \in \{\ell, r, c\}$ induce exact equivalences of $t\colon (\mathcal{C}, \mathcal{S}_t) \to \operatorname{Im} t$ where $\operatorname{Im} t$ is seen as a fully exact subcategory of $\mathcal{B}$, the functor e restricts in all three cases to an exact functor $\operatorname{Im} t \to (\mathcal{C}, \mathcal{S}_t)$ which is the quasi-inverse. So we directly conclude $e\colon \operatorname{Im} t \to \mathcal{C}$ is n-homologically faithful for $n=1$ and for all n such that $\mathcal{C}_t \to \mathcal{C}$ is n-homologically faithful. Furthermore, we remark the following easy properties:

Remark 5.4. Let $\eta\colon X \rightarrowtail Y \twoheadrightarrow Z$ be a short exact sequence in $\mathcal{C}$.

(1) The functor ℓ is right exact.
Then η is exact in $\mathcal{C}_\ell$ iff $\ell(X) \to \ell(Y)$ is a $\mathcal{B}$-inflation. We have: $\operatorname{Im} \ell$ is inflation-closed in $\mathcal{B}$ (i.e. if $\ell(C) \rightarrowtail \ell(D) \twoheadrightarrow B$ is a $\mathcal{B}$-exact sequence then $B \in \operatorname{Im} \ell$).
(2) The functor r is left exact.
Then η is exact in $\mathcal{C}_r$ iff $r(Y) \to r(Z)$ is a deflation in $\mathcal{B}$. We have: $\operatorname{Im} r$ is deflation-closed in $\mathcal{B}$.
(3) The functor c preserves inflations and deflations.
Then η is exact in $\mathcal{C}_c$ iff $c(X) \to c(Y) \to c(Z)$ is exact in the middle in $\mathcal{B}$. We have: $\operatorname{Im} c$ is closed under taking images of morphisms in $\mathcal{B}$.

We have the following trivial special case.

Corollary 5.5. *Let $t \in \{\ell, c, r, i\}$.*
If $\operatorname{gldim} \mathcal{B} \leq 1$ then $t\colon (\mathcal{C}, \mathcal{S}_t) \to \mathcal{B}$ or resp. $t=i\colon \mathcal{A} \to \mathcal{B}$ is homologically exact.
If $\operatorname{gldim} \mathcal{B} \leq 2$ this functor is homologically faithful.

5.0.1. *A quick survey of the knwon results.* We summarize some results from [**155**], [**75**] and [**11**] regarding to homological exactness the functors ℓ, r, c, i.

The essential images of ℓ, r, c: The functor q is right exact and the functor p is left exact. Assume that the first derived functors $L_1 q$ and $R^1 p$ exist, then from [**75**, section 4.3] we have

$$\operatorname{Im} \ell = \ker q \cap \ker L_1 q$$
$$\operatorname{Im} c = \ker q \cap \ker p$$
$$\operatorname{Im} r = \ker p \cap \ker R^1 p$$

Instead of using derived functors we choose to restrict to situations with $\mathcal{C}$ having enough projectives and or injectives and use the following alternative descriptions (following [**11**]):
For a subcategory of injectives $\mathcal{I}$ in $\mathcal{B}$, $j \geq 0$ we write $\operatorname{cogen}^j \mathcal{I}$ for the full subcategory given by all objects X such that that there exists an exact sequence

$$X \rightarrowtail I^0 \to \cdots \to I^j \twoheadrightarrow Y$$

with I^s in $\mathcal{I}$, $0 \leq s \leq j$. For $j=\infty$ we require an injective coresolution with all terms in $\mathcal{I}$. For $j=0$ we leave out the supscript. We define $\operatorname{gen}_j(\mathcal{P})$ for a subcategory of projectives $\mathcal{P}$ in $\mathcal{B}$ dually. By the horseshoe Lemma they are all extension-closed in $\mathcal{B}$.

Lemma 5.6. *We have:*

(1) *If $\mathcal{C}$ has enough projectives $\mathcal{P}$, then $\ell(\mathcal{P}) \subset \mathcal{P}(\mathcal{B})$ and $\operatorname{Im} \ell = \operatorname{gen}_1(\ell(\mathcal{P}))$, $\ker q = \operatorname{gen}(\ell(\mathcal{P}))$. Furthermore, for $j \geq 1$ the restriction*
$e\colon \operatorname{gen}_j(\ell(\mathcal{P})) \to \mathcal{C}$ is s-homologically exact for $0 \leq s \leq (j-1)$.
(2) *If $\mathcal{C}$ has enough injectives $\mathcal{I}$, then $r(\mathcal{I}) \subset \mathcal{I}(\mathcal{B})$ and $\operatorname{Im} r = \operatorname{cogen}^1(r(\mathcal{I}))$, $\ker p = \operatorname{cogen}(r(\mathcal{I}))$. Furthermore, for $j \geq 1$ the restriction*
$e\colon \operatorname{cogen}^j(r(\mathcal{I})) \to \mathcal{C}$ is s-homologically exact for $0 \leq s \leq (j-1)$.

(3) *If $\mathcal{C}$ has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$, then we have* $\operatorname{Im} c = \operatorname{gen}(\ell(\mathcal{P})) \cap \operatorname{cogen}(r(\mathcal{I}))$. *For $j+k \geq 0$, the restriction* $e\colon \operatorname{gen}_j(\ell(\mathcal{P})) \cap \operatorname{cogen}^k(r(\mathcal{I})) \to \mathcal{C}$ *is s-homologically exact for $0 \leq s \leq j+k$.*

Observe that these restricted functors factor over an exact functor to $\mathcal{C}_t$ for $t \in \{\ell, r, c\}$ respectively in case (1),(2),(3). As a consequence one obtains that their essential image is an extension-closed subcategory $\mathcal{X}$ of $\mathcal{C}_t$ such that the composition $\mathcal{X} \to \mathcal{C}_t \to \mathcal{C}$ is s-homologically exact, s in the range specified as in the lemma respectively in case (1),(2),(3). If $\mathcal{X} = \mathcal{C}_t$ and the inclusion of an exact substructure is 1-homologically exact, it is the identity. So in that case $\mathcal{C}_t = \mathcal{C}$ (so t is an exact functor). The same in epic detail:

Corollary 5.7. (a) *If $\mathcal{C}$ has enough projectives $\mathcal{P}$ then the following are equivalent:*
- (a1) *$\ell\colon \mathcal{C} \to \mathcal{B}$ is an exact functor.*
- (a2) $\operatorname{gen}_1(\ell(\mathcal{P})) = \operatorname{gen}_2(\ell(\mathcal{P})$.
- (a3) $\operatorname{gen}_1(\ell(\mathcal{P})) = \operatorname{gen}_\infty(\ell(\mathcal{P}))$.
- (a4) $\operatorname{Im}\ell$ *has enough projectives given by $\ell(\mathcal{P})$ (and is partially resolving, Def 2.21).*
- (a4') $\operatorname{Im}\ell$ *is deflation-closed in $\mathcal{B}$.*
- (a5) *$\ell\colon \mathcal{C} \to \mathcal{B}$ is a homologically exact functor.*

(b) *If $\mathcal{C}$ has enough injectives $\mathcal{I}$ then the following are equivalent*
- (b1) *$r\colon \mathcal{C} \to \mathcal{B}$ is an exact functor.*
- (b2) $\operatorname{cogen}^1(r(\mathcal{I})) = \operatorname{cogen}^2(r(\mathcal{I}))$.
- (b3) $\operatorname{cogen}^1(r(\mathcal{I})) = \operatorname{cogen}^\infty(r(\mathcal{I}))$.
- (b4) $\operatorname{Im} r$ *has enough injectives given by $r(\mathcal{I})$ and is partially coresolving.*
- (b4') $\operatorname{Im} r$ *is inflation-closed in $\mathcal{B}$*
- (b5) *$r\colon \mathcal{C} \to \mathcal{B}$ is a homologically exact functor.*

(c) *The following are equivalent*
- (c1) *$c\colon \mathcal{C} \to \mathcal{B}$ is exact*
- (c2) $\operatorname{Im} c$ *is inflation-closed or deflation-closed*
- (c3) $\operatorname{Im} c$ *is inflation- and deflation-closed*

(d) *If $\mathcal{C}$ has enough injectives $\mathcal{I}$ and enough projectives $\mathcal{P}$ the following are equivalent*
- (d1) $\operatorname{gen}(\ell(\mathcal{P})) \cap \operatorname{cogen}(r(\mathcal{I})) = \operatorname{gen}_j(\ell(\mathcal{P})) \cap \operatorname{cogen}^k(r(\mathcal{I}))$ *for at least one* $(j,k) \in \{(1,0),(0,1)\}$
- (d2) $\operatorname{gen}(\ell(\mathcal{P})) \cap \operatorname{cogen}(r(\mathcal{I})) = \operatorname{gen}_j(\ell(\mathcal{P})) \cap \operatorname{cogen}^k(r(\mathcal{I}))$ *for at least one* $(j,k) \in \{(\infty,0),(0,\infty)\}$
- (d3) $\operatorname{Im} c = \operatorname{Im}\ell$ *has enough projectives given by $c(\mathcal{P}) = \ell(\mathcal{P})$ or* $\operatorname{Im} c = \operatorname{Im} r$ *has enough injectives given by $c(\mathcal{I}) = r(\mathcal{I})$*
- (d4) $\operatorname{Im} c$ *is deflation-closed and contains $\ell(\mathcal{P})$ or inflation-closed and contains $r(\mathcal{I})$ or both.*
- (d5) *$c = \ell\colon \mathcal{C} \to \mathcal{B}$ or $c = r\colon \mathcal{C} \to \mathcal{B}$ is homologically exact.*
- (d6) $p\ell = 0$ *or* $qr = 0$

We do not know if the conditions in (c) already imply (d) but it does not seem to be the case. Do there exist recollements with $p\ell = 0$?

Proof. For part (a) (and (b)), look at [**11**], (c) is very easy and (d) is a corollary from case (3) in the previous Lemma. □

The functor i. The functor i is already exact (as it is a functor between abelian categories which has a left and right adjoint) and following [**155**] we call a recollement **k-homological** if the functor i is m-homologically exact for $0 \leq m \leq k$.
Observe that it is always 1-homological.

Lemma 5.8. *If $\mathcal{C}$ has enough projectives $\mathcal{P}$ resp. enough injectives $\mathcal{I}$, then we have*

$$\operatorname{gen}_k(\ell(\mathcal{P})) = \bigcap_{j=0}^{k} \ker \operatorname{Ext}^j_{\mathcal{B}}(-, \operatorname{Im} i)(= \bigcap_{j=0}^{k} \ker \operatorname{Ext}^j_{\mathcal{B}}(-, i(\mathcal{J})) \text{ if } \mathcal{A} \text{ w.e.i. } \mathcal{J}) \quad \text{resp.}$$

$$\operatorname{cogen}^k(r(\mathcal{I})) = \bigcap_{j=0}^{k} \ker \operatorname{Ext}^j_{\mathcal{B}}(\operatorname{Im} i, -)(= \bigcap_{j=0}^{k} \ker \operatorname{Ext}^j_{\mathcal{B}}(i(\mathcal{Q}), -) \text{ if } \mathcal{A} \text{ w.e.p. } \mathcal{Q})$$

the following conditions are equivalent to i being m-homologically exact for $0 \leq m \leq k$

(1) *for $\mathcal{C}$ with enough projectives $\mathcal{P}$:* $\operatorname{Im} i \subseteq \operatorname{gen}_k(\ell(\mathcal{P}))$
(2) *for $\mathcal{C}$ with enough injectives $\mathcal{I}$:* $\operatorname{Im} i \subseteq \operatorname{cogen}^k(r(\mathcal{I}))$
(3) *for $\mathcal{A}$ with enough projectives $\mathcal{Q}$:* $\operatorname{Ext}^j_{\mathcal{B}}(i(Q), i(A)) = 0$ *for all* $0 \leq j \leq k$, *A in $\mathcal{A}$*, $Q \in \mathcal{Q}$
(4) *for $\mathcal{A}$ with enough injectives $\mathcal{J}$:* $\operatorname{Ext}^j_{\mathcal{B}}(i(A), i(J)) = 0$ *for all* $0 \leq j \leq k$, *A in $\mathcal{A}$*, $J \in \mathcal{J}$

Example 5.9. We recall for an artin algebra Λ the following result [**11**, Thm 2.1']: An idempotent e gives rise to a k-homological recollement if and only if $\Lambda e \Lambda \in \operatorname{gen}_{k-1}(\ell(\mathcal{P}))$

5.0.2. *But this is not all....* One observes that Corollary 5.7 is much stronger than what we asked for (they can only occur for already exact functors t).
For example as a corollary of 5.7, $\operatorname{Im} \ell$ is right cofinal iff it is deflation-closed iff $\ell \colon \mathcal{C} \to \mathcal{B}$ is homologically exact.
We know that there are many more positive answers which are not covered by the previous corollary. As $\operatorname{Im} \ell$ is already inflation-closed we are already on the way to left cofinal.

Remark 5.10. The unfortunate fact is that we do not have a characterization of homologically exactness but only sufficient criteria so a satisfying complete answer can not be expected.

Remark 5.11. (1) If $\operatorname{Im} \ell$ is cogenerating, then $\operatorname{Im} \ell$ is coresolving in $\mathcal{B}$.
(2) If $\operatorname{Im} r$ is generating , then $\operatorname{Im} r$ is resolving in $\mathcal{B}$.

Example 5.12. A module M over a ring A is called faithfully balanced if the natural ring homomorphism $A \to \operatorname{End}_{\operatorname{End}_A(M)}(M)$ is an isomorphism. Let $e \in A$ be an idempotent such that Ae is a faithfully balanced projective left A-module. This means that $A = \operatorname{End}_{eAe}(eA)^{op}$ is endomorphism of a generator (in left eAe-modules). In particular $r(eA) = A \in \operatorname{Im} r$ is resolving (in finitely generated A-modules).
In fact, this is giving a complete answer, at least for module categories of artin algebras see below (this is partly true for more general rings we just have not studied this very much).

Lemma 5.13. *Given a finite-dimensional algebra A and an idempotent $e \in A$ and $D = \operatorname{Hom}_K(-, K)$, the following are equivalent for the recollement:*

(a) $DA \in \operatorname{Im} \ell$
(b) $A \in \operatorname{Im} r$
(c) ${}_A Ae$ *is faithfully balanced*

Proof. (b) $\Rightarrow$ (c): $A = r(X) \Rightarrow eA \cong X \Rightarrow \operatorname{End}_{eAe}(eA) = r(eA) \cong r(X) = A$
(c) $\Rightarrow$ (b): ${}_A A = {}_A \operatorname{End}_{eAe}(eA) = r(eA) \in \operatorname{Im} r$
(c) $\Rightarrow$ (a): Now, observe with Hom-tensor adjunction gives an isomorphisms of bimodules $D\ell e(DA) = \operatorname{End}_{eAe}(Ae)$. So if Ae is faithfully balanced then $\ell(eA) = DA \in \operatorname{Im} \ell$.
(a)$\Rightarrow$ (c): $DA = \ell(X) \Rightarrow eDA \cong X \Rightarrow D \operatorname{End}_{eAe}(Ae) = \ell(eDA) \cong \ell(X) = DA$ and apply D. □

Example 5.14. We also studied examples of recollements such that $\operatorname{Im} c$ is self-orthogonal (and therefore homologically exact in $\mathcal{B}$). In this case $\mathcal{C}_c$ is the split exact structure. We give here only the reference, look at [**61**, Thm 4.4, Lem. 4.7].

Example 5.15. Can we also find cases where $\operatorname{Im}\ell$ is only partially coresolving (still assuming that $\mathcal{C}$ has enough projectives)? This means we are looking for pairs $(\mathcal{P}, \mathcal{I})$ of a projective and an injective subcategory in $\mathcal{B}$ such that

$$\mathcal{I} \subseteq \operatorname{gen}_1(\mathcal{P}) \subseteq \operatorname{cogen}(\mathcal{I})$$

Then since $\operatorname{gen}_1(\mathcal{P})$ is inflation-closed, it follows that $\operatorname{gen}_1(\mathcal{P})$ has enough injectives given by $\mathcal{I}$. Here is a very simple example where this occurs: Take $\mathcal{B} = KQ \operatorname{mod}$ for K a field and Q the quiver $1 \to 2 \to 3 \to 4$. We choose $P = S_4 \oplus P_1$ and $I = P_1 \oplus I_3$, then in this case $\operatorname{gen}_1(P) = \operatorname{add}(S_4 \oplus P_1 \oplus I_3) = \operatorname{cogen}^1(I)$ is partially coresolving.

Now, we give some negative answers.

Example 5.16. This gives an example for which $\operatorname{Im}\ell$ is not homologically exact: Let Λ be the following path algebra (over a field) with radical square zero relations

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4, \qquad a_2a_1 = 0 = a_3a_2$$

and $e = e_1 + e_2 + e_4$. Then $\operatorname{Im}\ell = \operatorname{add}(S_4 \oplus P_2 \oplus P_4 \oplus S_1)$ is as fully exact subcategory of $\Lambda \operatorname{mod}$ semi-simple, therefore $\operatorname{Ext}^2_{\operatorname{Im}\ell}(S_1, S_4) = 0$. But $\operatorname{Ext}^2_\Lambda(S_1, S_4) \neq 0$. We conclude $\operatorname{Im}\ell$ is not homologically exact in this case. But it is nevertheless homologically faithful.

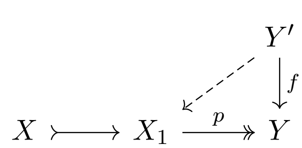
Example 5.17. We look at a recollement for module categories of finite dimensional algebras with $\operatorname{gldim}\mathcal{C} = \infty = \operatorname{gldim}\mathcal{A}$ and $\operatorname{gldim}\mathcal{B} = 2$ (then i is not homologically exact): Let Λ be a self-injective algebra of finite representation-type (e.g. $\Lambda = K[X]/(X^n)$ or a more general self-injective Nakayama algebra), let Γ be its Auslander algebra and $e \in \Gamma$ the projection onto the projective-injective summand, so $\Lambda = e\Gamma e$. In this case we have $\operatorname{gldim}\Gamma/(e) = \infty = \operatorname{gldim}\Lambda$ and $\operatorname{gldim}\Gamma = 2$.
In this case we have that the projective-injective summand in Γ is a faithfully balanced module and therefore $\operatorname{Im}\ell$ is coresolving and $\operatorname{Im}r$ is resolving. This means we find that $\operatorname{gldim}\mathcal{C}_\ell \leq 2$ and $\operatorname{gldim}\mathcal{C}_r \leq 2$ even though $\operatorname{gldim}\mathcal{C} = \infty$.

Example 5.18. We look at a recollement for module categories of finite-dimensional algebras with $\operatorname{gldim}\mathcal{B} = \infty = \operatorname{gldim}\mathcal{A}$ and $\operatorname{gldim}\mathcal{C} = 2$: Let Π_{2n} be the preprojective algebra of Dynkin type A_{2n} and $e = e_1 + \cdots + e_n$, then $e\Pi_{2n+1}e$ is the Auslander algebra of $K[X]/(X^n)$ and $\Pi_{2n}/(e) \cong \Pi_n$. In particular, we have $\operatorname{gldim}\Pi_{2n} = \infty = \operatorname{gldim}\Pi_n$ and $\operatorname{gldim} e\Pi_{2n}e = 2$. In this case we find that all three exact substructures $\mathcal{C}_\ell, \mathcal{C}_r, \mathcal{C}_c$ are Frobenius exact (by [**46**, Thm II 2.6]) of infinite global dimension even though $\operatorname{gldim}\mathcal{C} = 2$.

6. Appendix: n-split exact sequences

Set $\sigma \colon X \rightarrowtail X_1 \to \cdots \to X_n \twoheadrightarrow Y$ be an n-exact sequence and $[\sigma] \in \operatorname{Ext}^n_{\mathcal{E}}(Y, X)$ be the representing class. We want to characterize when is $[\sigma] = 0$, we say then that σ is (n-)**split**. Warning: Only for $n = 1$ is this equivalent to being an exact sequence in the split exact structure.
Secondly we also ask for a characterization when $f^*\sigma$ is split for a morphism $f \colon Y' \to Y$.

n=1: $[\sigma] = 0$ is equivalent to σ is a short exact sequence in the split exact structure, i.e. the inflation is a split monomorphism and the deflation is a split epimorphism.
$f^*\sigma$ is split if and only if f factors over the deflation d



n=2: In this case we also defined $[\sigma] = [\alpha]$ where $\alpha\colon X_1 \to X_2$.
σ is 2-split iff and only if the left hand side diagram can be filled in with exact sequences in the rows and columns as a commuting diagram and $f^*\sigma$ is 2-split if so for the right hand side.

$$\begin{array}{ccccc} X & \rightarrowtail & X_1 & \twoheadrightarrow & Z \\ \| & & \downarrow & & \downarrow \\ X & \rightarrowtail & A & \twoheadrightarrow & X_2 \\ & & \downarrow & & \downarrow \\ & & Y & = & Y \end{array} \qquad \begin{array}{ccccc} X & \rightarrowtail & X_1 & \twoheadrightarrow & Z \\ & & \downarrow & & \downarrow \\ & & A & \longrightarrow & X_2 \\ & & \downarrow & & \downarrow \\ & & Y' & \xrightarrow{f} & Y \end{array}$$

Equivalently: Fill-in with exact rows

$$\begin{array}{ccccccc} X & \rightarrowtail & X_1 & \longrightarrow & X_2 & \twoheadrightarrow & Y \\ & & \| & & \uparrow & & \| \\ & & X_1 & \rightarrowtail & A & \twoheadrightarrow & Y \end{array} \qquad \begin{array}{ccccccc} X & \rightarrowtail & X_1 & \longrightarrow & X_2 & \twoheadrightarrow & Y \\ & & \| & & \uparrow & & \uparrow f \\ & & X_1 & \rightarrowtail & A & \twoheadrightarrow & Y' \end{array}$$

n>2. Then σ is n-split iff the following diagram can be filled in as a commuting diagram with exact rows.

$$\begin{array}{ccccccccccccccc} X & \rightarrowtail & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & \cdots & \longrightarrow & X_n & \twoheadrightarrow & Y \\ & & \| & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \| \\ & & X_1 & \rightarrowtail & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots & \longrightarrow & A_{n-1} & \twoheadrightarrow & Y \end{array}$$

Figure 1

and then $f^*\sigma$ is n-split iff the following diagram can be filled in as a commuting diagram with exact rows.

$$\begin{array}{ccccccccccccccc} X & \rightarrowtail & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & X_4 & \longrightarrow & \cdots & \longrightarrow & X_n & \twoheadrightarrow & Y \\ & & \| & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow f \\ & & X_1 & \rightarrowtail & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots & \longrightarrow & A_{n-1} & \twoheadrightarrow & Y' \end{array}$$

Figure 2

Proof. $n = 1$ We only show: $f^*\sigma$ splits iff f factors as $f = ph$ where p is the deflation in σ. Let us complete the diagramm, the upper line is $f^*\sigma$

$$\begin{array}{ccccc} X & \rightarrowtail & X_1' & \xrightarrow{p'} & Y' \\ \downarrow & & \downarrow f' & & \downarrow f \\ X & \rightarrowtail & X_1 & \xrightarrow{p} & Y \end{array}$$

If $f^*\sigma$ is split then there exists a section s of p', (i.e. $p's = \mathrm{id}$). Then we have $pf's = fp's = f$, so f factors over p.
If $f = ph$ for some morphism h, we use the universal property of the pullback to find a morphism s as indicated

$$\begin{array}{ccccc} Y' & & \xrightarrow{\quad 1 \quad} & & Y' \\ & \overset{s}{\searrow} & & & \\ & & X_1' & \xrightarrow{p'} & Y' \\ & {\scriptstyle h}\searrow & \downarrow f' & & \downarrow f \\ & & X_1 & \xrightarrow{p} & Y \end{array}$$

Then s is a section of p' and therefore $f^*\sigma$ splits.
For general $n > 1$, by [**76**, Thm 6.39] the condition $[\sigma] = 0$ is equivalent to the fill-in of the diagram in Figure 1 (cf. condition (v) in loc. cit.).
It is straight-forward to see that the pullback $f^*\sigma$ has a diagram fill-in as in Figure 1 if and only the fill-in in Figure 2 holds.
For $n = 2$: We ignore the $f^*\sigma$-split-description as it is a trivial special case of the general n case. We need to see the equivalence of the two diagram fill-ins. Clearly from first diagram we can easily deduce the second type of diagram. Conversely, from the second we arrive at a fill-in of the form

$$\begin{array}{ccccc} X & \rightarrowtail & X_1 & \twoheadrightarrow & Z \\ & & \downarrow & & \downarrow \\ & & A & \xrightarrow{a} & X_2 \\ & & \downarrow & & \downarrow \\ & & Y & = & Y \end{array}$$

By [**49**, Prop. 2.12], we deduce that the commuting square in the upper corner is bicartesion (i.e. a pullback and pushout square). Now, by [**49**, Prop. 2.15] the pushout of a deflation along an inflation is a deflation, so a is a deflation. Again by [**49**, Prop. 2.12] we conclude that $\ker a$ is the composition $X \rightarrowtail X_1 \to A$ and therefore the claim follows.

□

Remark 6.1. You find in [**76**, Thm 6.39] several other variants of these diagram fill-ins.
In [**126**, in section 4.2] (section before Prop. 4.2.11) there is also claimed a very simple criterion: $[\sigma] = 0$ is equivalent to there exists a morphism of exact sequences $\sigma \to \sigma'$ with fixed end terms such that σ' is an n-exact sequence with the last deflation being a split epimorphism.

CHAPTER 2

The poset of exact structures

1. Synopsis

We survey the theory of exact structures on an essentially small idempotent complete additive category. We focus on explicit answers and examples. But we also collect/recall several lattice isomorphisms for the lattice of all exact structures. Several of these isomorphisms are induced by equivalences of 2-categories which we collect in an Appendix.
What is new here? The description of exact structures with enough projectives. The equivalence of 2-categories with tf-Auslander categories (i.e. the subcategory of torsionfree objects in the Auslander exact category) is new. Apart from the Auslander correspondence none of the equivalences of 2-categories are formulated as such in the literature that I know (we treat them for time reasons also somewhat sketchy). Furthermore, we look at all exact substructures in examples (e.g. finitely generated abelian groups) and establish some of their global dimensions.

2. Introduction

Now, we fix one essentially small, idempotent complete exact category $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ and introduce the following posets of exact substructures

$$\mathrm{ex}(\mathcal{E}) = \text{ exact substructures of } \mathcal{E}.$$

Here the poset structure is given by inclusion on the collection of short exact sequences, i.e. $\mathcal{E}_1 \leq \mathcal{E}_2$ means the identity functor $\mathcal{E}_1 \to \mathcal{E}_2$ is an exact functor. Rump showed [**167**] that for every essentially small additive category there always exists a maximal structure (independently this had been shown by Crivei [**62**] under the assumption that the underlying additive category is weakly idempotent complete). Therefore, we may as well define $\mathrm{ex}(\mathcal{A}) := \mathrm{ex}(\mathcal{E}_{max})$ where $\mathcal{E}_{max}$ is the maximal exact structure on the additive category $\mathcal{A}$.
As it is very easy to see that arbitrary intersections of exact structures give an exact structure, we have a complete meet semi-lattice. Using the existence of a maximal exact structure, this implies that $\mathrm{ex}(\mathcal{E})$ is a complete lattice, cf. [**29**]. We are interested in the following types of results:

(1) Explicit parametrizations and constructions of exact structures
(2) Lattice isomorphisms for $\mathrm{ex}(\mathcal{E})$

We survey three explicit answers in sections 2,3,4 respectively. Firstly, the easiest construction are exact substructures induced by subcategories, these include all exact structures with enough projectives (or resp. with enough injectives). They have been introduced by Auslander and Solberg in [**15**]. Secondly, in the representation-finite case, we obtain the very easy Boolean lattice of generators - first observed by Enomoto, cf. Enomoto's theorem 4.4.
Thirdly, for essentially small additive categories with weak cokernels, there exists a topological space called the Ziegler spectrum consisting of certain indecomposables in the ind-completion. The indecomposable injectives in the ind-completion of the maximal exact structure define a Ziegler-closed subset $\mathcal{U}_{max}$. Then there is a bijection between Ziegler-closed subsets containing $\mathcal{U}_{max}$ and exact structures. This connection has been observed by Schlegel (in [**173**]), cf. Theorem 5.6. We apply this result in some examples with known Ziegler spectrum to have an understanding all exact substructures.

Furthermore, we study lattice isomorphisms which are not leading (us) to explicit answers. First of all, all four equivalences of 2-categories from the Appendices A,B,C give such lattice isomorphisms. The most classical lattice isomorphism is the Butler-Horrock's theorem (Appendix A) which identifies $\mathrm{ex}(\mathcal{E})$ with the lattice of closed sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{E}_{max}}$. Auslander correspondence can be seen as an equivalence of 2-categories (cf. [**90**]) - as a companion we add the tf-Auslander correspondence (Appendix B). Ind-completion gives the forth equivalence of 2-categories (Appendix C).

(2a,2b,2c) We follow Auslander's idea to study associated functor categories. An exact structure is determined by three different classes of morphisms, the i) admissible morphisms, ii) the deflations, iii) the inflations. When looking at functors represented by either of these three classes we obtain three correspondences ,respectively, i) the Auslander correspondence (cf. also Appendix B), ii) Enomoto's correspondence and iii) a new one which we call tf-Auslander correspondence. The first and third are induced by the equivalences of 2-categories (cf. Appendix B). These lead to three further poset isomorphisms.

What remains open:

(2*)(Q1) Assuming the additive category has weak cokernels, [**173**] found several other lattice isomorphisms to $\mathrm{ex}(\mathcal{A})$ (with certain Ziegler-closed mentioned before, with fp-idempotent ideal, with torsion classes etc.). Can (some of it) be generalize to arbitrary small additive categories?

(Q2) Let $\mathcal{E}$ be an essentially small exact category and $\mathcal{I} = \mathcal{I}(\overrightarrow{\mathcal{E}})$ the injectives in the Ind-completion. Properties of the Ind-completion imply

$$\operatorname{gldim} \mathcal{E} \leq \operatorname{gldim} \overrightarrow{\mathcal{E}} \leq \operatorname{gldim} \mathcal{I} \operatorname{mod}_\infty$$

where $\mathcal{I} \operatorname{mod}_\infty$ is the category of all additive functors $F\colon \mathcal{I} \to (Ab)$ such that there exists an exact sequence

$$\cdots \to \mathrm{Hom}(I_2, -) \to \mathrm{Hom}(I_1, -) \to \mathrm{Hom}(I_0, -) \to F \to 0$$

with $I_n \in \mathcal{I}$. This is always an exact category with set-valued Ext-groups (even though $\mathcal{I}\,\mathrm{Mod}$ may not be).

Assume that there is a correspondence of exact structures with subsets of the Ziegler spectrum Zg (see Q1) by assigning $\mathcal{E} \to \mathcal{U}_{\mathcal{E}} = \mathcal{I} \cap \mathrm{Zg}$. Is there an upper bound for $\operatorname{gldim} \mathcal{E}$ using $\mathcal{U}_{\mathcal{E}}$?

3. Elementary constructions of exact substructures

Lemma 3.1. *([**66**], Section 1.2) Let $(\mathcal{A}, \mathcal{S})$ be an exact categeory. We have an obvious bijection between the following two sets*

(a) *(additive) subfunctors $F \subset \mathrm{Ext}^1_{\mathcal{S}}$*
(b) *subclasses $\mathcal{S}'$ of $\mathcal{S}$ closed under isomorphisms (and direct sums of short exact sequences), pullback and pushout of short exact sequences, i.e. (Ex2) holds for $\mathcal{S}'$.*

given by $F \mapsto \mathcal{S}_F$ where $\mathcal{S}_F$ consists of all exact pairs $Y \to E \to X$ in $\mathcal{S}$ such that its equivalence class is in $F(X,Y)$. Conversely, $\mathcal{S}' \mapsto F'$ with $F'(X,Y)$ consists of all equivalence classes of exact sequences in $\mathcal{S}'$.

As indicated by the brackets, the property of being an additive subfunctor translates into the property that the short exact sequences are closed under direct sums. To study the structures corresponding to additive sub(bi)functors the notion of **weakly exact structure** (i.e. those classes of kernel-cokernel pairs which fulfill (b) in the previous theorem) has been introduced and studied by [**29**].

Since exact structures are always closed under direct sums of short exact sequences, we will restrict to consider additive functors.

Definition 3.2. Given an exact category $(\mathcal{A}, \mathcal{S})$ and a sub(-bi)functors $F \subset \mathrm{Ext}^1_{\mathcal{S}}$. We call F **closed** if it is additive and $F(X, -)$ and $F(-, Y)$ are half exact for all objects X and Y in $\mathcal{A}$ (*here*: A functor is half exact if applied to a short exact sequence it gives a sequence which is exact in the middle).

Definition 3.3. We say an exact sequence $0 \to X \xrightarrow{i} E \xrightarrow{d} Y \to 0$ is **F-exact** if the equivalence class of (i, d) in $\mathrm{Ext}^1_{\mathcal{S}}(Y, X)$ lies in $F(Y, X)$. So $\mathcal{S}_F$ in Lemma 3.1 consists of F-exact sequences.

Then we have

THEOREM 3.4. *(**Butler-Horrock's Theorem**, [**66**, Prop.1.4]) Let $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignment $F \mapsto \mathcal{S}_F$ from Lemma 3.1 is a bijective map from*

(1) *closed sub(bi)functors of* $\mathrm{Ext}^1_{\mathcal{S}}$ *to*
(2) *exact structures $\mathcal{S}'$ on the additive category $\mathcal{A}$ with $\mathcal{S}' \subset \mathcal{S}$.*

Remark 3.5. Theorem 3.4 has been generalized to n-exangulated categories in [**95**], section 3.2. One can also assume that it was part of the inspiration to the definition of an extriangulated category.

Corollary 3.6. *If $\mathcal{A}$ is an additive category. Let $\mathcal{S}_{max}$ be its maximal exact structure. Then, the bijection of the Theorem 3.4 gives a $1-1$ correspondence between*

(1) *closed sub(bi)functors of* $\mathrm{Ext}^1_{\mathcal{S}_{max}}$ *and*
(2) *exact structures on $\mathcal{A}$.*

Continuing to ignore set-theoretic issues, we have the following:

Corollary 3.7. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category. The class of all closed sub(bi)functors of* $\mathrm{Ext}^1_{\mathcal{S}}$ *forms a* poset *with respect to inclusion of functors. It is even a lattice which is isomorphic via the bijection in Theorem 3.4 to the full sublattice of all exact structures which are contained in $\mathcal{S}$.*

Definition 3.8. Let F be an additive closed sub(bi)functor F of $\mathrm{Ext}^1_{\mathcal{S}}$. We write $\mathcal{P}(F)$ (resp. $\mathcal{I}(F)$) for the category of projectives (resp. injectives) in $(\mathcal{A}, \mathcal{S}_F)$ We will say that a closed sub(bi)functor F of $\mathrm{Ext}^1_{\mathcal{S}}$ **has enough projectives** (resp. **has enough injectives**) whenever $\mathcal{S}_F$ has. Instead of the index $\mathcal{S}_F$ we write just F, e.g. $\mathrm{Ext}^1_F := \mathrm{Ext}^1_{\mathcal{S}_F}$ etc.

Lemma 3.9. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category.*

(a) *If $F \subset \mathrm{Ext}^1_{\mathcal{S}}$ has enough projectives, then an exact sequence (i, d) is F-exact if and only if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ applied to it gives a short exact sequence in abelian groups for every $P \in \mathcal{P}(F)$.*
(b) *If $F \subset \mathrm{Ext}^1_{\mathcal{S}}$ has enough injectives, then an exact sequence (i, d) is F-exact if and only if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ applied to it gives a short exact sequence in abelian groups for every $I \in \mathcal{I}(F)$.*

PROOF. The proof of [**15**], Prop. 1.5, also works for exact categories. □

Remark 3.10. One can prove a stronger statement than the previous lemma, see [**49**], Ex. 11.10: Let $(\mathcal{A}, \mathcal{S})$ with enough projectives. Given *any two composable* morphisms (i, d), then this is an exact sequence if and only if $\mathrm{Hom}(P, -)$ applied to it gives a short exact sequence of abelian groups for all $P \in \mathcal{P}(\mathcal{S})$.

3.1. Subfunctors from subcategories.

We continue to look at an exact category $(\mathcal{A}, \mathcal{S})$. Let $\mathcal{X} \subseteq \mathcal{A}$ be a full subcategory of $\mathcal{A}$. We define two subfunctors $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ of $\mathrm{Ext}^1_{\mathcal{S}}$ for X, Z in $\mathcal{A}$

$$F_{\mathcal{X}}(Y, Z) := \{0 \to Z \to E \to Y \to 0 \text{ in } \mathrm{Ext}^1_{\mathcal{S}}(Y, Z) \mid \mathrm{Hom}_{\mathcal{A}}(X, -) \text{ exact on it for all } X \text{ in } \mathcal{X}\}$$

$$F^{\mathcal{X}}(Y, Z) := \{0 \to Z \to E \to Y \to 0 \text{ in } \mathrm{Ext}^1_{\mathcal{S}}(Y, Z) \mid \mathrm{Hom}_{\mathcal{A}}(-, X) \text{ exact on it for all } X \text{ in } \mathcal{X}\}$$

These are (the standard examples of) closed sub(bi)functors (closedness is proven in [**66**, Prop. 1.7]). The generalization of these functors to n-exangulated categories can be found in [**95**], Def. 3.16.

Definition 3.11. For two additive subcategories $\mathcal{C}$ and $\mathcal{D}$ of $\mathcal{A}$ we write $\mathcal{C} \vee \mathcal{D}$ for the smallest additive subcategory containing $\mathcal{C}$ and $\mathcal{D}$. We call this the **join** of $\mathcal{C}$ and $\mathcal{D}$.

Remark 3.12. We remark that we have the obvious inclusions: $\mathcal{X} \vee \mathcal{P}(\mathcal{S}) \subset \mathcal{P}(F_{\mathcal{X}})$ (resp. dually $\mathcal{X} \vee \mathcal{I}(\mathcal{S}) \subset \mathcal{I}(F^{\mathcal{X}})$). Furthermore, it is clear that $F_{\mathcal{X}} = F_{\mathcal{X} \vee \mathcal{P}(\mathcal{S})}$ (resp. $F^{\mathcal{X}} = F^{\mathcal{X} \vee \mathcal{I}(\mathcal{S})}$). Also, one can see easily that any sub(bi)functor F of $\mathrm{Ext}^1_{\mathcal{S}}$ is also a sub(bi)functor of $F_{\mathcal{P}(F)}$ (resp. of $F^{\mathcal{I}(F)}$) since an F-exact sequence η fulfills that $\mathrm{Hom}_{\mathcal{A}}(P, \eta)$ is exact for any $P \in \mathcal{P}(F)$.

Remark 3.13. Let $(\mathcal{A}, \mathcal{S})$ be an exact category. It is obvious that the inclusion of two additive subcategories $\mathcal{X} \subset \mathcal{X}'$ of $\mathcal{A}$ implies $F_{\mathcal{X}} \supset F_{\mathcal{X}'}$ and $F^{\mathcal{X}} \supset F^{\mathcal{X}'}$.
There are two trivial examples

(1) $\mathcal{X} = \mathcal{P}(\mathcal{S})$, in this case $\mathcal{S}_{F_{\mathcal{X}}} = \mathcal{S}$ and $\mathrm{Ext}^1_{F_{\mathcal{X}}} = \mathrm{Ext}^1_{\mathcal{S}}$. This is the unique maximal element in the poset of exact structures induced by closed sub(bi)functors of $\mathrm{Ext}^1_{\mathcal{S}}$.
(2) $\mathcal{X} = \mathcal{A}$, in this case, the exact structure is the split exact structure and $\mathrm{Ext}^1_{F_{\mathcal{A}}} = 0$. This is the unique minimal element in the lattice of all exact structures.

One can ask now: When is an exact structure $\mathcal{S}' \subset \mathcal{S}$ on an exact category $(\mathcal{A}, \mathcal{S})$ is of the form $\mathcal{S}_{\mathcal{X}}$ for an additive subcategory $\mathcal{X} \subset \mathcal{A}$?

Definition 3.14. We call a subcategory $\mathcal{X}$ of $\mathcal{A}$ **projectively saturated** (resp. **injectively saturated**) if $\mathcal{P}(\mathcal{S}_{\mathcal{X}}) = \mathcal{X}$ (resp. if $\mathcal{I}(\mathcal{S}_{\mathcal{X}}) = \mathcal{X}$). We call an exact structure $\mathcal{S}' \subset \mathcal{S}$ **projectively determined** (resp. **injectively determined**) if it is of the form $\mathcal{S}_{\mathcal{X}}$ (resp. $\mathcal{S}^{\mathcal{X}}$) for some additive subcategory $\mathcal{X} \subset \mathcal{A}$.

Lemma 3.15. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category and $\mathcal{X} \subset \mathcal{A}$ an additive category. We have the following properties*

(1) *$\mathcal{P}(\mathcal{S}_{\mathcal{X}})$ is the smallest projectively saturated subcategory that contains $\mathcal{X}$.*
(2) *If $\mathcal{S}' \subset \mathcal{S}$ is an exact structure with enough projectives, then $\mathcal{S}'$ is projectively determined.*

PROOF. (1) It is straight-forward to see that $F_{\mathcal{P}(\mathcal{S}_{\mathcal{X}})} = F_{\mathcal{X}}$ (since $\mathcal{X} \subset \mathcal{P}(\mathcal{S}_{\mathcal{X}})$ implies $F_{\mathcal{X}} \supset F_{\mathcal{P}(\mathcal{S}_{\mathcal{X}})}$ and conversely an $\mathcal{S}_{\mathcal{X}}$-exact sequence fulfills by definition of the projectives that it is $\mathcal{S}_{\mathcal{P}(\mathcal{S}_{\mathcal{X}})}$-exact). This implies that $\mathcal{P}(\mathcal{S}_{\mathcal{X}})$ is projectively saturated. If we have $\mathcal{X} \subset \mathcal{Y}$ with $\mathcal{Y}$ projectively saturated, then $\mathcal{S}_{\mathcal{X}} \supset \mathcal{S}_{\mathcal{Y}}$ and therefore $\mathcal{P}(\mathcal{S}_{\mathcal{X}}) \subset \mathcal{P}(\mathcal{S}_{\mathcal{Y}}) = \mathcal{Y}$.
(2) Follows from Lemma 3.9. □

Proposition 3.16. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignments $\mathcal{X} \mapsto \mathcal{S}_{\mathcal{X}}$ and $\mathcal{S}' \mapsto \mathcal{P}(\mathcal{S}')$ give inverse bijections between*

(1) *projectively saturated subcategories $\mathcal{X} \subset \mathcal{A}$*
(2) *projectively determined exact structures $\mathcal{S}' \subset \mathcal{S}$ on $\mathcal{A}$*

The proof is obvious. We leave the trivial dual statements to the imagination of the reader.

In [**45**], section 5 one can find an example of an exact structure on category of finite-dimensional modules over the Kronecker algebra which is not projectively determined.

3.2. Exact structures with enough projectives.

Definition 3.17. Let $\mathcal{A}$ be an additive category. We call a subcategory $\mathcal{M}$ **contravariantly** (resp. **covariantly**) **finite** in $\mathcal{A}$ if every object X in $\mathcal{A}$ admits a **right** (resp. **left**) **$\mathcal{M}$-approximation**, that is a morphism $\alpha \colon M \to X$ (resp. $\beta \colon X \to M$) with $M \in \mathcal{M}$ such that every $f \colon M' \to X$ with M' in $\mathcal{M}$ factors over α (resp. such that every $g \colon X \to M'$ factors over β). We say $\mathcal{M}$ is **functorially finite** if it is co- and contravariantly finite.

We remark that intersections of two contravariantly finite (resp. covariantly finite) subcategories do not necessarily have this property. We start with the following easy observation.

Lemma 3.18. *Let $\mathcal{A}$ be an additive category and $\mathcal{B}, \mathcal{C}$ two additive subcategories, we write $\mathcal{M} = \mathcal{B} \vee \mathcal{C}$ for their join. Then we have*

(1) *If $\mathcal{B}$ and $\mathcal{C}$ are contravariantly finite (resp. covariantly finite), then $\mathcal{M}$ too.*
(2) *If $\mathcal{M}$ is contravariantly finite (resp. covariantly finite) and $\operatorname{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{C}) = 0$, then $\mathcal{B}$ contravariantly finite (resp. $\mathcal{C}$ covariantly finite).*

Proof. (1) Let X be an object and assume we have a right $\mathcal{B}$-approximation $b_X : B_X \to X$ and a right $\mathcal{C}$-approximation $c_X : C_X \to X$. Then we get an induced morphism $m_X := (b_X, c_X) : B_X \oplus C_X \to X$. One can check that (b_X, c_X) is a right approximation for $\mathcal{B} \vee \mathcal{C}$.
(2) Let $m_X = (b_X, c_X) : M_X = B_X \oplus C_X \to X$ be a right $\mathcal{M}$-approximation and $\operatorname{Hom}(\mathcal{B}, \mathcal{C}) = 0$. Then we have $b_X : B_X \to X$ is a right $\mathcal{B}$-approximation.

□

For the later part we need to understand what it means that right approximations of a contravariantly finite subcategory are deflations. So we look at this special situation.

Lemma 3.19. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category with enough projectives (resp. enough injectives). Let $\mathcal{X}$ be a contravariantly finite (resp. covariantly finite) additive subcategory. Then the following are equivalent:*

(a) *Any right (resp. left) $\mathcal{X}$-approximation is a deflation (resp. inflation).*
(b) *$\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$ (resp. $\mathcal{I}(\mathcal{S}) \subset \mathcal{X}$)*

In particular, if $(\mathcal{A}, \mathcal{S})$ has enough projectives and $\mathcal{X}$ is contravariantly finite with $\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$, then any right $\mathcal{X}$-approximation also admits a kernel in $\mathcal{A}$.

Proof. (a) implies (b) is clear. So assume (b). Let $d : X \to Z$ be a right $\mathcal{X}$-approximation of an object Z in $\mathcal{A}$. Let $\pi : P \to Z$ be a deflation with $P \in \mathcal{P}(\mathcal{S})$. Since, by assumption, $P \in \mathcal{X}$ the map $\operatorname{Hom}_{\mathcal{A}}(P, X) \to \operatorname{Hom}_{\mathcal{A}}(P, Z)$ is surjective because d is a right approximation. Therefore, there exists a $\widetilde{\pi} : P \to X$ such that $d \circ \widetilde{\pi} = \pi$. Since π is a deflation, it follows that d is a deflation by axiom $E2$ of an exact category. □

Remark 3.20. If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A}, \mathcal{S})$ an exact category. Then $\mathcal{P}(\mathcal{S})$ is closed under direct sums and summands (cf. [49], Rem 11.5, Cor 11.6).

Theorem 3.21. *Let $\mathcal{A}$ be weakly idempotent complete additive category and $(\mathcal{A}, \mathcal{S})$ be an exact category. The assignments $\mathcal{X} \mapsto F_{\mathcal{X}} \mapsto S_{F_{\mathcal{X}}}$ gives a bijections from*

(1) *additively closed, contravariantly finite subcategories $\mathcal{X}$ of $\mathcal{A}$, closed under direct summands and whose right approximations are deflations to*
(2) *closed sub(bi)functors of $\operatorname{Ext}^1_{\mathcal{S}}$ with enough projectives and to*
(3) *exact structures $\mathcal{S}' \subset \mathcal{S}$ which have enough projectives.*

We consider the dual statement of the previous Proposition as obvious and leave it to the reader.

Proof. The bijection from (2) to (3) is clear from Theorem 3.4 and by definition of *having enough projectives*. The map from (1) to (2) is well-defined since $\mathcal{P}(F_{\mathcal{X}})$ contains by definition $\mathcal{X}$ and for any A in $\mathcal{A}$ we have a deflation $X \to A$ with $X \in \mathcal{X}$ given by the right $\mathcal{X}$-approximation. Now, the assignment $F \mapsto \mathcal{P}(F)$ goes from (2) to (1). We need to see that this is inverse to the previous map. By Lemma 3.9 we know $F = F_{\mathcal{P}(F)}$ since F has enough projectives. On the other hand, let $\mathcal{X}$ be as in (1). We clearly have $\mathcal{X} \subset \mathcal{P}(F_{\mathcal{X}})$. Let $P \in \mathcal{P}(F_{\mathcal{X}})$, we take the right $\mathcal{X}$-approximation $X \to P$ which is a deflation. By [49], Prop. 11.3, this map splits and P is a summand of X. Since $\mathcal{X}$ is closed under direct summands, we have $P \in \mathcal{X}$. □

Example 3.22. There exist projectively saturated categories which are not contravariantly finite. For example, take as $\mathcal{A}$ the category of finite-dimensional modules over the Kronecker algebra and consider the category $\mathcal{X}$ given by all the preprojective modules. This is projectively saturated but not contravariantly finite.

Given an additive category $\mathcal{A}$, we write $\mathrm{ex}^{ep}(\mathcal{A})$ (resp. $\mathrm{ex}_{ei}(\mathcal{A})$, resp. $\mathrm{ex}_{ei}^{ep}(\mathcal{A})$) for the subposet of all exact structures which have enough projectives (resp. enough injectives, resp. both).
For an interval J in the poset $\mathrm{ex}(\mathcal{A})$ we write $\mathrm{ex}^{ep}(\mathcal{A})_J$ (resp. $\mathrm{ex}_{ei}(\mathcal{A})_J$, resp. $\mathrm{ex}_{ei}^{ep}(\mathcal{A})_J$) for the intersection of these (respective) posets with the interval J.

Corollary 3.23. *Let $(\mathcal{A}, \mathcal{S})$ be an exact category with enough projectives. Let $i\colon \mathcal{B} \to \mathcal{A}$ be an inclusion of a full additively closed, contravariantly finite subcategory which contains $\mathcal{P}(\mathcal{S})$. Then, $\mathcal{S}_{\mathcal{X}} \mapsto \mathcal{S}_{i(\mathcal{X})}$ is an isomorphism of posets*

$$\mathrm{ex}^{ep}(\mathcal{B}, \mathcal{S}^{\leq \mathcal{S} \cap \mathcal{B}}) \to \mathrm{ex}^{ep}(\mathcal{A})_{\mathcal{S}_{i(\mathcal{B})} \leq * \leq \mathcal{S}}$$

Proof. The inclusion functor $i\colon \mathcal{B} \to \mathcal{A}$ gives a natural bijection between

(1) contravariantly finite, additive, summand-closed subcategories $\mathcal{X}$ of $\mathcal{B}$ with $\mathcal{P}(\mathcal{S}) \subset \mathcal{X}$.
(2) contravariantly finite, additive, summand-closed subcategories $\mathcal{X}$ of $\mathcal{A}$ with $\mathcal{P}(\mathcal{S}) \subset \mathcal{X} \subset i(\mathcal{B})$.

The rest of the claim follows from Prop.3.21 and Lem.3.19. □

Example 3.24. Let Λ be an artin algebra and $\mathcal{A} = \Lambda$-mod be the category of finitely generated left Λ-modules. Let C be a cotilting Λ-module (i.e. $\mathrm{id}\, C < \infty$, $\mathrm{Ext}^{>0}(C, C) = 0$ and there is an exact sequence $0 \to \mathrm{D}\,\Lambda \to C_0 \to \cdots \to C_r \to 0$ with $C_i \in \mathrm{add}(C)$). Then $\mathcal{B} = {}^{\perp}C := \bigcap_{i \geq 1} \ker \mathrm{Ext}^i(-, C)$ is full, extension-closed, summand-closed, contravariantly finite subcategory which contains Λ.

3.3. A classical situation. Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between exact categories $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$. Then we have maps natural in X and Y

$$\varphi_{X,Y}\colon \mathrm{Ext}^1_{\mathcal{S}}(X, Y) \to \mathrm{Ext}^1_{\mathcal{T}}(\varphi(X), \varphi(Y)).$$

This gives an additive sub(bi)functor $F := \ker \varphi_{*,*} \subset \mathrm{Ext}^1_{\mathcal{S}}$. It is closed by [66], Prop. 1.10. The F-exact sequences are the exact sequences in $(\mathcal{A}, \mathcal{S})$ which are split exact once we apply the functor φ.

Remark 3.25. If λ is a left adjoint functor to φ, then the counit $\lambda\varphi(X) \to X$ for an object X in $\mathcal{A}$ provides a right $\lambda(\mathcal{B})$-approximation of X. In particular, $\lambda(\mathcal{B})$ is contravariantly finite in $\mathcal{A}$.

Lemma 3.26. *If the functor φ has a left adjoint λ then*

(1) $F = F_{\lambda(\mathcal{B})} = F_{\lambda(\mathcal{B}) \vee \mathcal{P}(\mathcal{S})}$.
(2) *If all counits $\lambda\varphi(X) \to X$ are deflations in $(\mathcal{A}, \mathcal{S})$, then F has enough projectives and furthermore, $\mathcal{P}(F)$ consists of all direct summands of objects in $\lambda(\mathcal{B})$.*
(3) *If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A}, \mathcal{S})$ has enough projectives, then F has enough projectives and $\mathcal{P}(F)$ consists of direct summands of $\lambda(\mathcal{B}) \vee \mathcal{P}(\mathcal{S})$.*

Dually, if the functor φ has a right adjoint ρ then

(1') $F = F^{\rho(\mathcal{B})}$
(2') *If all units $X \to \rho\varphi(X)$ are inflations in $(\mathcal{A}, \mathcal{S})$, then F has enough injectives, and furthermore, $\mathcal{I}(F)$ consists of all direct summands of objects in $\rho(\mathcal{B})$.*

(3') *If $\mathcal{A}$ is weakly idempotent complete and $(\mathcal{A},\mathcal{S})$ has enough injectives, then F has enough injectives and $\mathcal{I}(F)$ consists of direct summands of $\rho(\mathcal{B})\vee\mathcal{I}(\mathcal{S})$.*

Proof. (1) Let η be an exact sequence in $(\mathcal{A},\mathcal{S})$. We have the adjunction property $\mathrm{Hom}_{\mathcal{A}}(\lambda(V),W)\cong\mathrm{Hom}_{\mathcal{B}}(V,\varphi(W))$ for all V in $\mathcal{B}$ and W in $\mathcal{A}$. Therefore, exactness of $\mathrm{Hom}_{\mathcal{A}}(\lambda(V),\eta)\cong\mathrm{Hom}_{\mathcal{B}}(V,\varphi(\eta))$ for all V in $\mathcal{B}$ means that $\varphi(\eta)$ is $F_{\mathcal{B}}$-exact is. But $F_{\mathcal{B}}$-exactness is the same as split exactness.

(2) By (1) we have $\lambda\mathcal{B}\subset\mathcal{P}(F)$. Let X be in $\mathcal{A}$. By assumption, the counit $\lambda\varphi(X)\to X$ is a deflation in $(\mathcal{A},\mathcal{S})$ and as just observes $\lambda\varphi(X)\in\mathcal{P}(F)$. We want to see, that this map is split when we apply the functor φ. But by the second triangle identity of the adjunction, we have that $\varphi\lambda\varphi(X)\to\varphi(X)$ has a section and therefore it splits (true?). Now given any $P\in\mathcal{P}(F)$, we have just constructed an F-epimorphism $\lambda\varphi(P)\to P$ and so this map has to be split, i.e. P is a summand of an object in $\lambda\mathcal{B}$. Since $\lambda\mathcal{B}\subset\mathcal{P}(F)$ by (1) and $\mathcal{P}(F)$ is closed under summands, equality follows.

(3) Since $\lambda(\mathcal{B})$ is contravariantly finite (cf. Rem. 3.25) we have that $\mathrm{add}(\lambda(\mathcal{B}))$ is contravariantly finite. By Lemma 3.18 we also have $\mathrm{add}(\lambda(\mathcal{B}))\vee\mathcal{P}(\mathcal{S})$ is contravariantly finite. Therefore, the claim follows from Theorem 3.21 and Lemma 3.19.

The dual statement can be proven analogously. □

Example 3.27. Let $f\colon B\to A$ a ring homomorphism and $\varphi\colon$ A-Mod $\to$ B-Mod, $X\mapsto {}_BX$ the functor given by restriction of scalars along f.
Then, there is a left adjoint given by the following tensor functor $\lambda(X):=A\otimes_B X$ called the **induced module** and a right adjoint given by the following Hom-functor $\rho(X):=\mathrm{Hom}_B(A,X)$ called the **co-induced module**. The counits $\lambda\varphi(X)=A\otimes_B X\to X$ are epimorphisms since their restrictions of scalars are surjective maps, this follows from the triangle identity. The units $X\to\mathrm{Hom}_B(A,{}_BX)$ are monomorphisms since their restrictions of scalars are injective maps by the triangle identity. Therefore, by the previous lemma we have for $F=\ker\varphi_{*,*}$ the following

(1) $F=F_{A\otimes_B\text{B-Mod}}=F^{\mathrm{Hom}_B(A,\text{B-Mod})}$
(2) F has enough projectives and enough injectives. The F-projectives are the direct summands of $A\otimes_B$ B-Mod, the F-injectives are the direct summands of $\mathrm{Hom}_B(A,\text{B-Mod})$.

This exact structure on A-Mod has been introduced by Hochschild in [**97**] in 1956. In loc. cit. this has been used to define relative Hochschild homology, a Tor and Ext functor have been defined for this setup. A very nice application of the classical situation is the finite representation type classification for group algebras, cf. [**26**], chapter III, section 3. A recent application to Han's conjecture can be found in [**56**].

Example 3.28. Let Γ be a ring and $e\in\Gamma$ an idempotent, we define $\Lambda:=e\Gamma e$. Then, the restriction functor $e\colon\Gamma\text{-Mod}\to\Lambda\text{-Mod}, X\mapsto eX$ has a left adjoint $\ell=\Gamma e\otimes_\Lambda(-)$ and right adjoint $r=\mathrm{Hom}_\Lambda(e\Gamma,-)$. Therefore, we have for $F=\ker e_{*,*}$ the following description (numbered by the parts of the lemma 3.26 that are used)

(1) $F=F_{\Gamma e\otimes_\Lambda\Lambda\text{-Mod}}=F^{\mathrm{Hom}_\Lambda(e\Gamma,\Lambda\text{-Mod})}$.
(3) Since Γ-Mod is abelian, it is weakly idempotent complete. It has enough projectives and enough injectives. So, it follows that F has enough projectives and enough injectives. We have $\mathcal{P}(F)$ consists of direct summands of $(\Gamma e\otimes_\Lambda\Lambda\text{-Mod})\vee\mathrm{Add}(\Gamma)$ and $\mathcal{I}(F)$ consists of direct summands of $\mathrm{Hom}_\Lambda(e\Gamma,\Lambda\text{-Mod})\vee\mathcal{I}(\Gamma\text{-Mod})$.
If we take a noetherian ring Γ and consider the abelian Γ-mod category given by finitely generated Γ-modules, then this category has not in general enough injectives but it has enough projectives given by $\mathrm{add}(\Gamma)$. Assume that $\Lambda=e\Gamma e$ is again noetherian, then the restriction functor $e\colon\Gamma\text{-mod}\to\Lambda\text{-mod}$ has a well-defined left adjoint functor $\ell=\Gamma e\otimes_\Lambda(-)$. We conclude that in this case F has enough projectives given by the direct summands of $(\Gamma e\otimes_\Lambda\Lambda\text{-Mod})\vee\mathrm{add}(\Gamma)$.

4. The representation-finite case - Enomoto's result

Definition 4.1. For a ring Γ, we denote by $\mathrm{proj}(\Gamma)$ the category of finitely generated projective left Γ-modules. We say an idempotent complete additive category $\mathcal{A}$ is **representation-finite** if it is equivalent to $\mathrm{proj}(\Gamma)$ for some ring Γ.

By definition, $\mathcal{A} = \mathrm{proj}(\Gamma)$ is Krull-Schmidt if and only if Γ is semi-perfect.

We recall Enomoto's results (without proof).

Lemma 4.2. *Let $\mathcal{A} = \mathrm{proj}(\Gamma)$ a Krull-Schmidt category and $\mathcal{E}$ an exact structure on $\mathcal{E}$. Then there exists an idempotent $e \in \Gamma$ such that $\mathcal{P}(\mathcal{E}) = \mathrm{add}(\Gamma e)$. Then $\mathcal{E}$ has enough projectives if and only if $\Gamma/\Gamma e\Gamma$ is a finite length left Γ-module.*

Assume additionally that we have a commutative artinian ring R and Γ is a finitely generated R-algebra with $R \subset Z(\Gamma)$. This is saying that $\mathcal{A} = \mathrm{proj}(\Gamma)$ is Hom-noetherian R-linear. Then every exact structure on $\mathcal{A} = \mathrm{proj}(\Gamma)$ has enough projectives and enough injectives.

Definition 4.3. Let $\mathcal{E}$ be a an exact category and $\mathcal{M}$ a full subcategory. We say $\mathcal{M}$ is a **generator** if $\mathcal{M}$ is additively closed and for every X in $\mathcal{E}$ there exists a short exact sequence $Y \to M \to X$ with M in $\mathcal{M}$. A **cogenerator** is a generator in $\mathcal{E}^{op}$.
If $\mathcal{A}$ is an additive category and $\mathcal{M}$ a full subcategory, then we call $\mathcal{M}$ a generator (resp. cogenerator) if it is one in the maximal exact structure on $\mathcal{A}$.

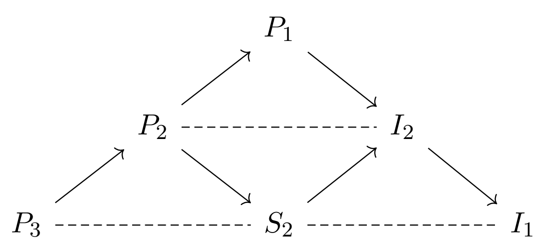
Then

THEOREM 4.4. *(Enomoto's Theorem) Let R be a commutative artinian ring. Let $\mathcal{A}$ idempotent complete, representation-finite, Hom-noetherian R-linear. Let $\mathcal{P} = \mathrm{add}(\Gamma e)$ be the projectives in the maximal exact structure on $\mathcal{A}$. Then generators in $\mathcal{A}$ are given by the Boolean lattice of all additively closed subcategories containing $\mathcal{P}$, we denote it by* $\mathrm{Generators}(\mathcal{A})$. *Then*

$$\begin{aligned} \mathrm{ex}(\mathcal{A}) &\to \mathrm{Generators}(\mathcal{A}), \\ \mathcal{E} &\mapsto \mathcal{P}(\mathcal{E}) \end{aligned}$$

is an isomorphism of lattices.

Example 4.5. We look at the quiver $1 \to 2 \to 3$ and at its Auslander-Reiten quiver



To see the generators, fix the projectives P_1, P_2, P_3 and add any subset of $\{S_2, I_2, I_1\}$. So this is just the power set of this set with three elements. More interesting is to observe that we have seven hereditary exact substructures and one exact substructure of global dimension 2, corresponding to the generator $P_1 \oplus P_2 \oplus P_3 \oplus I_2$.

More generally for type $\mathbb{A}_n$-equioriented quivers the maximal global dimension is $n-1$, cf...

Remark 4.6. We think that substructures of finite-dimensional Dynkin quiver representations are always of finite global dimension but we have not worked this out (except for type $\mathbb{A}$-equioriented). This should be mainly due to the Auslander-Reiten quiver has no oriented cycles. Then endomorphism rings of modules can be realized as upper triangular rings and should admit a quasi-hereditary structure which implies that they have finite global dimension.

If we are looking at representation-finite finite-dimensional algebras of global dimension 2, then we can already find examples of exact substructures of infinite global dimension.

Example 4.7. Let Γ be the Auslander algebra of $K[X]/(X^3)$. This has global dimension 2 and is representation-finite. Then look at the Frobenius exact substructure described in [**12**] on the Auslander algebra of a self-injective algebra.

5. Parametrization using the Ziegler spectrum - Schlegel's result

We refer to Appendix C for Ind-completion of additive and exact categories. For an essentially small additive category $\mathcal{C}$ we call $\overrightarrow{\mathcal{C}} =: \mathcal{A}$ its Ind-completion. Let $\mathcal{E}$ be an exact structure and $\overrightarrow{\mathcal{E}}$ be its ind-completion and $i\colon \mathcal{E} \to \overrightarrow{\mathcal{E}}$ the Yoneda embedding. We call X in $\overrightarrow{\mathcal{E}}$ is fp-$\overrightarrow{\mathcal{E}}$-injective if $\mathrm{Ext}^1_{\overrightarrow{\mathcal{E}}}(i(E), X) = 0$ for all E in $\mathcal{E}$.

Definition 5.1. An object in an additive category M is called **indecomposable** if $M = N \oplus L$ implies $N = 0$ or $L = 0$.
Let $\mathcal{A} = \overrightarrow{\mathcal{C}}$ be a locally finitely presented additive category. We define the **pure exact structure** to be $\mathcal{A}_p := \overrightarrow{\mathcal{C}_{\mathbf{split}}}$ to be the ind-completion of the split exact structure. An object in $\mathcal{A}$ is called **pure injective** if it is an injective in $\mathcal{A}_p$. Then we define the following class

$$\mathrm{ZSp}(\mathcal{A}) := \{M \in \mathcal{A} \mid M \text{ indecomposable and pure injective}\}$$

Remark 5.2. For general locally finitely presented additive categories we do not know of $\mathrm{ZSp}(\mathcal{A})$ is a set or if it is one if it is non-empty.

Definition 5.3. Let $\mathcal{C}$ be an essentially small additive category, $\mathcal{A} = \overrightarrow{\mathcal{C}}$ and S a class of morphisms in $\mathcal{C}$. Let $\mathcal{X}(S)$ be the full subcategory of $\mathcal{A}$ of all objects I with the following property: For any map $s\colon M \to M'$ in S and any map $f\colon M \to I$ there exists $f'\colon M' \to I$ such that $f's = f$. Alternatively, one can descibe this as

$$\mathcal{X}(\mathcal{S}) = \{I \in \mathcal{A} \mid \operatorname{coker} \mathrm{Hom}_{\mathcal{A}}(s, I) = 0 \quad \forall s \in S\}$$

A full subcategory $\mathcal{X}$ of $\overrightarrow{\mathcal{A}}$ is called **definable** if there exists a class of morphisms S in $\mathcal{A}$ such that $\mathcal{X} = \mathcal{X}(S)$.
Assume that $\mathrm{ZSp}(\mathcal{A})$ is a set, then a subset $\mathcal{U} \subseteq \mathrm{ZSp}(\mathcal{A})$ is called **Ziegler-closed** if there exists a definable subcategory $\mathcal{X}$ such that $\mathcal{U} = \mathrm{ZSp}(\mathcal{A}) \cap \mathcal{X}$.

From now on, we impose the condition that $\mathcal{C}$ has weak cokernels, this means that for every morphism $f\colon X \to Y$ in $\mathcal{C}$ there exists a morphism $g\colon Y \to Z$ such that the following sequence is exact (in the middle) in the abelian category $\operatorname{Mod} \mathcal{C}^{op}$ (all covariant, additive functors $\mathcal{C} \to (Ab)$)

$$\mathrm{Hom}_{\mathcal{C}}(Z, -) \xrightarrow{\mathrm{Hom}(g,-)} \mathrm{Hom}_{\mathcal{C}}(Y, -) \xrightarrow{\mathrm{Hom}(f,-)} \mathrm{Hom}_{\mathcal{C}}(X, -)$$

This condition is equivalent to $\mathrm{mod}_1 \mathcal{C}^{op}$ is abelian, in which case it also has enough projectives. This has been used in [**59**] and [**173**] to embed $\mathcal{A} = \overrightarrow{\mathcal{C}}$ in a locally coherent abelian category called the *purity category*. We are not going to explain this construction here, as we hope that these results can be generalized (without using this embedding).

Theorem 5.4. *(combine* [**59**, section (3.5), Lem 1] *with* [**126**, Lem. 12.1.12]*) If $\mathcal{C}$ has weak cokernels then* $\mathrm{ZSp}(\mathcal{A})$ *is a set and we have a topology on* $\mathrm{ZSp}(\mathcal{A})$ *with closed sets given by Ziegler-closed subsets. This topological space is called the* ***Ziegler spectrum*** *of $\mathcal{A}$.*

Lemma 5.5. *(*[**173**]*, proof of Lem. 2.9) Let $\mathcal{C}$ be essentially small, idempotent complete with weak cokernels and $\mathcal{E}$ an exact structure on it. Let $\overrightarrow{\mathcal{E}}$ be its Ind-completion.*

(1) *Let $\mathcal{X}_{\mathcal{E}}$ be the full subcategory fp-$\overrightarrow{\mathcal{E}}$-injectives, then this is a definable subcategory since it can be written as*

$$\mathcal{X}_{\mathcal{E}} = \mathcal{X}(\mathrm{Infl}_{\mathcal{E}})$$

where $\mathrm{Infl}_{\mathcal{E}}$ denotes the $\mathcal{E}$-inflations.

(2) *Let $\mathcal{U}_{\mathcal{E}}$ be the set of indecomposable injectives in $\overrightarrow{\mathcal{E}}$. Then $\mathcal{U}_{\mathcal{E}}$ is Ziegler-closed because*

$$\mathcal{U}_{\mathcal{E}} = \mathrm{ZSp}(\mathcal{A}) \cap \mathcal{X}_{\mathcal{E}}$$

THEOREM 5.6. *([**173**], Thm B) Let $\mathcal{C}$ be an idempotent complete essentially small additive category with weak cokernels and $\mathcal{E}_{\max}$ its maximal exact structure. We write $\mathcal{X}_{max} := \mathcal{X}_{\mathcal{E}_{max}}, \mathcal{U}_{max} := \mathcal{U}_{\mathcal{E}_{max}}$. Then the assigment $\mathcal{E} \mapsto \mathcal{X}_{\mathcal{E}}$ and resp. $\mathcal{E} \mapsto \mathcal{U}_{\mathcal{E}}$ gives a an lattice isomorphism between $\mathrm{ex}(\mathcal{A})$ and*

(1) *the lattice of definable subcategories which contain $\mathcal{X}_{max}$ and resp.*
(2) *the lattice of Ziegler-closed subsets which contain $\mathcal{U}_{max}$.*

We want to understand the exact substructures in cases where the Ziegler spectrum is known. For this we need the following:

Proposition 5.7. *In the situation of the previous theorem, we have a map*

$$\{\mathcal{U} \text{ Ziegler-closed }, \mathcal{U}_{max} \subseteq \mathcal{U}\} \to \mathrm{ex}(\mathcal{A})$$
$$\mathcal{U} \mapsto (\mathcal{E}_{max})^{\mathcal{U}}$$

where $\mathcal{E}_{max}^{\mathcal{U}}$ consists of all $\mathcal{E}_{max}$-short exact sequences σ such that $\mathrm{Hom}_{\mathcal{A}}(\sigma, U)$ is exact for all $U \in \mathcal{U}$. This map is the inverse to the bijective map $\mathrm{ex}(\mathcal{A}) \to \{\mathcal{U} \text{ Ziegler closed }, \mathcal{U}_{max} \subseteq \mathcal{U}\}$ in Theorem 5.6.

We need the following easy lemma for the proof.

Lemma 5.8. *Let $\mathcal{C}$ be an idempotent complete essentially small category and $\mathcal{A} = \overrightarrow{\mathcal{C}}$ its Ind-completion. Assume we have an exact structure $\mathcal{E}$ on $\mathcal{C}$ and $\mathcal{U}$ some set of pure injective objects in $\mathcal{A}$. We denote $\mathcal{E}^{\mathcal{U}}$ the exact substructure of $\mathcal{E}$ consisting of $\mathcal{E}$-exact sequences σ such that $\mathrm{Hom}_{\mathcal{A}}(\sigma, U)$ exact for all $U \in \mathcal{U}$.*
Then all objects in $\mathcal{U}$ are $\mathrm{fp}(\overrightarrow{\mathcal{E}^{\mathcal{U}}})$-injectives.

PROOF. (of Lemma 5.8) Let U be in $\mathcal{U}$ and X be an $\mathcal{C}$ and we take a $\overrightarrow{\mathcal{E}^{\mathcal{U}}}$-short exact sequence

$$\sigma \colon U \rightarrowtail Y \twoheadrightarrow X$$

We need to see it splits. We write $\sigma = \operatorname{colim} \sigma_i$ as a filtered colimit of $\mathcal{E}^{\mathcal{U}}$-short exact sequences $U_i \rightarrowtail Y_i \twoheadrightarrow X_i$. Now, we factorize the canonical morphisms $\sigma_i \to \sigma$, $i \in I$ of short exact sequences following [**49**]..

$$\begin{array}{ccccccc}
\sigma_i & & U_i & \rightarrowtail & Y_i & \twoheadrightarrow & X_i \\
\downarrow & & \downarrow & & \downarrow & & \| \\
\eta_i & & U & \rightarrowtail & Z_i & \twoheadrightarrow & X_i \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\sigma & & U & \rightarrowtail & Y & \twoheadrightarrow & X
\end{array}$$

This means η_i is the push-out of σ_i along the canonical morphism $U_i \to U$. As $\mathrm{Hom}_{\mathcal{A}}(\sigma_i, U)$ is exact, it follows that η_i is split exact. Now, it is a straight forward observation to see that we have $\operatorname{colim}_I \eta_i = \sigma$. As η_i are split exact they are also pure exact sequences. Now, filtered colimits of pure exact sequences are again pure exact as the pure exact structure is a locally coherent exact structure (cf. Appendix..). In particular σ is pure exact and U is pure injective, it splits. $\square$

Let us come back to:

PROOF. (of Prop. 5.7) Let $\mathcal{E}$ be an exact structure on $\mathcal{C}$ and we set $\mathcal{U} := \mathcal{U}_{\mathcal{E}}$. As $\mathcal{E}$ is fully exact in $\overrightarrow{\mathcal{E}}$ we have that $\mathcal{E} \leq \mathcal{F} := \mathcal{E}_{max}^{\mathcal{U}}$ is an exact substructure. To see that they are equal, it is enough to see that $\mathcal{U} = \mathcal{U}_{\mathcal{F}}$. As $\mathcal{E} \leq \mathcal{F}$ we have that $\overrightarrow{\mathcal{E}} \leq \overrightarrow{\mathcal{F}}$ this implies that the subcategory of injectives fulfill $\mathcal{I}(\overrightarrow{\mathcal{E}}) \supseteq \mathcal{I}(\overrightarrow{\mathcal{F}})$ and therefore $\mathcal{U} \supseteq \mathcal{U}_{\mathcal{F}}$. Now, for the other inclusion we conclude from Lemma 5.8 that $\mathcal{U} \subseteq \mathcal{X}_{\mathcal{F}}$. This implies $\mathcal{U} \subseteq \mathcal{X}_{\mathcal{F}} \cap \mathrm{ZSp}(\mathcal{A}) = \mathcal{U}_{\mathcal{F}}$ by Lemma 5.5. $\square$

Let us also note the following corollary.

Corollary 5.9. *Let $\mathcal{C}$ be an idempotent complete essentially small category and $\mathcal{A} = \overrightarrow{\mathcal{C}}$ its Ind-completion. Assume we have an exact structure $\mathcal{E}$ on $\mathcal{C}$ and $\mathcal{U}$ some set of pure injective objects in $\mathcal{A}$. Then*

$$\mathcal{E}^{\mathcal{U}} = \mathcal{E}^{\overline{\mathcal{U}}}$$

where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ in the Ziegler spectrum.

Proof. By definition $\mathcal{E}^{\overline{\mathcal{U}}}$ is an exact substructure of $\mathcal{E}^{\mathcal{U}}$. This implies that $\mathcal{U} \subseteq \mathcal{X}_{\mathcal{E}^{\mathcal{U}}} \subseteq \mathcal{X}_{\mathcal{E}^{\overline{\mathcal{U}}}}$. This implies $\mathcal{U} \subseteq \mathcal{U}_{\mathcal{E}^{\mathcal{U}}} \subseteq \mathcal{U}_{\mathcal{E}^{\overline{\mathcal{U}}}} = \overline{\mathcal{U}}$ but as $\mathcal{U}_{\mathcal{E}^{\mathcal{U}}}$ is Ziegler-closed, it has to be equal to $\overline{\mathcal{U}}$. Since we have a bijection it follows that $\mathcal{E}^{\overline{\mathcal{U}}} = \mathcal{E}^{\mathcal{U}}$. □

Let Λ be a ring, then we define the (left) Ziegler spectrum of Λ as $\mathrm{Zg}_\Lambda := \mathrm{ZSp}(\Lambda\,\mathrm{Mod})$.

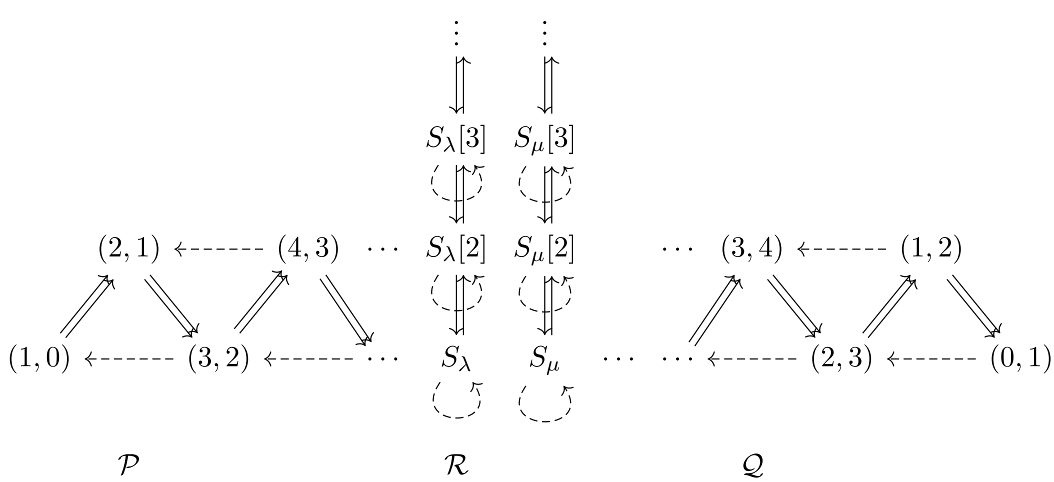
5.1. Examples.

Example 5.10. As a consequence of [**154**, Cor. 5.3.36, Cor. 5.3.37, Thm 5.1.12] one obtains: For a finite-dimensional algebra Λ the following are equivalent

(1) Λ is of finite representation-type
(2) Zg_Λ is a finite set
(3) Zg_Λ does not contain any infinite-dimensional modules.

In this case, Zg_Λ is a discrete topological space, $\mathcal{U}_{max}$ consists of the indecomposable injectives in $\Lambda\,\mathrm{mod}$. So, Ziegler-closed subsets containing $\mathcal{U}_{max}$ are in bijection with basic cogenerators in $\Lambda\,\mathrm{mod}$. This is easily seen to be an equivalent description to Enomoto's theorem in this case.

Example 5.11. Ziegler spectrum in tame hereditary case has been described by Ringel in [**160**], we just look here at the easiest case:
We define Q to be the Kronecker quiver $1 \leftleftarrows 2$ and $\Lambda = KQ$ for some field K. Its Auslander-Reiten quiver (see picture below) has as vertices the indecomposables in $\Lambda\,\mathrm{mod}$, they are divided in three types 1) $\mathcal{P}$ preprojectives (in the τ^--orbit of the projectives), they are denoted by their dimension vector $(n+1, n)$, 2) $\mathcal{R}$ regulars, they are determined by the regular simple which they contain and their dimension vector, the regular simples are denoted by $S_\lambda, \lambda \in K \cup \{\infty\} =: \Omega$, 3) $\mathcal{Q}$ preinjectives (in the τ-orbit of the injectives), they are denoted by their dimension vector $(n, n+1)$.



The arrows between the vertices indicate irreducible maps between the indecomposables and the dotted arrow the Auslander-Reiten translate, for every dotted arrow there is an almost split sequence. For more details look into [**159**].

Then Zg_Λ consists of the following points

(1) indecomposables in $\Lambda\,\mathrm{mod}$
(2) For every $\lambda \in \Omega$ a Prüfer module $S_\lambda[\infty]$, which is the filtered colimit (union) over $S_\lambda \rightarrowtail S_\lambda[2] \rightarrowtail S_\lambda[3] \rightarrowtail \cdots$
(3) For every $\lambda \in \Omega$ an adic module $\hat{S_\lambda}$, which is the limit over $\cdots \twoheadrightarrow S_\lambda[3] \twoheadrightarrow S_\lambda[2] \twoheadrightarrow S_\lambda$
(4) The generic module G, it is characterized by being an indecomposable module with $\mathrm{Hom}_\Lambda(G, S_\lambda) = 0 = \mathrm{Hom}_\Lambda(S_\lambda, G)$ for all $\lambda \in \Omega$

Now, $\mathcal{U}_{max} = \{(1,2),(0,1)\}$ only consists of the two indecomposable injectives in $\Lambda\,\mathrm{mod}$. Given a subset $\mathcal{U} \subseteq \mathrm{Zg}_\Lambda$ containing $\mathcal{U}_{max}$ we find $T, M \subseteq \Omega$

$$\mathcal{U}^{fin} := \{U \in \mathcal{U} \mid U \in \Lambda\,\mathrm{mod}, U \notin \mathcal{U}_{max}\}$$
$$\mathcal{U}_{T,M} := \mathcal{U}_{max} \cup \{S_t[\infty] \mid t \in T\} \cup \{\hat{S_m} \mid m \in M\} \cup \{G\}$$

such that $\mathcal{U} = \mathcal{U}_{max} \cup \mathcal{U}^{fin} \cup \mathcal{U}_{T,M}$ or $\mathcal{U} = \mathcal{U}_{max} \cup \mathcal{U}^{fin} \cup \mathcal{U}_{T,M} \setminus \{G\}$. Following Ringel's characterization in [**160**] we find that $\mathcal{U}$ is Ziegler-closed iff

(a) $\mathcal{U}^{fin}$ finite, then it and T, M can be arbitily chosen (also empty is allowed) and $\mathcal{U} = \mathcal{U}_{\max} \cup \mathcal{U}^{fin} \cup \mathcal{U}_{T,M}$ or if $T = M = \emptyset$ we can also have $\mathcal{U} = \mathcal{U}_{\max} \cup \mathcal{U}^{fin}$.
(b) $\mathcal{U}^{fin}$ infinite, then always $G \in \mathcal{U}$ but T, M must satisfy the following.
    (c1) If $\mathcal{U}^{fin} \cap \mathcal{P}$ is infinite, then $M = \Omega$ (all adics in)
    (c2) If $\mathcal{U}^{fin} \cap \mathcal{Q}$ is infinite, then $T = \Omega$ (all Prüfer in)
    (c3) For every $\lambda \in \Omega$, if $\mathcal{U}^{fin} \cap \{S_\lambda[n] \mid n \in \mathbb{N}\}$ is infinite, then $\lambda \in T \cap M$

Before we start we need to understand some properties of the functors $\mathrm{Hom}_\Lambda(-, U)$ and $\mathrm{Ext}^1_\Lambda(-, U)$ for points of type (2,),(3),(4). In [**161**, p. 46] and [**60**, section 3] we found the following vanishing where we set $S = S_\lambda$ and denote by $\mathcal{R}_\lambda$ be (the tube of) all regular modules with S as a submodule.

$$\mathrm{Hom}_\Lambda(\mathcal{R}, G) = \mathrm{Hom}_\Lambda(\mathcal{Q}, G) = 0 = \mathrm{Ext}^1_\Lambda(\mathcal{R}, G) = \mathrm{Ext}^1_\Lambda(\mathcal{P}, G)$$
$$\mathrm{Hom}_\Lambda(\mathcal{R}, \hat{S}) = \mathrm{Hom}_\Lambda(\mathcal{Q}, \hat{S}) = 0 = \mathrm{Ext}^1_\Lambda(\mathcal{P}, \hat{S}) = \mathrm{Ext}^1_\Lambda(\mathcal{R}_\mu, \hat{S}) \quad \mu \neq \lambda$$
$$\mathrm{Hom}_\Lambda(\mathcal{R}_\mu, S[\infty]) = \mathrm{Hom}_\Lambda(\mathcal{Q}, S[\infty]) = 0 = \mathrm{Ext}^1_\Lambda(\mathcal{R}, S[\infty]) = \mathrm{Ext}^1_\Lambda(\mathcal{P}, S[\infty]) \quad \mu \neq \lambda$$

As a consequence we see: If $U \in \{G, \hat{S}, S[\infty]\}$ and σ a short exact sequence in $\Lambda\,\mathrm{mod}$ with all three terms in either $\mathrm{add}(\mathcal{P})$, $\mathrm{add}(\mathcal{R})$ or $\mathrm{add}(\mathcal{Q})$, then $\mathrm{Hom}(\sigma, U)$ is exact.
As there are very many Ziegler-closed sets in this case, we focus on two types:

(I) Either $\mathcal{U}^{fin} = \emptyset$, these give exact structures containing all almost split sequences.
(II) $\mathcal{U} = \overline{\mathcal{U}^{fin}}$, these are so-called Auslander-Solberg exact structures. Here, this is still an Auslander-Reiten category, the almost split sequences are precisely the ones of $\Lambda\,\mathrm{mod}$ not starting in $\mathcal{U}^{fin}$.

Now, we look at the exact structures in these cases:

(I) For $\mathcal{U} = \overline{\{U\}}$, we set $\mathrm{Ext}^1_U := \mathrm{Ext}^1_{\mathcal{E}\mathcal{U}}$.
    We start with the unique maximal not abelian exact structure in this case $\mathcal{U} = \{G\}$, then $\mathrm{Ext}^1_G(\mathcal{X}, \mathcal{Y}) = \mathrm{Ext}^1_\Lambda(\mathcal{X}, \mathcal{Y})$ for all $(\mathcal{X}, \mathcal{Y}) \in \{\mathcal{P}, \mathcal{R}, \mathcal{Q}\}^2 \setminus \{(\mathcal{Q}, \mathcal{P})\}$ and $\mathrm{Ext}^1_G(\mathcal{Q}, \mathcal{P}) = 0$ (for this we leave the proof out). The interesting thing is that this is an exact substructure of global dimension ≥ 2 (probably $= 2$), since the following exact sequence σ is not zero in $\mathrm{Ext}^2_G(\mathcal{Q}, \mathcal{P})$: Let R be a regular module, take a projective Λ-module resolution and an injective Λ-module resolution of R and concatenate to an exact sequence σ

$$P_1 \rightarrowtail P_0 \to I_0 \twoheadrightarrow I_1$$

Observe, this implies for all not abelian exact structures of type (I) that $\mathrm{Ext}^1(\mathcal{Q}, \mathcal{P}) = 0$.

Next, we consider $\mathcal{U} = \overline{\{\hat{S}\}}$, we have $\operatorname{Ext}^1_{\hat{S}}(\mathcal{X}, \mathcal{Y}) = \operatorname{Ext}^1_G(\mathcal{X}, \mathcal{Y})$ for all $(\mathcal{X}, \mathcal{Y}) \neq (\mathcal{R}_\lambda, \mathcal{P})$ and $\operatorname{Ext}^1_{\hat{S}}(\mathcal{R}_\lambda, \mathcal{P}) = 0$.

Now, we consider $\mathcal{U} = \overline{\{S[\infty]\}}$. Then $\operatorname{Ext}^1_{S[\infty]}(\mathcal{Q}, \mathcal{R}_\lambda) = 0$ and $\operatorname{Ext}^1_{S[\infty]}(\mathcal{X}, \mathcal{Y}) = \operatorname{Ext}^1_G(\mathcal{X}, \mathcal{Y})$ for all $(\mathcal{X}, \mathcal{Y}) \in \{\mathcal{P}, \mathcal{Q}, \mathcal{R}_\mu, \mu \in \Omega\}^2 \setminus \{(\mathcal{P}, \mathcal{R}_\lambda)\}$.

In both cases $\mathcal{U} = \overline{\{\hat{S}\}}$ or $\mathcal{U} = \overline{\{S[\infty]\}}$ is the global dimension of $\mathcal{E}^{\mathcal{U}}$ still ≥ 2. Just look at the exact sequence σ as above. Choose in its definition R to be a regular module R with no summand in the tube λ, then for both exact substructures it gives an exact sequence which is not 2-split.

Now, we look at intersections of these exact structures and respectively unions of the Ziegler-closed sets.

When $M = \Omega$ and $T = \emptyset$ then the exact structure consists of ses $\sigma = \sigma_p \oplus \sigma_{rq}$ such that σ_p is an exact sequence in $\operatorname{add}(\mathcal{P})$ and σ_{rq} is an exact sequence in $\operatorname{add}(\mathcal{R} \cup \mathcal{Q})$. It is very easily seen to be hereditary exact.

When $M = \emptyset$ and $T = \Omega$ then the exact structure consists of ses $\sigma = \sigma_{pr} \oplus \sigma_q$ such that σ_{pr} is an exact sequence in $\operatorname{add}(\mathcal{P} \cup \mathcal{R})$ and σ_q is an exact sequence in $\operatorname{add}(\mathcal{Q})$. It is very easily seen to be hereditary exact.

The case $T = M = \Omega$, then this is the minimal exact structure containing all almost split sequences. The short exact sequences in this structure are $\sigma_p \oplus \sigma_r \oplus \sigma_q$ with σ_p is an exact sequence in $\operatorname{add}(\mathcal{P})$, σ_r is an exact sequence in $\operatorname{add}(\mathcal{R})$ and σ_q is an exact sequence in $\operatorname{add}(\mathcal{Q})$. Again, we easily see that this is hereditary exact.

(II) $\mathcal{U}^{fin} =: \mathcal{H}$. The exact structure corresponding to $\mathcal{U}$ is just given by all short exact sequences such that $\operatorname{Hom}(-, H)$ is exact on it for all H in $\mathcal{H}$, we write $\mathcal{E} = (\Lambda \operatorname{mod}, F^{\mathcal{H}})$. This case is well-studied in [**15**], [**13**], [**14**]. If $\mathcal{H}$ is finite, then the exact structure always has enough projectives and enough injectives given by $\operatorname{add}(\mathcal{H})$. Its global dimension can be characterized as follows $\operatorname{gldim} \mathcal{E} \leq k$ is equivalent to the following two conditions (i) $\operatorname{gldim} \operatorname{End}_\Lambda(\bigoplus_{H \in \mathcal{H}} H) \leq k + 2$ and (ii) $\operatorname{id}_{\mathcal{E}} \Lambda \leq k$.

For $\Lambda = KQ$ with Q the Kronecker quiver every global dimension can occur. This can be seen directly, just take $\mathcal{H} = \{(0, 1), (1, 2), (3, 4), (5, 6), (7, 8), \cdots, (2n - 1, 2n)\}$. Then injective coresolutions are calculated via left $\operatorname{add}(\mathcal{H})$-approximations and it can be easily seen that minimal injective coresolutions have at most $(n + 1)$-injective modules, e.g.

$$(2n, 2n + 1) \rightarrowtail (2n - 1, 2n)^{\oplus 2} \to (2n - 3, 2n - 2)^{\oplus 2} \to \cdots \to (1, 2)^{\oplus 2} \twoheadrightarrow (0, 1)$$

If you take $\mathcal{H} = \{S_\lambda, S_\lambda[3]\}$, then you find $\operatorname{id} S_\lambda[2] = \infty$ and therefore we have infinite global dimension.

Another class of examples always gives hereditary exact substructures, take $\mathcal{H} = \{(n, n + 1) \mid 0 \leq n \leq N\}$ for some $N \in \mathbb{N}$, then the cogenerator $\operatorname{add}(\mathcal{H})$ is closed under quotients, this is easily seen to imply that the corresponding exact structure is hereditary exact.

Once you take $\mathcal{H}$ infinite, it is also easy to find infinite global dimensions:

$$\mathcal{H} = \{(2n - 1, 2n) \mid n \in \mathbb{N}\}$$

Using minimal injective coresolutions for $(2n, 2n + 1)$ for all $n \in \mathbb{N}$, we find objects of injective dimension n for every $n \in \mathbb{N}$, this implies $\operatorname{gldim} = \infty$.

Example 5.12. We describe all exact substructures on finitely generated modules over a commutative discrete valuation ring R with maximal ideal P. We recall the description of the Ziegler spectrum from [**154**, Section 5.2]:
The points in Zg_R are:

(a) indecomposable modules of finite length R/P^n, $n \geq 1$
(b) the P-adic completion $\overline{R} = \lim R/P^n$ (this is the limit over $\cdots \to R/P^2 \to R/P = k$)
(c) The Prüfer module $R_{P^\infty} = \operatorname{colim} R/P^n$ (this is the colimit over $k = R/P \to R/P^2 \to \cdots$)
(d) the quotient division ring $Q = Q(R)$ of R

Now, $\mathcal{U}_{max} = \{Q, R_{P^\infty}\}$ is the Ziegler-closed set given by the indecomposable injective R-modules.
We also observe that $\mathrm{Zg}'_R := \{\overline{R}\} \cup \mathcal{U}_{max}$ is Ziegler-closed.
Next, all Ziegler-closed subsets containing $\mathcal{U}_{max}$ are given by:

(1) for $\emptyset \neq L \subseteq \mathbb{N}$ *finite*, we have

$$\mathcal{U}_L := \{R/P^n \mid n \in L\} \cup \mathcal{U}_{max}$$

(2) for $\emptyset \subseteq L \subseteq \mathbb{N}$ *arbitrary subset* we have

$$\mathcal{V}_L := \{R/P^n \mid n \in L\} \cup \mathrm{Zg}'_R$$

So let us describe the exact structures on $\mathcal{C} = R\,\mathrm{mod}$ the category of finitely generated left R-modules corresponding to these closed sets:

(max) Trivially $\mathcal{U}_{max}$ corresponds to the abelian structure on $\mathcal{C}$, this is hereditary and with enough projectives (but not with enough injectives)
(min) and Zg_R corresponds to the split exact structure.
(Zg') The Ziegler-closed set Zg'_R corresponds to the exact structure $\mathcal{E}'$ making the torsion functor exact. This is a hereditary exact structure , cp. example...
($\mathcal{U}_L$) This corresponds to the exact substructure $\mathcal{E}_L$ such that $\mathrm{Hom}_R(-, R/P^n)$, $n \in L$ are exact functors to abelian groups.
($\mathcal{V}_L$) This corresponds to the exact substructure $\mathcal{E}'_L$ such that the torsion functor and $\mathrm{Hom}_R(-, R/P^n)$, $n \in L$ are exact.

First, of all in general how can one see that for $\emptyset \neq L \subseteq \mathbb{N}$ finite: $\mathcal{E}_L$ and $\mathcal{E}'_L$ are different exact structures?
Take a short exact sequence $R \rightarrowtail R \twoheadrightarrow R/P^n$ and the pushout along $R \twoheadrightarrow R/P^m$, this gives an exact sequence $R/P^m \rightarrowtail R/P^{m+n} \twoheadrightarrow R/P^n$. But the cartesian square induces another exact sequence

$$R \rightarrowtail R/P^m \oplus R \twoheadrightarrow R/P^n$$

It is easily seen to be not exact in $\mathcal{E}'$ if $n \neq m$ and we conclude that $\mathrm{Ext}^1_{\mathcal{E}'}(R/P^n, R) = 0$. But if you apply $\mathrm{Hom}_R(-, R/P^\ell)$ using $\mathrm{Hom}_R(R/P^s, R/P^\ell) = R/P^{\min(s,\ell)}$ we see that this is exact for n, m both larger or equal than ℓ.
We say that L has *gaps* if there is an interval $[a, b]$ such that $[a, b] \cap L = \emptyset$ and $b + 1 \in L$ and $a - 1 \in L$ if $a > 1$. If L has gaps then $\mathrm{gldim}\, \mathcal{E}_L = \infty = \mathrm{gldim}\, \mathcal{E}'_L$ (for $\mathcal{E}'_L$ we also allow L to be an infinite subset).
In this case one can always find an infinite injective $\mathcal{E}^{(')}_L$-coresolution for an R/P^s some $s \in [a, b]$. We give them as sequence of short exact sequences.

($a = 1$) Take $s = b$ and $R/P^b \rightarrowtail R/P^{b+1} \twoheadrightarrow R/P$, then continue with $R/P \rightarrowtail R/P^{b+1} \twoheadrightarrow R/P^b$ and repeat with the first short exact sequence etc.
(2) If $a > 1$ and $b + a$ even then we take $s = \frac{1}{2}(a + b)$ and the short exact sequence

$$R/P^s \to R/P^{a-1} \oplus R/P^{b+1} \twoheadrightarrow R/P^s$$

and then continue with the same sequence.
(3) If $a > 1$ and $b + a$ uneven then we take $s = \frac{1}{2}(b + a - 1)$ and first $R/P^s \rightarrowtail R/P^{a-1} \oplus R/P^{b+1} \twoheadrightarrow R/P^{s+1}$ then $R/P^{s+1} \rightarrowtail R/P^{a-1} \oplus R/P^{b+1} \twoheadrightarrow R/P^s$ and then repeat with the first sequence.

If L has no gaps and $L \neq \mathbb{N}$ then $L = [1, n]$ for some $n \in \mathbb{N}$. The set $\{R/P^\ell \mid \ell \in L\}$ is closed under quotients, so injective coresolutions in this class of modules will always end after one short exact sequence. We show in the next Lemma that these exact structures are hereditary exact.

Lemma 5.13. *Let R denote a commutative discrete valuation ring with maximal ideal P and let $n \in \mathbb{N}$. We have a functor $\mathrm{rad}^n \colon R\,\mathrm{mod} \to R\,\mathrm{mod}$ defined by $\mathrm{rad}^n M = P^n M$. Let $L = [1, n] \subseteq \mathbb{N}$.*

(1) *Then we have* $\operatorname{Ext}^1_{\mathcal{E}_L}(-,-) = \operatorname{rad}^n \operatorname{Ext}^1_R(-,-)$ *and* $\operatorname{Ext}^1_{\mathcal{E}'_L}(-,-) = \operatorname{rad}^n_R \operatorname{Ext}^1_{\mathcal{E}'}(-,-)$
(2) *For every $\mathcal{E}_L$-exact sequence σ and every object X, the sequences $\operatorname{Ext}^1_{\mathcal{E}_L}(X,\sigma)$ is right exact. For every $\mathcal{E}'_L$-exact sequence σ and every object X, the sequences $\operatorname{Ext}^1_{\mathcal{E}'_L}(X,\sigma)$ is right exact.*

In particular, $\mathcal{E}_L$ and $\mathcal{E}'_L$ are hereditary exact.

PROOF. (1) We first describe the R-module structure on

(a) $\operatorname{Ext}^1_R(R/P^m, R/P^\ell) \cong R/P^s$ where $s = \min(m,\ell)$. For $a,b \in \mathbb{N}$ we write $\sigma_{a,b} \in \operatorname{Ext}^1_R(R/P^m, R/P^\ell)$ for an exact sequence with middle term $R/P^a \oplus R/P^b$ (whenever this exist). In R/P^s we pick $P = (p)$ and we have the following mult. by p $1 \mapsto p \mapsto p^2 \mapsto \cdots \mapsto p^{s-1} \mapsto 0$ this corresponds to the following on the Ext-group
 (a1) If $s = \ell \leq m$ we have
 $\sigma_{0,m+\ell} \mapsto \sigma_{1,m+\ell-1} \mapsto \cdots \mapsto \sigma_{\ell-1,m+1} \mapsto 0$
 Then we have $\operatorname{rad}^n \operatorname{Ext}^1_R(R/P^m, R/P^\ell) \cong R/P^{s-n}$ whenever $n < s$ and zero otherwise. For $n < s = \ell$ it is the image of p^n, i.e.
 $\sigma_{n,m+\ell-n} \mapsto \sigma_{n+1,m+\ell-n-1} \mapsto \cdots \mapsto \sigma_{\ell-1,m+1}$
 (a2) If $\ell > m = s$ we have
 $\sigma_{\ell+m,0} \mapsto \sigma_{\ell+m-1,1} \mapsto \cdots \mapsto \sigma_{\ell+1,m-1} \mapsto 0$
 Then we have $\operatorname{rad}^n \operatorname{Ext}^1_R(R/P^m, R/P^\ell) \cong R/P^{s-n}$ whenever $n < s$ and zero otherwise. For $n < s = \ell$ it contains the following elements
 $\sigma_{\ell+m-n,n} \mapsto \sigma_{\ell+m-n-1,n+1} \mapsto \cdots \mapsto \sigma_{\ell+1,m-1}$

(b) $\operatorname{Ext}^1_R(R/P^m, R) \cong R/P^m$ we write σ_a for the extension with $R \oplus R/P^a$ as middle term. Then the multiplication by p is given by
 $\sigma_0 \mapsto \sigma_1 \mapsto \cdots \mapsto \sigma_{m-1} \mapsto 0$
 The submodule $\operatorname{rad}^n \operatorname{Ext}^1_R(R/P^m, R)$ is of course zero is $n \geq m$ and if $n < m$ is given by the following
 $\sigma_n \mapsto \sigma_{n+1} \mapsto \cdots \mapsto \sigma_{m-1}$

We claim $\operatorname{Ext}^1_{\mathcal{E}_{[0,n]}} = \operatorname{rad}^n \operatorname{Ext}^1_R$. First observe that in $\mathcal{E}_L$: R/P^a $1 \leq a \leq n$ are injectives and they are also projectives. So for $s = \min(\ell,m) \leq n$ we have $\operatorname{Ext}^1_{\mathcal{E}_{[0,n]}}(R/P^m, R/P^\ell) = 0$ and for $m \leq n$ we have $\operatorname{Ext}^1_{\mathcal{E}_{[0,n]}}(R/P^m, R) = 0$.

So we may always assume wlog that $n < \min(\ell,m)$, then proceed by induction over n. For $n = 1$, a short exact sequence of in $R\,\mathrm{mod}$ is in $\mathcal{E}_{[0,1]}$ iff the indcomposable summands of number of the indec. summands of the outer terms add up to the indec summands of the middle term. This means in case (a) all exact sequence are in this exact structure except $\sigma_{0,m+n}$ and $\sigma_{m+n,0}$, in case (b) all except σ_0.

For $n > 1$, it is enough to observe that $\operatorname{Hom}(-, R/P^n)$ is
(ad a1) exact on $\sigma_{n,m+\ell-n}$ and not exact on $\sigma_{n-1,m+\ell-n+1}$
(ad a2) exact on $\sigma_{\ell+m-n,n}$ and not exact on $\sigma_{\ell+m-n+1,n-1}$
(ad b) exact on σ_n and not exact on σ_{n-1}

Then the rest follows by induction hypothesis.

Now, for $\mathcal{E}'_L$ one observes that $\operatorname{Ext}^1_{\mathcal{E}'_L}(X,Y) = \operatorname{Ext}^1_{\mathcal{E}_L}(X,Y)$ for all X,Y torsion, and for Y free it is zero.

As $\operatorname{Ext}^1_{\mathcal{E}'}(X,Y) = \operatorname{Ext}^1_R(X,Y)$ for all X,Y torsion and for Y free it is zero. Then the claim $\operatorname{Ext}^1_{\mathcal{E}'_L}$ follows from the proof for $\operatorname{Ext}^1_{\mathcal{E}_L}$.

(2) Taking n-th radical of an epimorphism in $R\,\mathrm{mod}$ is again an epimorphism - as the n-th radical can be described as the image of multiplication by p^n (making it also a quotient and not ony a submodule). As $\operatorname{Ext}^1_R(X,\sigma)$ (resp. $\operatorname{Ext}^1_R(\sigma,X)$) is right exact for all exact sequences σ, we conclude that $\operatorname{rad}^n \operatorname{Ext}^1_R(X,f)$ (resp. $\operatorname{rad}^n \operatorname{Ext}^1_R(g,X)$) are epimorphisms for f an epimorphism and g a monomorphism.

Of course, on general exact sequence rad^n is not a middle-exact functor (but this follows from (1) since for $\sigma \in \mathcal{E}_L$ the sequences $\operatorname{Ext}^1_{\mathcal{E}_L}(X,\sigma)$ and $\operatorname{Ext}^1_{\mathcal{E}_L}(\sigma,X)$ are middle exact). In particular, the right exactness claim follows.

For $\mathrm{Ext}^1_{\mathcal{E}'_L}(X,\sigma)$ we can restrict to X indecomposable torsion and as σ in $\mathcal{E}'$, it fits into an exact sequence of ses $\sigma_{tor} \rightarrowtail \sigma \twoheadrightarrow \sigma_{free}$ with σ_{tor} the torsion part and σ_{free} the free part, and we conclude that $\mathrm{Ext}^1_{\mathcal{E}'_L}(X,\sigma) = \mathrm{Ext}^1_{\mathcal{E}'_L}(X,\sigma_{tor}) = \mathrm{Ext}^1_{\mathcal{E}_L}(X,\sigma_{tor})$ is right exact.

□

After understanding exact substructures of $R\,\mathrm{mod}$ for R a commutative discrete valuation ring, we are ready to generalize this to commutative Dedekind domains:

Example 5.14. We describe all exact structures on finitely generated left modules over a commutative Dedekind domain.
The Ziegler spectrum is studied as a more general case of the discrete valuation ring, again we follow [**154**, Section 5.2] for its description: Let $\mathrm{mSpec}(R) := \{P \mid \text{ max. ideal in } R\}$. The points in Zg_R are:

(a) indecomposable modules of finite length R/P^n, $n \geq 1$, and $P \in \mathrm{mSpec}(R)$.
(b) the P-adic completion $\overline{R}_P = \lim R/P^n$ (this is the limit over $\cdots \to R/P^2 \to R/P$) for $P \in \mathrm{mSpec}(R)$.
(c) The Prüfer module $R_{P^\infty} = \mathrm{colim}\, R/P^n$ (this is the colimit over $R/P \to R/P^2 \to \cdots$) for $P \in \mathrm{mSpec}(R)$.
(d) the quotient division ring $Q = Q(R)$ of R

Now, $\mathcal{U}_{max} = \{Q\} \cup \{R_{P^\infty} \mid P \in \mathrm{mSpec}(R)\}$ is the Ziegler-closed set given by the indecomposable injective R-modules. We describe all Ziegler-closed subsets containing $\mathcal{U}_{max}$ (following loc. cit.).
First we fix some notation, let $L \subseteq \mathrm{mSpec}(R) \times \mathbb{N}$ always denote such a subset and for $P \in \mathrm{mSpec}(R)$ let $L_P := \{\ell \in \mathbb{N} \mid (\ell, P) \in L\}$. Subsets of indecomposable finite length modules are of the form $\mathcal{F}_L = \{R/P^\ell \mid (P,\ell) \in L\}$. We fix a closed subset $\mathcal{U}$ and denote by $\mathcal{F}_L$ its points of finite length.
(type 1) $L = \emptyset$. For every $M \subseteq \mathrm{mSpec}(R)$ we have a closed subset $\mathrm{Zg}'_M = \mathcal{U}_{max} \cup \{\overline{R}_P \mid P \in M\}$. We define $\mathrm{Zg}' := \mathrm{Zg}'_{\mathrm{mSpec}(R)}$.
(type 2) $0 < |L| < \infty$. Then $\mathcal{U} = \mathcal{F}_L \cup \mathrm{Zg}'_M$ for an arbitrary subset $M \subseteq \mathrm{mSpec}(R)$ (the sets L and M are independent from each other).
(type 3) $|L| = \infty$. We define $M_L := \{P \in \mathrm{mSpec}(R) \mid \exists\, n \in \mathbb{N}\colon (P,n) \in L\}$ and in this case $\mathcal{U} = \mathcal{F}_L \cup \mathrm{Zg}'_{M_L}$.
Let us look at the corresponding exact substructures:

(type 1) Make all P-torsion functors exact for all $P \in M$. As we are dealing with hereditary torsion pairs, the torsion functors are left exact and .. applies to show that these are hereditary exact structures.
(gaps) Let us assume we are in type 2 or type 3.
If for some $P \in \mathrm{mSpec}(R)$ we have L_P has a gap (see previous example), then we find an infinite injective coresolution as in the previous example and it follows $\mathrm{gldim} = \infty$.
(no gaps) Let us assume we are in type 2 or type 3. If all non-empty L_P have no gaps for every maximal ideal P, then we find $L_P = [1, n_P]$ for an $n_P \in \mathbb{N}$. We claim that we only have hereditary exact structures in this case. We give the proof in the next Lemma.

Now, let R be a commutative Dedekind domain. Observe that Ext^1_R only takes values in torsion modules. We write $()_P$ for its P-torsion submodule and $()_{tor}$ for the remaining torsion summands. So, for two finitely generated R-modules $X = R^t \oplus X_P \oplus X_{tor}, Y = R^s \oplus Y_P \oplus Y_{tor}$ we have

$$\mathrm{Ext}^1_R(X,Y) = \mathrm{Ext}^1_R(X,Y)_P \oplus \mathrm{Ext}^1_R(X,Y)_{tor}$$
$$\mathrm{Ext}^1_R(X,Y)_P = \mathrm{Ext}^1_R(X_P, Y_P \oplus R^s)$$
$$\mathrm{Ext}^1_R(X,Y)_{tor} = \mathrm{Ext}^1_R(X_{tor}, Y_{tor} \oplus R^s)$$

Then $\mathrm{Ext}^1_R = (\mathrm{Ext}^1_R)_P \oplus (\mathrm{Ext}^1_R)_{tor}$ is a direct sum decomposition of bifunctors (but this does not imply that these subfunctors are middle exact for the abelian structure on $R\,\mathrm{mod}$). Let

$F_P\colon R\,\mathrm{mod} \to R\,\mathrm{mod}$ be the functor $X \mapsto R^t \oplus X_P$ (resp. $F_{tor}(X) = R^t \oplus X_P$), induced by the projection into the torsionfree part (in the split hereditary torsion pairs considered in (type 1)). They preserve epimorphisms. In particular, if $f\colon A \twoheadrightarrow B$ is an epimorphism, then the following are also epimorphism as $\mathrm{Ext}^1_R(M,-)$ preserves epimorphisms for all objects M

$$\mathrm{Ext}^1_R(X,f)_P = \mathrm{Ext}^1_R(F_P(X), F_P(f))$$
$$\mathrm{Ext}^1_R(X,f)_{tor} = \mathrm{Ext}^1_R(F_{tor}(X), F_{tor}(f))$$

As it looks simpler let us look at the case of only one prime:

Lemma 5.15. *Let R be a commutative Dedekind domain. Let P be a fixed maximal prime ideal. We define for $M \in R\,\mathrm{mod}, P \in \mathrm{mSpec}(R), n \in \mathbb{N}$ the following $\mathrm{rad}^n_P M := P^n M$. If $L = \{P\} \times [1,n]$ we denote by $\mathcal{E}_L$ the exact structure corresponding to $\mathcal{F}_L \cup \mathcal{U}_{max}$.*

(1) *Then* $\mathrm{Ext}^1_{\mathcal{E}_L} = (\mathrm{rad}^n_P \mathrm{Ext}^1_R) \oplus (\mathrm{Ext}^1_R)_{tor}$
(2) *The exact structure $\mathcal{E}_L$ is hereditary exact.*

Proof. (1) As for different primes the torsion submodules are Hom_R- and Ext_R-orthogonal, the exactness of $\mathrm{Hom}(-, R/P^a)$ for some $a \in \mathbb{N}$ only depends on the P-torsion and the free module summand. The same proof as in Lemma 5.13 applies.
(2) By the discussion before the Lemma and knowing that rad^n preserves epimorphisms, it follows from (1) that $\mathrm{Ext}^1_{\mathcal{E}_L}$ preserves epimorphisms. Therefore $\mathcal{E}_L$ is hereditary exact.

□

But it actually is the same for an arbitrary subset of primes:

Lemma 5.16. *Let R be a commutative Dedekind domain.*
Let $L = \bigcup_{P \in M} \{P\} \times [1, n_P] \subseteq \mathrm{mSpec}(R) \times \mathbb{N}$ and $\mathcal{E} = \mathcal{E}^{\mathcal{U}}$ for a Ziegler-closed subset $\mathcal{U}$ with modules of finite length given by $\mathcal{F}_L$, then $\mathcal{E}$ is hereditary exact.

Proof. (of Cor. 5.16) $\mathcal{U} = \mathcal{F}_L \cup \mathrm{Zg}'_M$ for some subset $M \subseteq \mathrm{mSpec}(R)$ (type 2 or type 3). Let us denote by an index $tor(M^c)$ the torsion summand corresponding to the complement of M. The above Lemma generalizes to

$$\mathrm{Ext}^1_{\mathcal{E}_L} = \bigoplus_{P \in M} (\mathrm{rad}^{n_P}_P \mathrm{Ext}^1_R) \oplus (\mathrm{Ext}^1_R)_{tor(M^c)}$$

This is clear as intersection of exact substructures correspond to intersecting the corresponding Ext^1-subfunctors.
This functor is still preserving epimorphisms as before. □

6. The functorial point of view

Let $\mathcal{E}$ be an essentially small exact category. We consider three classical assignments for $\mathcal{F} \in \mathrm{ex}(\mathcal{E})$

(i) the Auslander category
(ii) its category of inflation represented functors
(iii) the category of deflation represented functors (called effaceable functors)

all will be considered fully exact subcategories in $\mathrm{mod}_1 \mathcal{A}$ (where $\mathcal{A}$ is the underlying additive category). By a results of [**90**] and [**72**], we have characterizations of the subcategories when we look only at exact substructures of $\mathcal{E}$.

Recall a Serre subcategory is a full additive subcategory $\mathcal{F}$ in an exact category $\mathcal{E}$ with the following property: For every $\mathcal{E}$-short exact sequence $X \to Y \to Z$ we have $Y \in \mathcal{F}$ if and only if $X, Z \in \mathcal{F}$.

Definition 6.1. We denote by $\mathcal{P}^2(\mathcal{A})$ the full subcategory of $\operatorname{Mod}\mathcal{A}$ given by all functors F such that there exists an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(-,X) \to \operatorname{Hom}_{\mathcal{A}}(-,Y) \to \operatorname{Hom}_{\mathcal{A}}(-,Z) \to F \to 0$$

for some X, Y, Z in $\mathcal{A}$.

$$\mathcal{G}^2(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d),(j,p) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{j}\circ \mathrm{d})\}$$
$$\mathcal{C}^2(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{d})\}$$
$$\mathcal{J}^1(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (j,p) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{j})\}$$

Apriory these are additive categories. The **grade** of $F \in \operatorname{Mod}\mathcal{A}$ is defined as the supremum of all natural numbers $i \geq 0$ such that $\operatorname{Ext}^j_{\operatorname{Mod}\mathcal{A}}(F, \operatorname{Hom}_{\mathcal{A}}(-,A)) = 0 \forall A \in \mathcal{A}$ for all $j < i$ (of course, only if this exists, else we define it to be ∞). Categories defined in terms of grade equalities are by definition extension-closed in $\operatorname{Mod}-\mathcal{A}$. We have

$$\mathcal{G}^2(\mathcal{A}) \subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} \in \{0,2\}\}$$
$$\mathcal{C}^2(\mathcal{A}) = \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} = 2\}$$
$$\mathcal{J}^1(\mathcal{A}) \subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} = 0\}$$

The two inclusions follow from the definition. The equality in the middle is proven in [72], Lemma 2.3, so $\mathcal{C}^2(\mathcal{A})$ is extension-closed in $\mathcal{P}^2(\mathcal{A})$. (These subcategories $\mathcal{G}^2(\mathcal{A}), \mathcal{J}^1(\mathcal{A})$ can be fully characterized using the Auslander-Bridger transpose, if one is keen to avoid $\mathrm{KC}(\mathcal{A})$ - look at [90], chapter 5 and the HKR-bijection for more details). We do not mind working with $\mathrm{KC}(\mathcal{A})$ and use these not always extension-closed subcategories.

Now we define our categories of interest.

Definition 6.2. Let $\mathcal{E}$ be essentially small exact. The **Auslander exact category** of $\mathcal{E}$ is defined as the full subcategory of $\mathcal{P}^2(\mathcal{A})$ given by

$$\mathrm{AE}(\mathcal{E}) = \{\mathrm{F} \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{F} = \operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{f})), \text{ with f } \mathcal{E}\text{-admissible}\}$$

It has as a subcategory

$$\mathrm{eff}(\mathcal{E}) = \mathrm{AE}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}) = \{\operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{d}) \mid \mathrm{d}\ \mathcal{E}\text{-deflation}\}$$

called the subcategory of **effaceable functors** and another subcategory

$$\mathrm{H}(\mathcal{E}) := \{\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, i) \mid i\ \mathcal{E}\text{-inflation}\}$$

which we refer to as **tf-Auslander category** (cf. Appendix B).

Obviously: $\mathrm{AE}(\mathcal{E}) \subseteq \mathcal{G}^2(\mathcal{A}), \mathrm{eff}(\mathcal{E}) \subseteq \mathcal{C}^2(\mathcal{A}), \mathrm{H}(\mathcal{E}) \subseteq \mathcal{J}^1(\mathcal{A})$.
In [90, Prop. 3.5, Prop.3.6, Prop.5.4], it is shown that $\mathrm{AE}(\mathcal{E})$ is extension-closed in $\operatorname{mod}_1\mathcal{A}$, even resolving in $\mathcal{P}^2(\mathcal{A})$, and $(\mathrm{H}(\mathcal{E}), \mathrm{eff}(\mathcal{E}))$ is a torsion pair in $\mathrm{AE}(\mathcal{E})$. In particular, $\mathrm{H}(\mathcal{E})$ is also a resolving subcategory of $\mathrm{AE}(\mathcal{E})$.
To state the results that we look at two (different) dualities. Enomoto found the following duality between $\mathcal{C}^2(\mathcal{A})$ and $\mathcal{C}^2(\mathcal{A}^{op})$

Theorem 6.3. *(Enomoto) Let $\mathcal{A}$ be an idempotent complete small additive category. There exists a duality $E\colon \mathcal{C}^2(\mathcal{A})^{op} \to \mathcal{C}^2(\mathcal{A}^{op})$ such that*
$E(\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,d)) \cong \operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(i,-))$ *for every kernel-cokernel pair (i,d) in $\mathcal{A}$.*

Then he can characterize exact structures as follows

Theorem 6.4. *(Enomoto's bijection) Given a small idempotent complete additive category $\mathcal{A}$. Then the assignments $\mathcal{E} \mapsto \mathrm{eff}(\mathcal{E})$ and $\mathcal{C} \mapsto \mathcal{S} := \{(i,d) \in \mathrm{KC}(\mathcal{A}) \mid \operatorname{coker}\operatorname{Hom}(-,\mathrm{d}) \in \mathcal{C}\}$ give inverse bijections between*

(1) *exact structures on $\mathcal{A}$*

(2) *full subcategories $\mathcal{C} \subseteq \mathcal{C}^2(\mathcal{A})$ with $\mathcal{C}$ is a Serre subcategory in $\operatorname{mod}_1 \mathcal{A}$ and $E(\mathcal{C})$ is a Serre subcategory in $\operatorname{mod}_1 \mathcal{A}^{op}$*

If we denote by $\mathcal{E}_{max}$ the maximal exact structure on $\mathcal{A}$ with corresponding Serre subcategory $\mathcal{C}_{max}$ then (2) coincides with the following.

(2') *Serre subcategories $\mathcal{C}$ of $\mathcal{C}_{max}$*

Where (2') is an observation of Kevin Schlegel (cf. [**173**, Cor. 2.3]).

The second duality is Auslander-Bridger transpose. We need the ideal quotient with respect to the projectives (called the stable category) - this exists even if the category has not enough projectives. As there are no grade 0 objects in $\mathcal{C}^2(\mathcal{A})$ the composition

$$\mathcal{C}^2(\mathcal{A}) \to \operatorname{mod}_1 \mathcal{A} \to \underline{\operatorname{mod}_1}\mathcal{A}$$

is still fully faithful.

THEOREM 6.5. *Let $\mathcal{A}$ be an idempotent complete small additive category. Then we have a duality $\operatorname{Tr}\colon (\underline{\operatorname{mod}_1}\mathcal{A})^{\mathrm{op}} \to \underline{\operatorname{mod}_1}(\mathcal{A}^{\mathrm{op}})$ which maps $\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,f)$ to $\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(f,-)$.*

Remark 6.6. On objects $(\mathrm{Tr}\circ\Omega)(\mathrm{C}) \cong \mathrm{E}(\mathrm{C})$ for C in $\mathcal{C}^2(\mathcal{A})$ but in general Ω is not an endofunctor on the stable category. (But on stable categories of exact categories *with* enough projectives, Ω defines an endofunctor.)

For a subcategory $\mathcal{X} \subseteq \operatorname{mod}_1 \mathcal{A}$, we denote by $\mathrm{Tr}(\mathcal{X})$ the full subcategory of $\operatorname{mod}_1 \mathcal{A}^{op}$ consisting of objects X such that $X \cong \mathrm{Tr}(\mathrm{X}')$ in $\underline{\operatorname{mod}_1}\mathcal{A}^{op}$ for some X' in $\mathcal{X}$.

Remark 6.7. By definition

$$\mathrm{Tr}(\mathcal{G}^2(\mathcal{A})) = \mathcal{G}^2(\mathcal{A}^{\mathrm{op}}), \quad \Omega\mathrm{Tr}(\mathcal{J}^1(\mathcal{A})) = \mathcal{J}^1(\mathcal{A}^{\mathrm{op}})$$

For every $\mathcal{X} \subseteq \mathcal{G}^2(\mathcal{A})$ containing all representables: $\mathrm{Tr}\mathrm{Tr}(\mathcal{X}) = \mathcal{X}$.
For every $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A})$ containing all representables: $\Omega\mathrm{Tr}\Omega\mathrm{Tr}(\mathcal{J}) = \mathcal{J}$.

THEOREM 6.8. *(HKR-bijection) Given a small idempotent complete additive category $\mathcal{A}$. Then the assignments $\mathcal{E} \mapsto \mathrm{AE}(\mathcal{E})$ gives a bijection between*

(1) *exact structures on $\mathcal{A}$*
(2) *resolving subcategories $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A})$ with all objects have either grade 0 or 2 such that $\mathrm{Tr}(\mathcal{X}) \subseteq \mathcal{P}^2(\mathcal{A}^{\mathrm{op}})$ is resolving and all objects have either grade 0 or 2.*

Furthermore, in this case, the full subcategory of grade 2 objects is $\mathrm{AE}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}) = \mathrm{eff}(\mathcal{E})$ and the one of grade 0-objects is $\mathrm{AE}(\mathcal{E}) \cap \mathcal{J}^1(\mathcal{A}) = \mathrm{H}(\mathcal{E})$.

Open question 6.9. In (2) we could use $\mathcal{G}^2(\mathcal{A})$ as well. Furthermore, we can also use the maximal exact structure, let $\mathcal{X}_{max}$ be the resolving subcategory of $\mathcal{P}^2(\mathcal{A})$ corresponding to the maximal exact structure. Is the following (2') equivalent to (2)?
(2') Resolving subcategories $\mathcal{X}$ in $\mathcal{X}_{max}$.

We add the following third bijection.

THEOREM 6.10. *Given a small idempotent complete additive category $\mathcal{A}$. Then the assignments $\mathcal{E} \mapsto \mathrm{H}(\mathcal{E})$ gives a bijection between*

(1) *exact structures on $\mathcal{A}$*
(2) *full subcategories $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A})$ such that $\mathcal{J} \subseteq \mathcal{P}^1(\mathcal{A})$ and $\Omega\mathrm{Tr}(\mathcal{J}) \subseteq \mathcal{P}^1(\mathcal{A}^{\mathrm{op}})$ are both resolving.*

Open question 6.11. Again, we look at $\mathcal{J}_{max}$ to be the resolving subcategory of $\mathcal{P}^1(\mathcal{A})$ corresponding to the maximal exact structure, then can we describe the subcategories in (2) also as the following?
(2') Resolving subcategories $\mathcal{J}$ in $\mathcal{J}_{max}$.

Remark 6.12. As $\mathcal{P}^1(\mathcal{A})$ is an hereditary exact (i.e. gldim ≤ 1) with enough projectives we have that a full subcategory is resolving if and only if it is fully exact, closed under summands and contains the projectives.

We need the following lemma for the proof.

Lemma 6.13. *If i is a monomorphism in $\mathcal{A}$ such that $F = \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,i) \in \mathcal{J}^1(\mathcal{A})$ then there is $(i,p) \in \mathrm{KC}(\mathcal{A})$. Also, if $\mathcal{A}$ is idempotent complete then $\mathcal{J}^1(\mathcal{A})$ is closed under taking direct sums and summands (in $\mathcal{P}^1(\mathcal{A})$).*

Proof. By assumption we can find a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \xrightarrow{q} & Z \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \\ U & \xrightarrow{i} & V & & \end{array}$$

with $(j,q) \in \mathrm{KC}(\mathcal{A})$ and $X \xrightarrow{(a,j)^t} U \oplus Y \xrightarrow{(i,-b)} V$ (call this (*)) a split exact sequence (this implies in particular that the commuting square is a pullback-pushout diagramm in $\mathcal{A}$). Then there exists $p\colon V \to Z$ with $q = pb$ and $p = \operatorname{coker}(i)$ - see e.g. [62], Lemma 2.3 (or prove this directly). We claim that (using the split exact sequence) we can also show $i = \ker(p)$ (i.e. $(i,p) \in \mathrm{KC}(\mathcal{A})$.
Given $r\colon R \to V$ with $pr = 0$. We claim that r factors over i (as i is a monomorphism, such a factorization is unique). We form the pullback (see below) of (*) along r in the split exact category and consider the commutative diagramm:

$$\begin{array}{ccccc} X & \longrightarrow & E & \xrightarrow{d} & R \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle (u,y)^t} & & \downarrow{\scriptstyle r} \\ X & \longrightarrow & U \oplus Y & \xrightarrow{(i,-b)} & V \\ & & & \searrow{\scriptstyle (0,-q)} & \downarrow{\scriptstyle p} \\ & & & & Z \end{array}$$

As $(0,-q) \circ (u,y)^t = 0$ we find that $(u,y)^t$ factors uniquely through $(1,j)^t = \ker(0,-q)\colon U \oplus X \to U \oplus Y$, i.e. $y = jx$ for an $x\colon E \to X$. Therefore, we have (using the two commuting squares from before)

$$rd = iu - bjx = iu - iax = i(u - ax)$$

As d is a split epimorphism there exists an $s\colon R \to E$ with $ds = 1_R$ and therefore $r = i(u - ax)s$ as claimed.
Closed under direct sums: Straight forward using the horseshoe lemma and the fact that direct sums of kernel-cokernel pairs are again kernel-cokernel pairs.
Now assume $F \oplus G \in \mathcal{J}^1(\mathcal{A})$. As $\mathcal{P}^1(\mathcal{A})$ is closed under taking summands, choose monomorphisms i, j such that $F = \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,i), G = \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-,j)$ and by the horseshoe lemma we conclude $F \oplus G = \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, i \oplus j)$. By the previous part it follows that there exists $(i \oplus j, g) \in \mathrm{KC}(\mathcal{A})$. We look at projection onto and then inclusion of the summand i of the two-term complex given by $i \oplus j$. This induces an idempotent endomorphism e on the cokernel $Z \to Z$. By assumption this idempotent is split admissible, i.e. factors as $Z \xrightarrow{\pi} Z_1 \xrightarrow{\iota} Z$ with π split epimorphism and ι split monomorphism. Then, it is straight forward to see that $g = p \oplus q$ with $(i,p), (j,q) \in \mathrm{KC}(\mathcal{A})$ and therefore $F, G \in \mathcal{J}^1(\mathcal{A})$.

∎

Proof. (of Thm 6.10). Given an exact structure $\mathcal{E}$, the category $\mathcal{J} = \mathrm{H}(\mathcal{E}) \subseteq \mathcal{J}^1(\mathcal{A})$ and $\Omega\mathrm{Tr}(\mathcal{J}) = \mathrm{H}(\mathcal{E}^{\mathrm{op}}) \subseteq \mathcal{J}^1(\mathcal{A}^{\mathrm{op}})$ by definition. As observed before $\mathcal{J}$ is resolving in $\mathrm{AE}(\mathcal{E})$ which is resolving in $\mathcal{P}^2(\mathcal{A})$ by Thm 6.8, this implies that it is also resolving in $\mathcal{P}^1(\mathcal{A})$. Dually, we also have $\Omega\mathrm{Tr}(\mathcal{J})$ is resolving in $\mathcal{P}^1(\mathcal{A}^{op})$, so the map is well-defined.
Conversely given $\mathcal{J}$ as in (2), we consider $\mathcal{S} = \{(i, d) \in \mathrm{KC}(\mathcal{A}) \mid \operatorname{coker} \mathrm{Hom}_{\mathcal{A}}(-, \mathrm{i}) \in \mathcal{J}\}$ and claim that $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ is an exact structure. As $\mathcal{J}$ contains the projectives, all split exact sequences are contained in $\mathcal{S}$. For $(i, d) \in \mathcal{S}$ we call i an $\mathcal{E}$-inflation and d an $\mathcal{E}$-deflation. We claim the following

(e1) composition ji of two (composable) $\mathcal{E}$-inflations j,i are $\mathcal{E}$-inflations
(e2) for every morphism f and every $\mathcal{E}$-inflation i (starting at the same object), the morphism $\left(\begin{smallmatrix} i \\ -f \end{smallmatrix}\right)$ is an $\mathcal{E}$-inflation with cokernel (f', i') where i' is again an $\mathcal{E}$-inflation.

Together with the dual statements for the deflations implied by $\Omega\mathrm{Tr}(\mathcal{J})$ having the same properties it follows that $\mathcal{E}$ is an exact category.
We use the following notation $P_X = \mathrm{Hom}_{\mathcal{A}}(-, X)$ and for a morphism $f \colon X \to Y$ we have $P_f = \mathrm{Hom}_{\mathcal{A}}(-, f) \colon P_X \to P_Y$.

(e1) Let $i \colon X \to Y$ and $j \colon Y \to V$ be two $\mathcal{E}$-inflations in $\mathcal{A}$, we look at the commutative diagram where F, G, H are defined as $\operatorname{coker} P_i, \operatorname{coker} P_{ji}, \operatorname{coker} P_j$ respectively.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_X & \xrightarrow{P_i} & P_Y & \longrightarrow & F & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & {\scriptstyle P_j}\downarrow & & \downarrow & & \\
0 & \longrightarrow & P_X & \xrightarrow{P_{ji}} & P_V & \longrightarrow & G & \longrightarrow & 0 \\
 & & {\scriptstyle P_i}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & \\
0 & \longrightarrow & P_Y & \xrightarrow{P_j} & P_V & \longrightarrow & H & \longrightarrow & 0
\end{array}$$

In particular i, j, ji are monomorphisms and the three rows are exact (in $\mathcal{P}^1(\mathcal{A})$). Now, using the *ker-coker sequence* (e.g. [**49**], Prop. 8.11), in $\mathrm{Mod} - \mathcal{A}$, we can deduce that $0 \to F \to G \to H \to 0$ is a short exact sequence on $\mathcal{P}^1(\mathcal{A})$. As $\mathcal{J}$ is extension-closed, it follows that G is an object in $\mathcal{J}$. By Lemma 6.13 it follows that ji is an $\mathcal{E}$-inflation.
(e2) We have $\begin{pmatrix} i \\ -f \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and as composition and direct sums of $\mathcal{E}$-inflations are $\mathcal{E}$-inflations by (e1) and Lemma 6.13 we conclude that $\begin{pmatrix} i \\ -f \end{pmatrix}$ is again an $\mathcal{E}$-inflation. So there exists $(\begin{pmatrix} i \\ -f \end{pmatrix}, (g \quad j)) \in \mathrm{KC}(\mathcal{A})$, i.e. we have a pullback-pushout diagramm in $\mathcal{A}$

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
U & \xrightarrow{j} & V
\end{array}$$

We need to see that j is an $\mathcal{E}$-inflation. It is again a monomorphism in $\mathcal{A}$, therefore we have $F = \operatorname{coker} P_j \in \mathcal{P}^1(\mathcal{A})$. We will show $F \in \mathcal{J}$:

As we have an $(i, p \colon Y \to Z) \in \mathrm{KC}(\mathcal{A})$ we can find a cokernel $q = \operatorname{coker}(j)$ with $p = qg$. But $j = \ker q$ is not directly clear.

We look at the covariant functors $P^A = \mathrm{Hom}_{\mathcal{A}}(A, -)$ and define $G := \operatorname{coker} P_i$. We find a commutative diagram with rows exact in $\mathrm{Mod} - \mathcal{A}^{op}$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & P^Z & \longrightarrow & P^Y & \longrightarrow & P^X & \longrightarrow & \mathrm{Tr(G)} & \longrightarrow & 0 \\
 & & {\scriptstyle =}\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & P^Z & \longrightarrow & P^V & \longrightarrow & P^U & \longrightarrow & \mathrm{Tr(F)} & \longrightarrow & 0
\end{array}$$

We get an induced commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P^Z & \longrightarrow & P^Y & \longrightarrow & \Omega\mathrm{Tr}(\mathrm{G}) & \longrightarrow & 0 \\
 & & {=}\big\uparrow & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & P^Z & \longrightarrow & P^V & \longrightarrow & \Omega\mathrm{Tr}(\mathrm{F}) & \longrightarrow & 0
\end{array}$$

Therefore, the right hand square is a pullback-pushout square and we get an induced exact sequence

$$0 \to P^V \to P^Y \oplus \Omega\mathrm{Tr}(\mathrm{F}) \to \Omega\mathrm{Tr}(\mathrm{G}) \to 0$$

As $\Omega\mathrm{Tr}(\mathcal{J})$ is resolving, it follows that $\Omega\mathrm{Tr}(\mathrm{F}) \in \Omega\mathrm{Tr}(\mathcal{J})$. This implies $F \in \mathcal{J}$.

If we denote this exact category by $\mathcal{E} = \mathcal{E}_{\mathcal{J}}$, then we easily see by definition $\mathcal{J} = \mathrm{H}(\mathcal{E}_{\mathcal{J}})$.
Also, by definition we have for an exact structure $\mathcal{E}$ that $\mathcal{E} \leq \mathcal{E}_{\mathrm{H}(\mathcal{E})}$ is an exact substructure. For equality, we need to see: If $(i\colon X \to Y, p) \in \mathrm{KC}(\mathcal{A})$ such that $F := \operatorname{coker} P_i \in \mathrm{H}(\mathcal{E})$ then i is an $\mathcal{E}$-inflation. By definition there exist an $\mathcal{E}$-inflation $j\colon U \to V$ such that $\operatorname{coker} P_j = F$. Looking at the projective presentations we get a morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_U & \xrightarrow{P_j} & P_V & \longrightarrow & F & \longrightarrow & 0 \\
 & & {\scriptstyle P_u}\big\downarrow & & {\scriptstyle P_v}\big\updownarrow & & {=}\big\downarrow & & \\
0 & \longrightarrow & P_X & \xrightarrow{P_i} & P_Y & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

Since the right hand square has to be bicartesian it follows that we have a split exact sequence $P_U \rightarrowtail P_X \twoheadrightarrow P_V \to P_Y$ which has to be split as P_Y is projective. This means we have split exact sequence $U \rightarrowtail X \oplus V \twoheadrightarrow Y$, in particular we have a pullbak-pushout diagram in $\mathcal{A}$

$$\begin{array}{ccc}
U & \xrightarrow{j} & V \\
{\scriptstyle u}\big\downarrow & & {\scriptstyle v}\big\downarrow \\
X & \xrightarrow{i} & Y
\end{array}$$

As j is an $\mathcal{E}$-inflation it follows that i is also one. □

7. Appendix on equivalences of categories

We shortly review three equivalences of 2-categories for small exact categories.
We will only consider strict 2-categories, i.e. they are enhanced in the category of small categories. This means 1-morphisms will be certain functors between categories and 2-morphisms will be natural transformations between them. We only consider strict 2-functors, i.e. these are which preserve compositions of 1-morphisms and compositions of 2-morphisms.

(A) The Butler-Horrock theorem (seeing exact categories as extriangulated categories)
(B) The Auslander correspondence (going to functor categories)
(C) Ind-completion (passes from small exact to locally coherent exact structures)

8. Appendix A: Going into extriangulated - The Butler Horrocks Theorem

For a small additive category $\mathcal{A}$ we denote by $\mathcal{E}_{max}$ its maximal exact structure and we set $\mathrm{Ext}^1_{\mathcal{A}} := \mathrm{Ext}^1_{\mathcal{E}_{max}}$.
The Butler Horrock's theorem (Thm. 3.4) gives for a small additive category a one-to-one correspondence between closed sub-bifunctors of $\mathrm{Ext}^1_{\mathcal{A}}$ and exact structures on $\mathcal{A}$. By [**140**], the pair $(\mathcal{A}, \mathrm{Ext}^1_{\mathcal{E}})$ gives an extriangulated category.
We have $\mathbb{E}\mathbf{x}$ is the 2-category of small exact categories, 1-morphisms are exact functors and 2-morphisms are natural transformations.

In [37] the 2-category $\mathbb{E}$**xtri** is defined and exact categories are embedded via the Butler-Horrocks theorem as objects in it. Let $\mathbb{BH}$ be the full 2-subcategory of the 2-category of extriangulated categories with objects in exact categories. It can be described as follows:

(1) **Objects** are pairs $(\mathcal{A}, \mathbb{E})$ of a small additive category $\mathcal{A}$ together with an additive bifunctor $\mathbb{E}\colon \mathcal{A} \times \mathcal{A}^{op} \to (Ab)$ which is a closed sub-bifunctor of $\mathrm{Ext}^1_{\mathcal{A}}$.
(2) **1-Morphisms** $(\mathcal{A}, \mathbb{E}) \to (\mathcal{A}', \mathbb{E}')$ are pairs of an additive functor $f\colon \mathcal{A} \to \mathcal{A}'$ and a natural transformation $\varphi\colon \mathbb{E} \to \mathbb{E}'(f(-), f(-))$ of functors on $\mathcal{A}$ which satisfies the following *connecting-property*: Let $\mathcal{E}, \mathcal{E}'$ be the exact categories corresponding to $(\mathcal{A}, \mathbb{E}), (\mathcal{A}', \mathbb{E}')$. For every $\mathcal{E}$-exact sequence (i, p) we have an associated distinguished triangle in $D^b(\mathcal{E})$, with $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\sigma} X[1]$, then we have a distinguished triangle in $D^b(\mathcal{E}')$
$$f(X) \xrightarrow{f(i)} f(Y) \xrightarrow{f(p)} f(Z) \xrightarrow{\varphi(\sigma)} f(X)[1]$$
A morphism (f, φ) induces a functor $\mathrm{Ses}(f)\colon \mathrm{Ses}(\mathcal{E}) \to \mathrm{Ses}(\mathcal{E}')$ on the categories of short exact sequences.
(3) **2-Morphisms** between two 1-morphisms $(f, \varphi), (g, \psi)\colon (\mathcal{A}, \mathbb{E}) \to (\mathcal{A}', \mathbb{E}')$ are given by natural transformations $\Phi\colon f \to g$. These always induce natural transformations $\mathrm{Ses}(f) \to \mathrm{Ses}(g)$, i.e. for every $\mathcal{E}$-short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ we have a commutative diagram with $\mathcal{E}'$-exact rows
$$\begin{array}{ccccc} f(X) & \rightarrowtail & f(Y) & \twoheadrightarrow & f(Z) \\ \downarrow{\scriptstyle \Phi_X} & & \downarrow{\scriptstyle \Phi_Y} & & \downarrow{\scriptstyle \Phi_Z} \\ g(X) & \rightarrowtail & g(Y) & \twoheadrightarrow & g(Z) \end{array}$$
such that for every morphism of short exact sequences are mapped into the corresponding 3-dimensional diagram.

The connecting property is a reformulation of: The composition $\mathcal{E} \to \mathcal{E}' \to D^b(\mathcal{E}')$ is a δ-functor in the sense of Keller ([117]) - or in a modern language: We only want extriangulated functors as morphisms in $\mathbb{BH}$ in the sense of [38, Def. 2.32].

THEOREM 8.1. *(cf. Thm 3.4 together with* [38, Thm 2.34]*) The assignment* $\mathcal{E} = (\mathcal{A}, \mathcal{S}) \mapsto (\mathcal{A}, \mathrm{Ext}^1_{\mathcal{E}})$, *and mapping an exact functor to the underlying additive functor gives an equivalence of strict 2-categories* $\mathbb{E}\boldsymbol{x} \to \mathbb{BH}$.

As morphisms in $\mathbb{BH}$ are defined, we can obviously write down the inverse 2-functor.

9. Appendix B: Auslander correspondences as equivalence(s) of 2-categories

We recall the equivalence of 2-categories to Auslander exact categories from [90] and on the way the explain the similar equivalence of 2-categories to torsionfree subcategories in Auslander exact categories.
Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an essentially small exact category. We have the Auslander exact category and the tosionfree subcategory assigned to $\mathcal{E}$
$$\mathrm{AE}(\mathcal{E}) := \mathrm{mod}_{\mathrm{adm}}\,\mathcal{A} \quad \supseteq \quad \mathrm{mod}_{\mathrm{infl}}\,\mathcal{A} =: \mathrm{H}(\mathcal{E})$$
Here H stands for **hereditary** (i.e. gldim ≤ 1)[1]. We endow both with the fully exact substructure restricted from $\mathrm{Mod}\,\mathcal{A}$. As we have no suitable name for it, we call $\mathrm{H}(\mathcal{E})$ the **tf-Auslander category** (tf stands for torsionfree).

Definition 9.1. We call an additive functor between to exact categories $f\colon \mathcal{E} \to \mathcal{F}$ **inflation-preserving** if it maps $\mathcal{E}$-inflation to $\mathcal{F}$-inflation. We call it **left exact** if every $\mathcal{E}$-short

[1] please do not confuse this with the Hall algebra of the exact category

exact sequence (i,p) is mapped to a pair $(f(i), f(p) = j \circ q)$ with $f(i), j$ $\mathcal{F}$-inflations and $q = \operatorname{coker} f(i)$ an $\mathcal{F}$-deflation.

Obviously, every left exact functor is inflation-preserving.

We define the following 2-categories

$$\mathbb{E}\mathbf{x} \subseteq \mathbb{E}\mathbf{x}_L \subseteq \mathbb{E}\mathbf{x}_{\text{inf}}$$

with

(a) $\mathbb{E}\mathbf{x}$ is the 2-category of small exact categories, 1-morphisms are exact functors and 2-morphisms are natural transformations.
(b) $\mathbb{E}\mathbf{x}_L$ is the 2-category of small exact categories, 1-morphisms are left exact functors and 2-morphisms are natural transformations.
(c) $\mathbb{E}\mathbf{x}_{\text{inf}}$ is the 2-category of small exact categories, 1-morphisms are inflation-preserving additive functors and 2-morphisms are natural transformations.

Lemma 9.2. (1) *The assignment* $\text{mod}_{\text{adm}} \colon \mathcal{E} \mapsto \text{AE}(\mathcal{E})$ *defines a* 2*-functor which is left adjoint to* $\mathbb{E}\boldsymbol{x} \subseteq \mathbb{E}\boldsymbol{x}_L$
(2) *The assignment* $\text{mod}_{\text{inf}} \colon \mathcal{E} \mapsto \text{H}(\mathcal{E})$ *defines a* 2*-functor which is left adjoint to the inclusion* $\mathbb{E}\boldsymbol{x} \subseteq \mathbb{E}\boldsymbol{x}_{\text{inf}}$

For (1), look at [**90**, Cor 3.15] and (2) is completely analogue - just observe that the Yoneda embedding into the Auslander exact category is left exact and the Yoneda embedding into the tf-Auslander category is only inflation-preserving (and usually not left exact).
Recall the intrinsic definition of an Auslander exact category

Definition 9.3. An exact category $\mathcal{E}$ is called an **Auslander exact** category if it is an exact category with enough projectives $\mathcal{P}$ such that

(1) $({}^{\perp}\mathcal{P} =: \text{eff}, \text{cogen}(\mathcal{P}) =: \text{H})$ is a torsion pair (here H is the *torsionfree* subcategory)
(2) Every morphism to an object in eff is admissible with image also in this category
(3) $\text{Ext}^1_{\mathcal{E}}(\text{eff}, \mathcal{P}) = 0$
(4) $\text{gldim}\, \mathcal{E} \leq 2$

Now we define the following two 2-categories

(d) $\mathbb{AE}$ is the 2-category of Auslander exact categories with 1-morphsims are exact functors mapping projectives to projectives and 2-morphisms are natural transformations.
(e) $\mathbb{H}$ is the 2-category of tf-Auslander categories with 1-morphisms are a exact functors mapping projectives to projectives and 2-morphisms are natural transformations.

Observe that we have a 2-functor

$$\text{Res} \colon \mathbb{AE} \to \mathbb{H}$$

assigning to an Auslander exact category its torsionfree subcategory. Exact functors preserving projectives restrict to the torsionfree subcategory (as it can be presented as objects which admit an inflation to a projective).

Theorem 9.4. *The* 2*-functors* $\text{mod}_{\text{adm}}, \text{mod}_{\text{inf}}$ *induce equivalences of* 2*-categories*

$$\text{mod}_{\text{adm}} \colon \mathbb{E}\boldsymbol{x}_L \to \mathbb{AE} \qquad \text{mod}_{\text{inf}} \colon \mathbb{E}\boldsymbol{x}_{\text{inf}} \to \mathbb{H}$$

These fit into a diagram which commutes up to

$$\begin{array}{ccc} \mathbb{E}\boldsymbol{x}_L & \xrightarrow{\text{mod}_{\text{adm}}} & \mathbb{AE} \\ {\scriptstyle\subseteq}\downarrow & & \downarrow{\scriptstyle\text{Res}} \\ \mathbb{E}\boldsymbol{x}_{\text{inf}} & \xrightarrow{\text{mod}_{\text{inf}}} & \mathbb{H} \end{array}$$

This is all adapted from [**90**, Thm 4.8]. For the second equivalence of 2-categories, we just define the inverse 2-functor $\mathbb{H} \to \mathbb{E}\mathbf{x}_{\mathrm{inf}}$. On objects assign to a tf-Auslander category H its category of projectives $\mathcal{P}(H)$ and the exact structure such that the inflations are the H-inflations in $\mathcal{P}(H)$.
On morphisms, an exact functor $f\colon H \to H'$ which restricts to projectives, is restricted to projectives $f|_{\mathcal{P}}\colon \mathcal{P}(H) \to \mathcal{P}(H')$ an inflation-preserving functor.
A natural transformation $\Phi\colon f \to g$ between two exact functors which preserve the projectives, restricts to a natural transformation $\Phi|_{\mathcal{P}}\colon f|_{\mathcal{P}} \to g|_{\mathcal{P}}$ between the restricted functors.

9.0.1. *Properties of tf-Auslander algebras.*

Remark 9.5. The category $\mathcal{H} = \mathrm{H}(\mathcal{E})$ is always hereditary exact with enough projectives $\mathcal{P}$. Every object admits a monomorphism (in $\mathcal{H}$) $m\colon X \to P_X$ with P_X in $\mathcal{P}$ such that $\mathrm{Hom}_{\mathcal{H}}(m, P)$ is an isomorphism for all P in $\mathcal{P}$.
If $X = \operatorname{coker} i$ for a $\mathcal{P}$-monomorphism $i\colon P_1 \to P_0$, then a $\mathcal{P}$-cokernel for i is obtained by the composition $p\colon P_0 \to X \to P_X$. This way we find the short exact sequences (i, p) for the exact structure on $\mathcal{P}$.

Observe in a hereditary exact category with enough projectives: If a monomorphism $X \to P$ is an inflation then X has to be projective as well (this follows since the category is hereditary exact). As a consequence we see.

Remark 9.6. If $\mathrm{H}(\mathcal{E})$ is abelian then it is semi-simple and this implies $\mathcal{E}$ is split exact.

We link this with the following concept.

Definition 9.7. ([**82**], Def.1.1) An exact category is called a 0**-Auslander category** if it is a hereditary exact category with enough projectives and for every projective P there exists a short exact sequence

$$P \to I \to X$$

with I projective-injective.
We say an exact category is **torsionfree** 0**-Auslander category** if it is a 0-Auslander category which is also hereditary torsionfree.

Remark 9.8. We recall [**179**], Thm B: Let $\mathcal{Q}$ be a quasi-abelian category and $(\mathcal{T}, \mathcal{F})$ a torsion-pair in $\mathcal{Q}$, then $\mathcal{T}$ and $\mathcal{F}$ are also quasi-abelian.

We also easily deduce the following special cases.

Lemma 9.9. *If $\mathcal{E}$ is abelian then $\mathrm{H}(\mathcal{E})$ is quasi-abelian.*
If $\mathcal{E}$ has enough injectives then $\mathrm{H}(\mathcal{E})$ is a 0-Auslander exact category.

This is particularly interesting as 0-Auslander exact categories have a very strong mutation theory for tilting subcategories, cf. [**82**].

Open question 9.10. We are missing an intrinsic characterization of tf Auslander categories.

10. Appendix C: Ind-Completion of (small) exact categories

These notes are based on the recent preprint of Positselski [**152**] - but we prefer the construction using the Gabriel-Quillen embedding (this way, we extend Crawley-Boeveys classical dictionary to exact structures [**59**]).

Locally finitely presented additive categories. Here, we give a quick summary of the *correspondence* from [**59**].
Let $\mathcal{C}$ an essentially small additive category. We define $\hat{\mathcal{C}} := \mathrm{Mod}-\mathcal{C}$ to be the category of all additive functors $\mathcal{C}^{op} \to (Ab)$, we call this the category of (left) $\mathcal{C}$-**modules**. It is easily seen to be an abelian category.We have the (covariant) Yoneda embedding

$$\mathbb{Y}: \mathcal{C} \to \hat{\mathcal{C}}, \quad X \mapsto (-, X) := \mathrm{Hom}_{\mathcal{C}}(-, X)$$

this is fully faithful, the essential image consists of (some) projective objects which we call **representable functors**.
Every object in $\hat{\mathcal{C}}$ is as a small colimit of representables - for $F \in \hat{\mathcal{C}}$ define the *slice* category $\mathcal{C}/F$ for $F \in \hat{\mathcal{C}}$ (objects: (X, x), $X \in \mathcal{C}, x \in F(X)$, morphisms $f: X \to X'$ in $\mathcal{C}$ such that $F(f)(x') = x$), then we have a small category and a functor $\Phi: \mathcal{C}/F \to \hat{\mathcal{C}}, (X, x) \mapsto (-, X)$. Its colimit is $F = \mathrm{colim}_{\mathcal{C}/F} \Phi$.

Definition 10.1. ([**8**], Expose I) We define the **ind-completion** $\overrightarrow{\mathcal{C}}$ (in the literature denoted as $\mathrm{Ind}(\mathcal{C})$) as the closure of $\mathcal{C}$ under arbitrary directed colimits: Objects are functors $D: I \to \mathcal{C}$ from small filtered categories I. Morphisms are defined as

$$\begin{aligned}\mathrm{Hom}(D: I \to \mathcal{C}, E: J \to \mathcal{C}) &:= \mathrm{Hom}_{\mathrm{Mod}-\mathcal{C}}(\operatorname*{colim}_I \mathbb{Y}D, \operatorname*{colim}_J \mathbb{Y}E) \\ &= \lim_{i \in I} \operatorname*{colim}_{j \in J} \mathrm{Hom}_{\mathcal{C}}(D(i), E(j))\end{aligned}$$

Observe that the Yoneda embedding factors over $\overrightarrow{\mathcal{C}}$. Via the Yoneda embedding, we can identify this with the following full subcategory of $\hat{\mathcal{C}}$

$$\overrightarrow{\mathcal{C}} := \{\operatorname*{colim}_{i \in I}(-, X_i) \mid \quad (X_i)_{i \in I}\ I-\text{shaped diagram in } \mathcal{C} \text{ with } I \text{ directed set } \}$$

Remark 10.2. The second description uses that closure under small filtered colimits it the same as closure under small directed colimits, cf. [**3**], Thm 1.5.

Proposition 10.3. *([**8**], Expose I, Prop. 8.6.4) Ind-completion is a 2-functorial.*
An additive functor f is faithful (resp. fully faithful) if and only if $\overrightarrow{f}$ is faithful (resp. fully faithful). Furthermore the ind-completion $\overrightarrow{f}$ of an additive functor $f: \mathcal{C} \to \mathcal{D}$ is an equivalence if and only it is fully faithful and the essential image inclusion $\mathrm{Im}\, f \to \mathcal{D}$ induces an equivalence on idempotent completions.

Let $\mathcal{D}$ be an additive category, we denote by $\mathrm{Add}(\mathcal{C}, \mathcal{D})$ the category of additive functors from $\mathcal{C}$ to $\mathcal{D}$ and $\mathrm{Add}_{fc}(\mathcal{C}, \mathcal{D})$ the subcategory of functors which preserve directed colimits (whenever these exist).

Ind-completion can be defined for arbitrary additive and even arbitary categories categories and can be characterized by a universal property such as:

Lemma 10.4. *(**Universal property of ind-completion**, cf. [**8**], Expose I, Prop. 8.7.3) Let $\mathcal{C}$ be a small additive category, then $\overrightarrow{\mathcal{C}}$ has all directed colimits.*
Assume that $\mathcal{D}$ is an additive category which is closed under arbitrary directed colimits. Precomposition with $\mathcal{C} \to \overrightarrow{\mathcal{C}}$ is an equivalence of categories

$$\mathrm{Add}_{fc}(\overrightarrow{\mathcal{C}}, \mathcal{D}) \to \mathrm{Add}(\mathcal{C}, \mathcal{D})$$

Furthermore, it also has the following property

Lemma 10.5. *([**59**], Lem. 1) $\overrightarrow{\mathcal{C}}$ is idempotent complete.*

For small additive categories, we have an alternative description of the ind-completion found in [**59**].

Definition 10.6. Let $\mathcal{C}$ be a small additive category. We say that an object F in $\hat{\mathcal{C}}$ is **flat** if the tensor functor $F \otimes_{\mathcal{C}} -: \hat{\mathcal{C}^{op}} \to (Ab)$ is exact. We denote by $\mathrm{Flat}(\mathcal{C}^{op}, Ab)$ the full subcategory of flat functors.

Theorem 10.7. *([59], Thm p. 1646)*
$\overrightarrow{\mathcal{C}} = \mathrm{Flat}(\mathcal{C}^{op}, Ab)$ *and* $F \in \overrightarrow{\mathcal{C}}$ *is equivalent to*

(1) $\mathcal{C}/F$ *is filtered*
(2) *Every natural transformation* $\mathrm{coker}(-, f) \to F$ *factors over a representable.*

Definition 10.8. Let $\mathcal{A}$ be an additive category. We say an object X in $\mathcal{A}$ is **finitely presented** if $\mathrm{Hom}_{\mathcal{A}}(X, -)$ commutes with arbitrary filtered colimits. We denote by $\mathrm{fp}(\mathcal{A})$ the full subcategory of finitely presented objects in $\mathcal{A}$.
The additive category $\mathcal{A}$ is called **locally finited presented** if $\mathrm{fp}(\mathcal{A})$ is essentially small and $\mathcal{A}$ is equivalent to $\overrightarrow{\mathrm{fp}(\mathcal{A})}$.

Remark 10.9. If $\mathcal{A}$ is locally finitely presented then $\mathrm{fp}(\mathcal{A})$ is essentially small, closed under direct sums and summands. In particular by Lemma 4.2, it is idempotent complete.

Lemma 10.10. *(cf. [59], part of Thm on p.1647)*
For an essentially small category $\mathcal{C}$ *we have* $\mathrm{fp}(\overrightarrow{\mathcal{C}}) \cong \mathcal{C}^{ic}$ *is equivalent to the idempotent completion of* $\mathcal{C}$.

Theorem 10.11. *([59], Thm. in (1.2), p.1645) If* $\mathcal{C}$ *is essentially small, then*

$$\mathrm{fp}(\hat{\mathcal{C}}) = \{F \in \hat{\mathcal{C}} \mid F \cong \mathrm{coker}(-, f),\ f \in \mathrm{Mor}(\mathcal{C})\} \quad =: \mathrm{mod}_1 \mathcal{C}$$

Furthermore, $\hat{\mathcal{C}}$ *is locally finitely presented.*

Example 10.12. Locally finitely presented abelian categories are **Grothendieck categories** (i.e. (1) abelian, (2) with arbitrary small coproducts, (3) directed colimits are exact, (4) has a generating object G). Here, the generator can be chosen as

$$G = \bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} (-, C) \quad \in \overrightarrow{\mathcal{C}}$$

For the converse: If a Grothendieck category admits a set of finitely presented objects whose coproduct is a generator, then it is locally finitely presented.
Grothendieck categories always have enough injectives (often they are hard to find), have arbitrary small limits and colimits.

Remark 10.13. For a not necessarily small category $\mathcal{C}$ we can still define its ind-completion. If $\mathcal{C}$ is abelian, this is an abelian category - but it may not have enough injectives (cf. [114], Prop. 15.1.2).

The following is a consequence of Lem. 10.4 together with [59], Thm in (1.4), p. 1647.

Theorem 10.14. *(equivalence of (2-)categories)*
The assignments $\mathcal{C} \mapsto \overrightarrow{\mathcal{C}}$ *and* $\mathcal{A} \mapsto \mathrm{fp}(\mathcal{A})$ *are* $2-$*functorial and give rise to an equivalence of (strict)* $2-$*categories between*

(1) *essentially small, idempotent complete additive categories* $\mathcal{C}$ *with additive functors*
(2) *Locally finitely presented additive categories* $\mathcal{A}$ *with additive functors that preserve arbitrary filtered colimits and restrict to the subcategories of finitely presented functors.*

Let $\mathcal{C}$ be idempotent complete, essentially small additive category and $\mathcal{A}$ a locally finitely presented (additive) category. We assume $\mathcal{C} = \mathrm{fp}(\mathcal{A})$ and $\mathcal{A} = \overrightarrow{\mathcal{C}}$. Then the following holds (by restricting further and further):

(i) $\mathcal{C}$ left abelian $\Leftrightarrow$ $\mathcal{A}$ abelian
the definition of **left abelian** (cf. [59], (2.4)): Every morphisms has a cokernel, every epi is a cokernel and whenever $A \xrightarrow{f} B \xrightarrow{c} C$ with $c = \mathrm{coker}(f)$ and $g \colon D \to B$, $cg = 0$ then there exists an epi $d \colon E \to D$ such that gd factors over f.
(ii) $\mathcal{C}$ abelian $\Leftrightarrow$ $\mathcal{A}$ locally coherent

(iii) $\mathcal{C}$ abelian and all objects noetherian $\Leftrightarrow$ $\mathcal{A}$ locally noetherian abelian
(iv) $\mathcal{C}$ is length abelian $\Leftrightarrow$ $\mathcal{A}$ is locally finite abelian

Example 10.15. R a ring:
(i) $R-\mathrm{Mod}$ abelian and $R-\mathrm{mod}_1$ is left abelian
(ii) for R left coherent
(iii) for R left noetherian
(iv) e.g. for R left artinian (with Loewy length)

10.1. Gabriel Quillen embedding. We review this well-known embedding of an essentially small exact category as a fully exact subcategory in an abelian category.
Let $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ be an essentially small exact category.

Definition 10.16. We define the category $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ to be the category of all additive functors $F: \mathcal{C}^{op} \to (Ab)$ which map short exact sequences $X \xrightarrow{i} Y \xrightarrow{d} Z$ in $\mathcal{E}$ to left exact sequences $0 \to F(Z) \xrightarrow{F(d)} F(Y) \xrightarrow{F(i)} F(X)$ in abelian groups. We will call this the category of **left exact functors on $\mathcal{E}$**. We define the category of **locally effaceble functors** $\mathrm{Eff}_{\mathcal{E}}$ to be the full subcategory of $\hat{\mathcal{C}}$ of objects F such that for every pair (X, x) of an object X in $\mathcal{C}$ and $x \in F(X)$ there exists an $\mathcal{E}$-deflation $d: Z \to X$ with $F(d)(x) = 0$.

Lemma 10.17. [126], *Prop. 2.3.7 (1),(2) (with intermediate steps)*

(i) *(Prop 2.2.16) $\mathcal{D} = \mathrm{Eff}_{\mathcal{E}}$ is a Serre subcategory of $\hat{\mathcal{C}}$ closed under coproducts. Therefore, the Serre quotient functor $Q\colon \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}/\mathcal{D}$ admits a right adjoint.*
(ii) *(Lem. 2.2.10) Let Q_ρ be the right adjoint. It factors as $\hat{\mathcal{C}}/\mathcal{D} \xrightarrow{\Phi} \mathcal{D}^\perp \xrightarrow{I} \hat{\mathcal{C}}$ with Φ an equivalence of categories and I the inclusion functor. The quasi-inverse of Φ is given by $Q \circ I$.*
(iii) $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}) = \mathcal{D}^\perp := \{\mathrm{Y} \in \hat{\mathcal{C}} \mid \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{C}}(\mathrm{E}, \mathrm{Y}) = 0 = \mathrm{Ext}^1_{\mathrm{Mod}\,\mathcal{C}}(\mathrm{E}, \mathrm{Y}) \forall \mathrm{E} \in \mathcal{D}\}$.

Remark 10.18. $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ is a Grothendieck category (as it is the localization of a Grothendieck category by a Serre subcategory?). In the abelian structure it has a generator

$$G = \bigoplus_{X \in \mathrm{Ob}(\mathcal{C})} (-, X)$$

As $(-, X)$ are (some) finitely presented objects in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, it follows that $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$ is locally finitely presented abelian.
It also has an exact substructure as fully exact category in $\hat{\mathcal{C}}$ but these two exact structures usually do not coincide.

Remark 10.19. The inclusion $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}) \to \hat{\mathcal{C}}$ is not an exact functor (if we consider $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ with its abelian structure). Yet it reflects exactness in the following sense: If $0 \to F \to G \to H \to 0$ in $\hat{\mathcal{C}}$ with F, G, H in $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$, then this is a short exact sequence in $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.

Corollary 10.20. $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}) \subseteq \hat{\mathcal{C}}$ *is a deflation-closed subcategory: Given a short exact sequence* $0 \to F \to G \to H \to 0$ *in* $\mathrm{Mod}\,\mathcal{A}$.
If $G, H \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$*, then also* $F \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.
If $F, G \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ *and* $\mathrm{Ext}^2(E, F) = 0$ *for all E effaceable, then* $H \in \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.

In general we can characterize short exact sequences in $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ (in the abelian structure) as follows:

Lemma 10.21. *Given two composable maps* $0 \to F \xrightarrow{i} G \xrightarrow{p} H \to 0$ *in* $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$. *TFAE*

(1) *(i, p) are an exact sequence in* $\mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$
(2) *In $\hat{\mathcal{C}}$ we have a exact sequence $0 \to F \xrightarrow{i} G \xrightarrow{p} H$ and* $\mathrm{coker}(p)$ *is locally effaceable.*

THEOREM 10.22. *Let $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ be an exact category. The Yoneda functor gives a functor*

$$i\colon \mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab}), \quad \mathrm{X} \mapsto (-, \mathrm{X}) = \mathrm{Hom}_{\mathcal{C}}(-, \mathrm{X})$$

with the following properties

(1) *i is exact, reflects exactness and the essential image of i is extension-closed, its idempotent completion is deflation-closed.*
(2) *i induces isomorphisms on all extension-groups.*

PROOF. (1) [**49**], Appendix A and [**126**], Prop. 2.3.7.(3). The last statement follows from [**152**], Prop. 6.1, (e).
(2) [**126**] in Lem 4.2.17 it is shown that $\mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ is right cofinal (Keller's definition) that implies the statement. □

The following result is also relevant for us:

Proposition 10.23. *([**152**], Prop. 6.2) For every X in the category $\mathcal{E}$, the functor $\mathrm{Ext}^n_{\mathrm{Lex}(\mathcal{E}^{op}, Ab)}((-, X), -)$ preserves filtered colimits.*

10.2. Locally coherent exact. The ind-completion of a small exact category $\mathcal{E}$ has a natural exact structure (namely as fully exact in the Gabriel-Quillen embedding), this exact structure can also be described as directed colimits of short exact sequences in $\mathcal{E}$ and is called **locally coherent exact structure**.
The CB-correspondence (section 1) is extended to i.c. small exact categories.

We begin with the following observation.

Lemma 10.24. *Let $\mathcal{C}$ be a small additive category, F a flat functor and $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g = \mathrm{coker}(f)$ in $\mathcal{C}$ then $0 \to F(Z) \to F(Y) \to F(X)$ is exact in abelian groups.*

PROOF. Define $(X, -) := \mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to (Ab)$ and the contravariant Yoneda embedding $\mathcal{C}^{op} \to \hat{\mathcal{C}^{op}}, X \mapsto (X, -)$. By assumption we have an exact sequence $0 \to (Z, -) \to (Y, -) \to (X, -)$. Since F is flat, the functor $F \otimes_{\mathcal{A}}$ is exact and we have $F \otimes_{\mathcal{C}} (E, -) \cong F(E)$. Therefore, we obtain the exact sequence $0 \to F(Z) \to F(Y) \to F(X)$. □

Now for an exact category $\mathcal{E} = (\mathcal{C}, \mathcal{S})$, by the previous lemma $\mathrm{Flat}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \subseteq \mathrm{Lex}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$.

Lemma 10.25. *(and definition.) $\mathrm{Flat}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is closed under extensions in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$.*
*We define $\overrightarrow{\mathcal{E}}$ to be the fully exact structure on $\overrightarrow{\mathcal{C}}$ and call this the **ind-completion of the exact category $\mathcal{E}$**.*

PROOF. (sketch) (of Lemma 10.25) Given a short exact sequence $\sigma\colon F \to G \to H$ in (the abelian structure on) $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$ with F, H flat.
First assume $H = (-, X)$, write F as filtered colimit and use Prop 10.23 and the fact that the essential image of $\mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{op}, Ab)$ is extension-closed to conclude the claim.
In general, we use Thm 10.7, (2). Given a morphism $\theta\colon \mathrm{coker}(-, f) \to G$. Postcompose to $\mathrm{coker}(-, f) \to H$. As H is flat, this factors over a morphism $g\colon (-, X) \to H$ for some X in $\mathrm{Ob}(\mathcal{E})$. Now form the pull-back of σ along g in the abelian category $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, say this is a short exact sequence $F \to E \to (-, X)$. The universal property of the pull-back gives a morphism $\theta'\colon \mathrm{coker}(-, f) \to E$ and a morphism $u\colon E \to G$ with $\theta = u\theta'$. By the first case θ' factors over a representable, therefore θ does so too. □

Remark 10.26. (and definition) Let $\mathcal{E} = (\mathcal{C}, \mathcal{S})$ be an essentially small exact category, then $\mathcal{E}$ is fully exact in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$. As the latter is abelian, it is idempotent complete, therefore $\mathcal{E}^{ic}$ is also a fully exact subcategory in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$.

But this means $\mathrm{fp}(\overrightarrow{\mathcal{C}})$ is an extension-closed subcategory in $\overrightarrow{\mathcal{E}}$. We define $\mathrm{fp}(\overrightarrow{\mathcal{E}})$ to be the fully exact structure on $\mathrm{fp}(\overrightarrow{\mathcal{C}})$.

Definition 10.27. Let $\mathcal{F} = (\mathcal{A}, \mathcal{T})$ be an exact category. We say it is **locally coherent exact** if $\mathrm{fp}(\mathcal{A})$ is essentially small and extension-closed in $\mathcal{F}$ -in this case we denote by $\mathrm{fp}(\mathcal{F})$ the fully exact subcategory- and $\mathcal{F} = \overrightarrow{\mathrm{fp}(\mathcal{F})}$.

In an ess. small exact category $\mathcal{E}$, the category of short exact sequences $\mathrm{Ses}(\mathcal{E})$ is again an essentially small exact category (with degree-wise short exact sequences).

THEOREM 10.28. *([152], proof of Lemma 1.2) A filtered colimit of short exact sequences in $\overrightarrow{\mathcal{E}}$ is a short exact sequence in $\overrightarrow{\mathcal{E}}$.*
The universal property of the ind-completion yields an equivalence of categories

$$\overrightarrow{\mathrm{Ses}(\mathcal{E})} \to \mathrm{Ses}(\overrightarrow{\mathcal{E}})$$

We also observe the following:

Lemma 10.29. *For an essentially small exact category $\mathcal{E}$, we have that $\overrightarrow{\mathcal{E}}$ is a resolving subcategory in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, in particular it is homologically exact.*

PROOF. $\overrightarrow{\mathcal{E}}$ is extension-closed, idempotent complete and contains a generator of $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, so it is enough to see that it is also deflation-closed. Now given a short exact sequence $F \to G \to H$ in $\mathrm{Lex}(\mathcal{E}^{op}, Ab)$, it gives a 4-term exact sequence in $\hat{\mathcal{C}}$: $0 \to F \to G \to H \to E \to 0$ withe E locally effaceble. Assume that G, H are flat, we want to see that E is so too. As every finietly presented effaceable functor has projecive dimension 2, every locally effaceable has flat dimension 2 and this implies F is flat. □

Furthermore, since $(-, X) \in \mathrm{Flat}(\mathcal{E}^{\mathrm{op}}, \mathrm{Ab})$ for all X in $\mathcal{C}$, we get a fully faithful exact functor with extension-closed essential image

$$i \colon \mathcal{E} \to \overrightarrow{\mathcal{E}}, \quad X \mapsto (-, X)$$

Lemma 10.30. *The functor i is homologically exact.*

PROOF. As $\mathcal{E} \to \mathrm{Lex}(\mathcal{E}^{op}, Ab)$ is homologically exact and also $\overrightarrow{\mathcal{E}} \to \mathrm{Lex}(\mathcal{E}^{op}, Ab)$, this is immediate. □

We directly get the following from the previous corollary and Prop. 10.23.

Corollary 10.31. *For X in $\mathrm{Ob}(\mathcal{E})$, the functor $\mathrm{Ext}^n_{\overrightarrow{\mathcal{E}}}((-, X), -)$ commutes with filtered colimits.*

Let $\mathrm{Ex}(\mathcal{E}, \mathcal{F})$ be the category of exact functors between two exact categories and $\mathrm{Ex}_{fc}(\mathcal{E}, \mathcal{F})$ be the full subcategory of exact functors which preserve filtered colimits.

Lemma 10.32. *(**Universal property of ind-completion for exact categories**) Let $\mathcal{E}$ be an essentially small exact category. Then $\overrightarrow{\mathcal{E}}$ is closed under directed colimits and directed colimits are exact functors.*
Let $\mathcal{F}$ be an exact category closed under all directed colimits and they are exact functors. Then precomposition $\mathcal{E} \to \overrightarrow{\mathcal{E}}$ gives an equivalence

$$\mathrm{Ex}_{fc}(\overrightarrow{\mathcal{E}}, \mathcal{F}) \to \mathrm{Ex}(\mathcal{E}, \mathcal{F})$$

THEOREM 10.33. *(equivalence of 2-categories)*
The assignments $\mathcal{E} \mapsto \overrightarrow{\mathcal{E}}$ and $\mathcal{F} \mapsto \mathrm{fp}(\mathcal{F})$ are functorial and give rise to an equivalence of $(2-)$categories between

(1) *essentially small, idempotent complete exact categories $\mathcal{E}$ with exact functors*
(2) *Locally coherent exact categories $\mathcal{F}$ with exact functors that preserve arbitrary filtered colimits and restrict to the subcategories of finitely presented functors.*

We remark that a functor that preserves filtered colimits on objects, also preserves filtered colimits on morphism categories and this implies it also preserves filtered colimits of short exact sequences. This implies that as exact functor between essentially small exact categories $F\colon \mathcal{E}_1 \to \mathcal{E}_2$ extends with the universal property of the ind-completion uniquely to an exact functor $\overrightarrow{F}\colon \overrightarrow{\mathcal{E}_1} \to \overrightarrow{\mathcal{E}_2}$, this is a consequence of Thm 10.28. Recall: F fully faithful if and only $\overrightarrow{F}$ is fully faithful by Prop. 10.3.

Ignoring set-theory for a moment: Let $\mathcal{F}$ be an exact category, we consider $\mathrm{EX}(\mathcal{F})$ the lattice of all exact subcategories. For $\mathcal{F}$ locally coherent exact, we define $\mathrm{EX}_{fc}(\mathcal{F})$ to be the exact subcategories in the category (2) above, i.e. exact functors $i\colon \mathcal{F}' \to \mathcal{F}$ such that i is fully faithful, $\mathcal{F}'$ is locally coherent exact and i preserves filtered colimits.

Corollary 10.34. *Let $\mathcal{E}$ be an essentially small category. We have mutually inverse, isomorphisms of posets*

$$\overrightarrow{(-)} \quad : \mathrm{EX}(\mathcal{E}) \leftrightarrow \mathrm{EX}_{fc}(\overrightarrow{\mathcal{E}}): \quad \mathrm{fp}(-)$$

It restricts to all the usual subposets such as extension-closed, exact substructures etc.

Positselski found the maximal and minimal locally coherent exact structure on a locally finitely presented category particular interesting. The minimal exact structure is the ind-completion of the split exact structure (on an essentially small additive category) and is called **pure exact structure** on a locally finitely presented category.

Example 10.35. Let R be a ring. The abelian exact structure on $R-\mathrm{Mod}$ is the maximal locally coherent exact structure, it corresponds to the left abelian structure on $R-\mathrm{mod}_1$ (fp R-modules). The category of flat R-modules $R-\mathrm{Mod}_{fl}$ is extension-closed in $R-\mathrm{Mod}$. Its subcategory of finitely presented objects is $(\mathrm{add}(R))^{ic}$ with the split exact structure is the fully exact substructure. By a Thm of Govorov-Lazard, $\overrightarrow{\mathrm{add}(R)} = R-\mathrm{Mod}_{fl}$. In this case: The fully exact structure is the pure exact structure.

Stovicek generalized the notion of a Grothendieck category to an exact category of Grothendieck type.

THEOREM 10.36. *([**152**, Cor. 5.4]) Locally coherent exact categories are exact categories of Grothendieck type (in the sense of Stovicek).*

In particular, all established properties of exact categories of Grothendieck type hold true.

Corollary 10.37. *(also [**152**, Cor. 5.4]) $\overrightarrow{\mathcal{E}}$ has enough injectives.*

This implies that the unbounded derived category $\mathrm{D}(\overrightarrow{\mathcal{E}})$ is locally small (i.e. has Hom-sets), cf. Chapter 6.

11. Open problems

Here is my personal (naive) list of problems

(0) Describe $\overrightarrow{\mathcal{E}}$ for exact categories of the form $\mathrm{mod}_S\,\mathcal{M}$ and for Auslander-Soberg exact structures.
(1) If $\mathcal{E}$ has enough projectives/injectives what are the corresponding properties in $\overrightarrow{\mathcal{E}}$?
(2) Which conditions on the exact category imply $\mathrm{gldim}(\mathcal{E}) = \mathrm{gldim}(\overrightarrow{\mathcal{E}})$?

(3) When is $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\overrightarrow{\mathcal{E}})$ fully faithful? (for some answers, cf [**153**])
(4) Are there situations when derived equivalence is preserved/reflected by ind-completion of exact categories?

11.1. Literature.

11.1.1. *For ind-completion.* Ind-categories have been introduced for arbitrary categories by Grothendieck in [**84**], and more thoroughly studied by Grothendieck-Verdier in [**8**], Expose I. The concept of *Finitely presented/presentable categories* is due to [**79**].
In [**59**] it had been observed that in the *additive category*-setup, ind-completion for small additive functors can be realized as categories of flat functors.
A (multiply) more general approach can be found in [**3**], where the more general analogue of finitely presented categories is called finitely accessable categories.

11.1.2. *For the Gabriel-Quillen embedding.* References: Bühler *exact categories*, Appendix A contains a historical discussion of the origins. Further reference [**126**] and [**152**], section 5.

11.1.3. *For locally coherent exact categories.* This has been introduced in [**152**]. In the special case that $\mathcal{C}$ has weak cokernels an alternative construction is given using the embedding into the purity category by [**173**].

CHAPTER 3

The posets of exact subcategories

1. Synopsis

This is the quest to extend known descriptions of the lattice of exact structures on a given additive category to the much bigger lattice of all exact subcategories.
What is new? This question is usually not considered, so everything in this chapter.

2. Introduction

Now, we fix one essentially small, idempotent complete exact category $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ and introduce the following four posets of exact subcategories (the first poset is just auxiliary):

$$\boxed{\begin{aligned} \mathrm{ADD}(\mathcal{A}) = &\text{ all additively closed subcategories } \mathcal{A}' \text{ of } \mathcal{A} \\ \mathrm{EX}(\mathcal{E}) = &\text{ exact categories } (\mathcal{A}', \mathcal{S}') \text{ such that } \mathcal{A}' \in \mathrm{ADD}(\mathcal{A}) \\ &\text{with } (\mathcal{A}', \mathcal{S}') \subseteq (\mathcal{A}, \mathcal{S}) \text{ is an exact functor} \\ \mathrm{Ex}(\mathcal{E}) = &\,(\mathcal{A}', \mathcal{S}') \in \mathrm{EX}(\mathcal{E}) \text{ such that } \mathcal{A}' \text{ is extension-closed in } \mathcal{E} \\ \mathrm{ext}(\mathcal{E}) = &\text{ fully exact subcategories } \mathcal{A}' \text{ in } \mathrm{ADD}(\mathcal{A}) \\ \mathrm{ex}(\mathcal{E}) = &\text{ exact substructures of } \mathcal{E} \end{aligned}}$$

here, *fully exact* extension-closed means that the exact structure $\mathcal{S}'$ on $\mathcal{A}'$ is all kernel-cokernel pairs (i, p) in $\mathcal{A}'$ such that $(i, p) \in \mathcal{S}$.
Observe that $\mathrm{Ex}(\mathcal{E})$ contains $\mathrm{ext}(\mathcal{E})$ and $\mathrm{ex}(\mathcal{E})$.
We have the following operation on $\mathrm{EX}(\mathcal{E})$

$$(\mathcal{A}', \mathcal{S}') \wedge (\mathcal{A}'', \mathcal{S}'') := (\mathcal{A}' \cap \mathcal{A}'', \mathcal{R} = \{\sigma \in \mathcal{S}' \cap \mathcal{S}'' \mid \text{ all three objects are in } \mathcal{A}' \cap \mathcal{A}''\})$$

with respect to this operation, $\mathrm{EX}(\mathcal{E})$ becomes a complete meet-semilattice and all three $\mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E}), \mathrm{ex}(\mathcal{E})$ are closed under this operation (making them complete meet-subsemilattices).
If a complete meet-semilattice $(X, \leq, \wedge)$ has a unique maximal element then one can define a join such that it becomes a complete lattice, so given a subset $\{x_i \mid i \in I\}$ of X its join is given by

$$\bigvee_{i \in I} x_i := \bigwedge_{y\colon x_i \leq y \forall i \in I} y.$$

Observe that the joins obtained this way for $\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E})$ usually differ.

We first make the following easy observation.

THEOREM 2.1. *(cf. Thm 3.3)* $\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E})$ *are complete lattices and* $\mathrm{ex}(\mathcal{E})$ *is a complete sublattice of* $\mathrm{EX}(\mathcal{E})$ *and of* $\mathrm{Ex}(\mathcal{E})$.

Open question 2.2. Considering the bijection between Ziegler-closed subsets (containing a given closed set) and exact structures on an idempotent complete small additive category with weak cokernels (cf. Chapter 2), we ask: Are the opposite lattices frames? Are they even coherent frames? (see e.g. [**129**] for the definitions)

As a corollary of Rump (cf. in Chapter 1, Cor. 4.4) we deduced that for every additive functor between small exact categories $f\colon \mathcal{F} \to \mathcal{E}$ there is a unique maximal exact substructure of $\mathcal{F}$ such that f becomes an exact functor on it.
Let now $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be a small exact category and $i\colon \mathcal{B} \subseteq \mathcal{A}$ a full additively closed subcategory. We write $\mathcal{S} \cap \mathcal{B}$ for the subset of $\mathcal{S}$ given by all short exact sequences such with all three objects in $\mathcal{B}$ (also denoted by $\mathcal{S}_i$ in Chapter 1).
Let $\mathcal{F} = \mathcal{F}_{max}$ the maximal exact structure on $\mathcal{B}$ and we look at the inclusion functor $i\colon \mathcal{F} \to \mathcal{E}$, then we denote by

$$\mathcal{F}_{\mathcal{B}} := (\mathcal{B}, \mathcal{S}^{\leq (\mathcal{S} \cap \mathcal{B})})$$

the maximal exact structure on $\mathcal{B}$ such that i is an exact functor on it.

Then our second result is the following simple corollary of this:

THEOREM 2.3. . *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an idempotent complete essentially small exact category. Then we have equalities of sets*

$$\mathrm{EX}(\mathcal{E}) = \bigsqcup_{\mathcal{B} \in \mathrm{ADD}(\mathcal{A})} \mathrm{ex}(\mathcal{F}_{\mathcal{B}})$$

$$\mathrm{Ex}(\mathcal{E}) = \bigsqcup_{\mathcal{E}' \in \mathrm{ext}(\mathcal{E})} \mathrm{ex}(\mathcal{E}')$$

To our knowledge, usually either only the lattice of exact structures is studied or a chosen subposet of $\mathrm{ext}(\mathcal{E})$ assuming extra-properties (such as Serre subcategories, torsion classes, thick subcategories, wide subcategories, resolving subcategories, tilting subcategories etc.).

We are interested in the following questions

(1) Explicit descriptions
(2) Lattice isomorphisms for $\mathrm{EX}(\mathcal{E})$
(3) Can we find homological properties which are preserved under forming meets?

In neither case we claim to have a good answer but we give some partial answers.
Representation-finite means *here*:
Krull-Schmidt, K-linear (for some field K), Hom-finite with only finitely many indecomposables.
The first task is even in the representation-finite complicated because of the size of the constructed lattice, see e.g. some enumerative example in the end. We remark that the Ziegler spectrum is not functorial and even in the representation-finite case we do not see how we can use it here. In this case the cogenerators (whose indecomposable summands) give Ziegler-closed subsets in the Ziegler spectrum of $\mathcal{B} \in \mathrm{ADD}(\mathcal{A})$ are those which contain the cogenerator from the dual of Lemma 6.1.

For the second task, we look again at Auslander's functorial point of view (cf. Chapter 2). We extend the three lattice isomorphisms from $\mathrm{ex}(\mathcal{E})$ to the whole lattice $\mathrm{EX}(\mathcal{E})$: Using the Auslander category, the tf Auslander category and the category of effaceble functors. This result is Theorem 4.15.

For the third question:
Is a given *homological*[1] condition preserved under taking meet in $\mathrm{EX}(\mathcal{E})$?
We have no systematic way of studying this, we just collect some answers (if you know more please let me know).
Some negative answers:

(1) homologically exactness (and also homologically faithfulness)
(2) gldim $= n$ (same fixed n)
(3) having enough projectives (or injectives)
(4) ((2) for $n = 1$ but more specifically:) hereditary exact substructures in an hereditary exact category

[1] i.e. defined by imposing conditions on the Ext-functors.

Also, the subposet of hereditary exact substructures may not have a unique maximal element.
Some positive answers:
This is what I found:

(1) 1-homologically exact (i.e. extension-closed).
Special case: If $\mathcal{E}$ is hereditary exact, all extension-closed subcategories are homologically exact (and also hereditary exact)
(2) n-rigid subcategories (all with the same n).
Special case: self-orthogonal subcategories (n-rigid for all n)
(3) resolving subcategories in an exact category with enough projectives
(4) (only wrt. finite meets!) exact substructures with enough projectives in an exact structure with enough projectives

There are probably many more. We also give the following examples (without further applications). Given an exact fully faithful functor $f\colon \mathcal{F} = (\mathcal{B}, \mathcal{S}') \to \mathcal{E}$ We say that the $f - \operatorname{gldim} \mathcal{F} \leq n$ if $\operatorname{Ext}^n_{\mathcal{E}}(f(X), f(\sigma))$ is right exact for all objects X in $\mathcal{B}$ and $\mathcal{F}$-exact short exact sequences σ.
This can be seen as a relative version of gldim, since for $f = \mathrm{id}\colon \mathcal{E} \to \mathcal{E}$ we have $f - \operatorname{gldim} \mathcal{E} \leq n$ iff $\operatorname{gldim} \mathcal{E} \leq n$ by Lemma 5.4.
We observe the following completely obvious:

Lemma 2.4. *Let $\mathcal{E}$ be essentially small idempotent complete exact category and $\mathcal{F}_j = (\mathcal{B}_j, \mathcal{S}_j)$ be in* $\mathrm{EX}(\mathcal{E})$*, $j \in J$. Then:*
If all $\mathcal{F}_j$ have relative global dimension $\leq n$ for all $j \in J$, then also $\bigwedge_{j \in J} \mathcal{F}_j$ has relative global dimension $\leq n$.

Here is another property one may consider: Let $\mathcal{E}$ be an exact category. Contravariantly-finite $\mathcal{E}$-generators $\mathcal{G}$ (i.e. for every object X in $\mathcal{E}$ there is an $\mathcal{E}$-deflation $d_X\colon G_X \to X$ which is also a right $\mathcal{G}$-approximation) which are also $\mathcal{E}$-subobject-closed (i.e. given an $\mathcal{E}$-exact sequence $X \rightarrowtail G \twoheadrightarrow Y$ with $G \in \mathcal{G}$, then X is also in $\mathcal{G}$) induce always hereditary exact substructures with enough projectives (by looking at the exact substructure such that $\operatorname{Hom}(G, -)$ becomes exact for all $G \in \mathcal{G}$). These give usually not all hereditary exact substructures. But we observe that $\mathcal{E}$-subobject-closedness is preserved under arbitrary intersections (but we do not know when such an intersection is contravariantly finite!).

Lemma 2.5. *Let $\mathcal{E}$ be an exact category such that the underlying additive category is representation-finite (see above). Then all generators are contravariantly finite. The set of all hereditary exact substructures with enough projectives given by an $\mathcal{E}$-subobject-closed generator is closed under taking arbitrary joins in* $\mathrm{EX}(\mathcal{E})$.

Proof. Given $\mathcal{G}_i$ subobject-closed generators with $\mathcal{E}_i$ corresponding exact substructures such that $\mathcal{P}(\mathcal{E}_i) = \mathcal{G}_i$. As we are in a finite-type situation the join is $\bigvee_i \mathcal{E}_i = \mathcal{F}$ with $\mathcal{P}(\mathcal{F}) = \bigcap \mathcal{G}_i =: \mathcal{G}$ which is again a subobject-closed generator. □

Example 2.6. Let $\mathcal{E} = \Lambda_n \operatorname{mod}$ with Λ_n the path algebra over $1 \to 2 \to \cdots \to n$. In this case, the subobject-closed generators are not only closed under intersections but also by taking direct sums and form a sublattice of $\mathrm{ex}(\mathcal{E})$. This is in bijection with subobject-closed subcategories in $\Lambda_{n-1} \operatorname{mod}$. In general, subobject-closed subcategories in $\Lambda \operatorname{mod}$ with Λ a Dynkin quiver has been studied, explicit bijections to the elements of the Weyl group of the corresponding (simply-laced) Dynkin-type have been found in [**146**].

What I do not know:
In the following we do not know the answer (cf. Chapter 1 for the definitions): Left (or right) cofinal, (co)resolving (in general exact category), partially (co)resolving

3. The lattice of exact subcategories in an exact category

Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category.
We will always assume $\mathcal{A}$ is idempotent complete to avoid confusion on the definition of an exact category. We also will assume that $\mathcal{A}$ is essentially small (else we would have to generalize the notion *poset* from sets to classes).
Let ADD($\mathcal{A}$) be the collection of all additively closed subcategories of $\mathcal{A}$ (this is a complete lattice). Observe that additively closed subcategories are also idempotent complete.
For $\mathcal{A}' \in \mathrm{ADD}(\mathcal{A})$ we denote bu $i_{\mathcal{A}',\mathcal{A}}$ (or if there is no confusion, just i) for the inclusion $\mathcal{A}' \subseteq \mathcal{A}$.

Definition 3.1. Now we define $\mathrm{EX}(\mathcal{E})$ as the collection of exact categories $(\mathcal{A}', \mathcal{S}')$ with $\mathcal{A}' \in \mathrm{ADD}(\mathcal{A})$ and $i_{\mathcal{A}',\mathcal{A}}$ an exact functor (wrt. $(\mathcal{A}', \mathcal{S}')$ and $(\mathcal{A}, \mathcal{S})$). We call this the **poset of exact subcategories** of $\mathcal{E}$.
We also consider $\mathrm{Ex}(\mathcal{E}) \subseteq \mathrm{EX}(\mathcal{E})$ consisting of all $(\mathcal{A}', \mathcal{S}')$ such that $\mathcal{A}'$ is also extension-closed in $\mathcal{E}$. This implies that $(\mathcal{A}', \mathcal{S}')$ is an exact substructure on $(\mathcal{A}', \mathcal{S} \cap \mathcal{A}')$. We call this the **poset of extension-closed exact subcategories** of $\mathcal{E}$.

In the literature two subposets of $\mathrm{Ex}(\mathcal{E})$ are studied:

(1) $\mathrm{ext}(\mathcal{E})$ consisting of all additively closed **fully exact subcategories** $\mathcal{E}$.
(2) $\mathrm{ex}(\mathcal{E})$ consisting of all **exact substructures of** $\mathcal{E}$ (cf. [44, Thm 5.3] for the lattice structure).

We have by definition

$$\mathrm{Ex}(\mathcal{E}) = \bigsqcup_{\mathcal{E}' \in \mathrm{ext}(\mathcal{E})} \mathrm{ex}(\mathcal{E}')$$

Remark 3.2. If $f\colon \mathcal{E}' \to \mathcal{E}$ is a fully faithful exact functor, then the additive closure of the essential image equipped with the exact structure induced from $\mathcal{E}'$ gives an element in $\mathrm{EX}(\mathcal{E})$. It lies in $\mathrm{Ex}(\mathcal{E})$ if and only if the essential image is extension closed in $\mathcal{E}$.

We have the obvious poset structure on $\mathrm{EX}(\mathcal{E})$:

$$\mathcal{E}_1 \leq \mathcal{E}_2 \quad \text{if} \quad \mathcal{E}_1 \in \mathrm{EX}(\mathcal{E}_2)$$

THEOREM 3.3. $\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E})$ *are complete lattices and* $\mathrm{ex}(\mathcal{E})$ *is complete sublattice of* $\mathrm{EX}(\mathcal{E})$ *and of* $\mathrm{Ex}(\mathcal{E})$.

All four posets have a unique maximal element $\mathcal{E}$ and a unique minimal element (which is $\{0\}$ except for $\mathrm{ex}(\mathcal{E})$ where it is the split exact structure).

The main ingredient is the following observation

Proposition 3.4. *Let $\mathcal{A}$ be a small additive category and $i_j\colon \mathcal{A}_j \to \mathcal{A}$, $j \in I$ (for some set I) inclusions of full, additively closed subcategories (recall these also closed under isomorphism). Assume, we have exact structures $\mathcal{E} = (\mathcal{A}, \mathcal{S}), \mathcal{E}_j = (\mathcal{A}_j, \mathcal{S}_j)$ such that i_j are exact functors. Then:*

$$\bigwedge \mathcal{E}_j := (\mathcal{B} := \bigcap_{j \in I} \mathcal{A}_j,\ \mathcal{R} := \{X_1 \xrightarrow{i} X_2 \xrightarrow{p} X_3 \colon X_i \in \bigcap_{j \in I} \mathcal{A}_j, (i,p) \in \mathcal{S}_j \forall j \in I\})$$

is an exact category such that the inclusion $\bigwedge_{j \in I} \mathcal{E}_j \to \mathcal{E}_i$ is an exact functor for all $i \in I$.

Remark 3.5. $I = \{1, 2\}$, the exact category $\mathcal{E}_1 \wedge \mathcal{E}_2$ is an exact category, it will fulfill the universal property of a pullback in the category of exact categories with exact functors (warning: This will fulfill the universal property of the strict 2-pullback, we are not considering other versions of 2-pullbacks here!).

Proof. Clearly, $(i,p) \in \mathcal{R}$ implies that (i,p) is a kernel-cokernel pair in $\mathcal{B}$ as this is the case in the bigger categories $\mathcal{A}_j$ for each $j \in I$.

Assume we have a cospan of objects in $\mathcal{B}$: $Y \xrightarrow{p} Z \xleftarrow{f} C$ such that p is an $\mathcal{E}_j$-deflation for all $j \in I$.

Then by [**49**], Prop.2.12: We have an $\mathcal{E}_j$-exact sequence $K_j \xrightarrow{k_j} Y \oplus C \xrightarrow{(p,f)} Z$ for $j \in I$. But as i_j are exact functors, we find all k_j are $\mathcal{A}$-kernels of the same map, so they are isomorphic and therefore $K := K_j \in \mathcal{B}$. Now, as the $\mathcal{A}_j$-pullback of p coincides with the $\mathcal{A}_{j'}$-pullback for every $j, j' \in I$, it is an $\mathcal{R}$-deflation $K \to C$.

For $C = 0$, we also get the $\mathcal{A}_j$-kernel and the $\mathcal{A}_{j'}$-kernel of p are isomorphic for all $j, j' \in I$, then it is obvious that composition of $\mathcal{R}$-deflations are $\mathcal{R}$-deflations.

Now, assume we have a span in $\mathcal{B}$: $D \xleftarrow{g} X \xrightarrow{i} Y$ with i an $\mathcal{E}_j$-inflation for $j \in I$. As above we observe that we have $Q = \operatorname{coker}(X \to D \oplus Y) \in \mathcal{B}$ and then the $\mathcal{A}_j$-pushout of i coincides with the $\mathcal{A}_{j'}$-pushout of i for all $j, j' \in I$ and is an $\mathcal{R}$-inflation. We also easily see that composition of $\mathcal{R}$-inflations are $\mathcal{R}$-inflations. □

Proof. (of Thm. 3.3) From the previous Proposition, we can conclude that $\mathrm{EX}(\mathcal{E})$ is a complete meet semi-lattice. As we have an obvious unique maximal element $\mathcal{E} \in \mathrm{EX}(\mathcal{E})$ it becomes a lattice via the following join described in the introduction. □

Remark 3.6. In given $\mathcal{B} \in \mathrm{ADD}(\mathcal{A})$, and $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ an exact category. We may look at $\mathcal{B} \cap \mathcal{S}$, i.e. all short exact sequences with all three terms in $\mathcal{B}$. To characterise when this is an exact structure is technical (and not very enlightning), see Lemma below. Here are some easy positive answers:
If $\mathcal{B}$ is extension-closed it is and in this case $\mathrm{Ext}^1_{\mathcal{S}\cap\mathcal{B}} = (\mathrm{Ext}^1_{\mathcal{E}})|_{\mathcal{B}}$.
If $\mathcal{B}$ is deflation- and inflation-closed (i.e. closed under kernels of arbitary deflations and cokernels of arbitrary inflations between objects in $\mathcal{B}$) then it also is.
A negative answer is provided below.

Example 3.7. (A negative answer) Consider the abelian category of finite-dimensional representations (over some field) of the quiver $1 \to 2 \to 3$. The indecomposables are the projectives P_i, injectives I_i (with $I_3 = P_1$) and S_2. We consider $\mathcal{B} = \mathrm{add}(P_3 \oplus P_1 \oplus I_2 \oplus S_2)$. We have an exact sequence $0 \to P_3 \to P_1 \to I_2 \to 0$. Which in $\mathcal{E}$ has a pull-back along $S_2 \to I_2$ given by P_2. But in $\mathcal{B} \cap \mathcal{S}$ there does not exist a kernel-cokernel pair to which it could pullback.

Example 3.8. Consider $\mathcal{E} = \mathrm{Mod}\, A$ for a ring A. Let I be a two sided ideal, then $\mathcal{A}' = \{X \in \mathrm{Mod}-A \mid IX = 0\}$ is extension-closed if and only if $I^2 = 0$. In either case the restriction of scalars $\mathrm{Mod}-(A/I) \to \mathrm{Mod}-A$ is a fully faithful exact functor with essential image $\mathcal{A}'$ and $\mathcal{A}'$ is inflation- and deflation-closed. So this gives an element $(\mathcal{A}', \mathcal{S}') \in \mathrm{EX}(\mathcal{E})$. It is easily see that this exact structure is abelian (since it is equivalent to the one on $\mathrm{Mod}-A/I$).

Lemma 3.9. *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category and $\mathcal{B} \in \mathrm{ADD}(\mathcal{A})$. Then, the following are equivalent:*

(1) *$(\mathcal{B}, \mathcal{S} \cap \mathcal{B})$ is an exact category*
(1') *$\mathcal{S} \cap \mathcal{B}$ are closed under pull-back and $\mathcal{S} \cap \mathcal{B}$ are closed under push-out.*
(In particular, these pull-back and push-out have to exist in $\mathcal{B}$).
(2) *For every inflation $i\colon B \rightarrowtail B'$ in $\mathcal{S} \cap \mathcal{B}$ we have: If i factors in $\mathcal{A}$ as $i = ba$, $a\colon B \to C$ with $\operatorname{coker} a$ in $\mathcal{B}$, then $C \in \mathcal{B}$.*
For every deflation $d\colon B' \twoheadrightarrow B''$ in $\mathcal{S} \cap \mathcal{B}$ we have: If d factors in $\mathcal{A}$ as $d = ef$, $e\colon D \to B''$ with $\ker e$ in $\mathcal{B}$ then $D \in \mathcal{B}$.

In this case, $i_{\mathcal{B},\mathcal{A}}$ is an exact functor, i.e. $(\mathcal{B}, \mathcal{S} \cap \mathcal{B}) \in \mathrm{EX}(\mathcal{E})$.

Proof. The equivalence (1) to (1') follows from [**66**, Lem. 1.9, Prop.1.10]. The equivalence (1') to (2) follows from the strong Obscure axiom [**49**, Prop. 7.6]. □

This is one of the few instances where more general exact subcategories than just extension-closed are considered:

Example 3.10. Not extension-closed exact subcategories in (the following exact categories) Hausdorff locally convex spaces, Frechet spaces and topological vector spaces respectively are studied in [**64**].

As a direct corollary of Cor. 4.4 we have the following: Every $\mathcal{F} \in \mathrm{EX}(\mathcal{E})$ we have $i\colon \mathcal{F} \to \mathcal{E}$ can be factorized either as

$$\mathcal{F} \leq \mathcal{F}_{\mathcal{B}} \xrightarrow{i} \mathcal{E}$$

where $\mathcal{F}_{\mathcal{B}}$ is the maximal exact structure making i exact (and $\leq$ means we have an inclusion of a substructure). Or it can be factorized as

$$\mathcal{F} \xrightarrow{i} \mathcal{E}' \leq \mathcal{E}$$

where $\mathcal{E}' = \bigwedge \mathcal{E}''$ where $\mathcal{E}''$ runs through all exact substructures such that $i\colon \mathcal{F} \to \mathcal{E}''$ is exact. From the first factorization we can conclude:

Theorem 3.11. *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an i.c. small exact category then we have*

$$\mathrm{EX}(\mathcal{E}) = \bigsqcup_{\mathcal{B} \in \mathrm{ADD}(\mathcal{A})} \mathrm{ex}(\mathcal{F}_{\mathcal{B}})$$

$$\mathrm{Ex}(\mathcal{E}) = \bigsqcup_{\mathcal{E}' \in \mathrm{ext}(\mathcal{E})} \mathrm{ex}(\mathcal{E}')$$

There are many other posets one can define here. Since we think fully faithful exact functors with extension-closed images are interesting, these two posets are our main interest. But for example we could also look at

$$\mathrm{Ex}'(\mathcal{E}) := \bigsqcup_{\mathcal{E}' \in \mathrm{ex}(\mathcal{E})} \mathrm{ext}(\mathcal{E}')$$

Example 3.12. Let $\mathcal{T} \subseteq \mathcal{E}$ be a self-orthogonal category, then $\mathrm{pres}^{\mathcal{E}}(\mathcal{T}) = \{X : \exists T \twoheadrightarrow X, \text{ with } T \in \mathcal{T}\}$ is extension closed in $\mathcal{E}$. If $\mathcal{E}'$ is an exact substructure of $\mathcal{E}$, we have $\mathcal{T} \subseteq \mathcal{E}'$ is still self-orthogonal and $\mathrm{pres}^{\mathcal{E}'}(\mathcal{T}) \subseteq \mathrm{pres}^{\mathcal{E}}(\mathcal{T})$. We get a subposet of $\mathrm{Ex}'(\mathcal{E})$

$$\{\mathrm{pres}^{\mathcal{E}'}(\mathcal{T}) \mid \mathcal{E}' \in \mathrm{ex}(\mathcal{E})\}$$

It has a unique maximal element $\mathrm{pres}^{\mathcal{E}}(\mathcal{T})$ and a unique minimal element $\mathcal{T}$.

Remark 3.13. Let $\mathcal{A}' \in \mathrm{ADD}(\mathcal{A})$ and $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ an exact category. Let $\mathcal{I}_{\mathcal{A}'} \subseteq \mathrm{ex}(\mathcal{E})$ be the exact substructures of $\mathcal{E}$ such that $\mathcal{A}'$ is extension-closed in it. I do not know anything on maximal elements in $I_{\mathcal{A}'}$.

4. The functorial point of view

Let $\mathcal{E}$ be an essentially small exact category. We consider three classical assignments (which are all 2-functorial on the category of small exact category with exact functors) for $\mathcal{F} = (\mathcal{B}, \mathcal{S}') \in \mathrm{EX}(\mathcal{E})$ the **Auslander exact category, tf Auslander category** and **effaceable functors** respectively

$$\mathrm{AE}(\mathcal{F}) = \{\mathrm{coker}(\mathrm{Hom}_{\mathcal{B}}(-, f)) \mid f \ \mathcal{F}\text{-admissible}\}$$

$$\mathrm{H}(\mathcal{F}) := \{\mathrm{coker}\,\mathrm{Hom}_{\mathcal{B}}(-, i) \mid i \ \mathcal{F}\text{-inflation}\}$$

$$\mathrm{eff}(\mathcal{F}) := \{\mathrm{coker}(\mathrm{Hom}_{\mathcal{B}}(-, d) \mid d \ \mathcal{F}\text{-deflation}\}$$

All will be considered fully exact subcategories in $\mathrm{mod}_1 \mathcal{A}$ (where $\mathcal{A}$ is the underlying additive category of $\mathcal{E}$). By a results of [**90**] and [**72**] (cf. chapter 2), we have characterizations of the subcategories when we look only at exact substructures of $\mathcal{E}$. Our aim is to extend these to the whole lattice $\mathrm{EX}(\mathcal{E})$.

4.1. Partially resolving subcategories. We start with some background definitions.

Definition 4.1. Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Let $\mathcal{F} \subseteq \mathcal{E}$ be a fully exact subcategory. We say that $\mathcal{F}$ is **partially resolving** if

(PR1) $\mathcal{F} = \operatorname{add}(\mathcal{F})$ (i.e. $\mathcal{F}$ is closed under taking direct summands in $\mathcal{E}$)
(PR2) For every $F \in \mathcal{F}$ we find an $\mathcal{E}$-short exact sequence $\Omega F \to P \to F$ with $P \in \mathcal{P} \cap \mathcal{F}$ and $\Omega F \in \mathcal{F}$ (we call this property: $\mathcal{F}$ is *closed under taking syzygies*)

Then $\mathcal{F}$ also has enough projectives with $\mathcal{P}(\mathcal{F}) = \mathcal{Q} \subseteq \mathcal{P}(\mathcal{E})$ and we say $\mathcal{F}$ is partial resolving with respect to $\mathcal{Q} \subseteq \mathcal{P}$.

Remark 4.2. A partially resolving subcategory $\mathcal{F}$ in an exact category with enough projectives $\mathcal{P}$ is resolving if and only if $\mathcal{P} \subseteq \mathcal{F}$.

Lemma 4.3. *([71], Lem. 2.5) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Let $\mathcal{F}$ be a fully exact subcategory which is closed under taking summands in $\mathcal{E}$, then the following are equivalent:*

(1) *$\mathcal{F}$ is partially resolving*
(2) *$\mathcal{F}$ is deflation-closed with enough projectives $\mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P}$.*

The proof is completely analogue to the given reference, we leave it to the reader.

Remark 4.4. If we are in a Krull-Schmidt category (say we have minimal projective covers) then the syzygies in (PR2) can be always taken with respect to the minimal projective cover and in this case we can find intersections of arbitrary partially resolving subcategories are partially resolving.

Remark 4.5. If $\mathcal{F}$ is partially resolving in $\mathcal{E}$ then it is homologically exact, cf. chapter 1.

For a small additive category $\mathcal{A}$, we denote by $\operatorname{mod}_1 \mathcal{A}$ the category of all additive functors $F\colon \mathcal{A}^{op} \to (Ab)$ with $F \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, f)$ for some morphism f in $\mathcal{A}$ ($f \in \operatorname{Mor} - \mathcal{A}$). We see this as a fully exact subcategory of the abelian category $\operatorname{Mod} -\mathcal{A}$ (all additive contravariant functors $\mathcal{A}^{op} \to (Ab)$).
For a full additive subcategory $\mathcal{B} \subseteq \mathcal{A}$ we define the full subcategory of $\operatorname{mod}_1 \mathcal{A}$

$$\operatorname{mod}_1(\mathcal{A}|\mathcal{B}) := \{F \in \operatorname{mod}_1 \mathcal{A} \mid F \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{A}}(-, f), f \in \operatorname{Mor} - \mathcal{B}\}$$

Then this is a fully exact subcategory of $\operatorname{mod}_1 \mathcal{A}$ by the horseshoe lemma.

Lemma 4.6. *The restriction functor*

$$\Phi\colon \operatorname{mod}_1(\mathcal{A}|\mathcal{B}) \to \operatorname{mod}_1 \mathcal{B}, \quad F \mapsto F|_{\mathcal{B}}$$

is an equivalence of additive categories which is also an exact functor.

The proof is straight-forward. This is usually not an equivalence of exact categories, the quasi-inverse functor is a not necessarily exact tensor functor - we see $\operatorname{mod}_1(\mathcal{A}|\mathcal{B})$ as an exact substructure of $\operatorname{mod}_1 \mathcal{B}$.
(Nevertheless it restricts to an exact equivalence of many smaller categories, e.g. on $\operatorname{mod}_2(\mathcal{A}|\mathcal{B}) \to \operatorname{mod}_2 \mathcal{B}$ it is already an exact equivalence, see other instances later).

Recall a Serre subcategory is a full additive subcategory $\mathcal{F}$ in an exact category $\mathcal{E}$ with the following property: For every $\mathcal{E}$-short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ we have $Y \in \mathcal{F}$ if and only if $X, Z \in \mathcal{F}$.

Definition 4.7. We denote by $\mathcal{P}^2(\mathcal{A})$ the full subcategory of $\operatorname{Mod} \mathcal{A}$ given by all functors F such that there exists an exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{A}}(-, X) \to \operatorname{Hom}_{\mathcal{A}}(-, Y) \to \operatorname{Hom}_{\mathcal{A}}(-, Z) \to F \to 0$$

for some X, Y, Z in $\mathcal{A}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be a full additively closed subcategory. We write $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ for the full subcategory of $\mathcal{P}^2(\mathcal{A})$ given by all functors F such that there exists an exact sequence as above with X, Y, Z in $\mathcal{B}$.

It is an easy horse-shoe-lemma argument to see that $\mathcal{P}^2(\mathcal{A}|\mathcal{B}) \subseteq \mathcal{P}^2(\mathcal{A}) \subseteq \operatorname{Mod}\mathcal{A}$ are inclusions of extension-closed subcategories. From now on, we equip them with the fully exact structure.

Lemma 4.8. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a full additive subcategory. Then, $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ is a partially resolving subcategory of $\mathcal{P}^2(\mathcal{A})$. Furthermore the restriction functor*

$$\mathcal{P}^2(\mathcal{A}|\mathcal{B}) \to \mathcal{P}^2(\mathcal{B}), \quad F \mapsto F|_{\mathcal{B}}$$

is an equivalence of exact categories (i.e. an equivalence of categories which is homologically exact).

Proof. As $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ is by definition closed under taking syzygies in $\mathcal{P}^2(\mathcal{A})$, it follows that it is an exact category with enough projectives given by $\operatorname{Hom}_{\mathcal{A}}(-, B)$, $B \in \mathcal{B}$. This implies it is partially resolving.
Restriction functors are exact functors on functor categories, therefore their restrictions to fully exact subcategories are still exact. By definition this functor is essentially surjective. Using the projective presentations one can see that this is an equivalence of additive categories which restricts to an equivalence on the category of projetives. Now, both are exact categories with enough projectives and have gldim ≤ 2, therefore the derived functor is a triangle equivalence. This implies that the funtor is homologically exact. ☐

In particular, (using the quasi-inverse of the equivalence) we will consider $\mathcal{P}^2(\mathcal{B})$ from now on as a partially resolving subcategory in $\mathcal{P}^2(\mathcal{A})$.

4.2. Short recap of definitions from chapter 2. The **grade** of $F \in \operatorname{Mod}\mathcal{A}$ is defined as the supremum of all natural numbers $i \geq 0$ such that
$\operatorname{Ext}^j_{\operatorname{Mod}\mathcal{A}}(F, \operatorname{Hom}_{\mathcal{A}}(-, A)) = 0 \forall A \in \mathcal{A}$ for all $j < i$ (of course, only if this exists, else we define it to be ∞). Let us denote by $KC(\mathcal{A})$ the **collection of all kernel-cokernel pairs** in $\mathcal{A}$

$$\begin{aligned}
\mathcal{G}^2(\mathcal{A}) &= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d),(j,p) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{j} \circ \mathrm{d})\} \\
&\subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} \in \{0,2\}\} \\
\mathcal{C}^2(\mathcal{A}) &= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (i,d) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{d})\} \\
&= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} = 2\} \\
\mathcal{J}^1(\mathcal{A}) &= \{F \in \mathcal{P}^2(\mathcal{A}) \mid \exists (j,p) \in \mathrm{KC}(\mathcal{A}), \mathrm{F} \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, \mathrm{j})\} \\
&\subseteq \{F \in \mathcal{P}^2(\mathcal{A}) \mid \mathrm{grade(F)} = 0\}
\end{aligned}$$

Enomoto's duality:

$$\begin{gathered}
E \colon \mathcal{C}^2(\mathcal{A})^{op} \to \mathcal{C}^2(\mathcal{A}^{op}) \\
E(\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, d)) \cong \operatorname{coker}(\operatorname{Hom}_{\mathcal{A}}(i, -)) \quad (i,d) \in \mathrm{KC}(\mathcal{A})
\end{gathered}$$

Auslander-Bridger transpose (also a duality):
The ideal quotient of $\operatorname{mod}_1\mathcal{A}$ with respect to the projectives is denoted by $\underline{\operatorname{mod}_1}\mathcal{A}$.

$$\begin{gathered}
\mathrm{Tr} \colon (\underline{\operatorname{mod}_1}\mathcal{A})^{\mathrm{op}} \to \underline{\operatorname{mod}_1}(\mathcal{A}^{op}) \\
\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, f) \mapsto \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(f, -).
\end{gathered}$$

Enomoto (in [**72**],..) observed the following:

Lemma 4.9. *If there is an exact structure $\mathcal{E}$ and we have a short exact sequence (i,d) and a kernel-cokernel pair (j,p) such that* $\operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, d) = F = \operatorname{coker}\operatorname{Hom}_{\mathcal{A}}(-, p)$*, then (j,p) is also an $\mathcal{E}$-short exact sequence.*

Following loc. cit. we say $\mathcal{E}$ short exact sequences are *closed under homotopy* (within kernel-cokernel presentations).

In a similar way, we showed in Lemma 4.9 that $\mathcal{E}$-inflations are closed under homotopy among all presentations.

4.3. Generalizations. Now, given a full additive subcategory $\iota\colon \mathcal{B} \subseteq \mathcal{A}$ and denote by $KC(\mathcal{A}|\mathcal{B})$ the **collection of all relative kernel-cokernel pairs** in $\mathcal{A}$, these are kernel-cokernel pairs $(\iota(j), \iota(p))$ in $\mathcal{A}$ such that (j, p) is a kernel-cokernel pair in $\mathcal{B}$.

$$\mathcal{G}^2(\mathcal{A}|\mathcal{B}) = \{F \in \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \mid \exists (i,d),(j,p) \in \mathrm{KC}(\mathcal{A}|\mathcal{B}), \mathrm{F} \cong \operatorname{coker} \mathrm{Hom}_{\mathcal{A}}(-, \mathrm{j} \circ \mathrm{d})\}$$
$$\mathcal{C}^2(\mathcal{A}|\mathcal{B}) = \{F \in \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \mid \exists (i,d) \in \mathrm{KC}(\mathcal{A}|\mathcal{B}), \mathrm{F} \cong \operatorname{coker} \mathrm{Hom}_{\mathcal{A}}(-, \mathrm{d})\}$$
$$\mathcal{J}^1(\mathcal{A}|\mathcal{B}) = \{F \in \mathcal{P}^1(\mathcal{A}|\mathcal{B}) \mid \exists (j,p) \in \mathrm{KC}(\mathcal{A}|\mathcal{B}), \mathrm{F} \cong \operatorname{coker} \mathrm{Hom}_{\mathcal{A}}(-, \mathrm{j})\}$$

For an exact category $\mathcal{F} = (\mathcal{B}, \mathcal{S}')$ in $\mathrm{EX}(\mathcal{E})$, we obviously have
$$\mathrm{AE}(\mathcal{F}) \subseteq \mathcal{G}^2(\mathcal{A}|\mathcal{B}), \quad \mathrm{eff}(\mathcal{F}) \subseteq \mathcal{C}^2(\mathcal{A}|\mathcal{B}), \quad \mathrm{H}(\mathcal{F}) \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B}).$$

The Auslander category $\mathrm{AE}(\mathcal{F}) \subseteq \mathcal{P}^2(\mathcal{B}) \cong \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \subseteq \mathrm{mod}_1 \mathcal{A}$ is an extension-closed subcategory. As $\mathcal{P}^2(\mathcal{A}|\mathcal{B})$ is partially resolving in $\mathcal{P}^2(\mathcal{A})$, we have that $\mathrm{AE}(\mathcal{F})$ is partially resolving in $\mathcal{P}^2(\mathcal{A})$ (and also $\mathrm{H}(\mathcal{F})$ is partially resolving in $\mathcal{P}^1(\mathcal{A})$). By Auslander correspondence we expect $\mathcal{F} \mapsto \mathrm{AE}(\mathcal{F})$ to be a bijection with certain partially resolving subcategories of $\mathcal{P}^2(\mathcal{A})$. But we have to modify the definition of the transpose category (because it always adds all projectives).
On the other hand, for $\mathrm{eff}(\mathcal{F})(\subseteq \mathrm{AE}(\mathcal{F})) \subseteq \mathrm{mod}_1 \mathcal{A}$ we have to expect to get very often the same subcategory (for example: The split exact categories $\mathcal{F}$ always have $\mathrm{eff}(\mathcal{F}) = 0$ no matter on which underlying category). In fact, the data which is lost, is given precisely by the additive category $\mathcal{B}$ on which the exact structure has to be defined.

Assume that we have a left exact functor $f\colon \mathcal{F} = (\mathcal{B}, \mathcal{T}) \to (\mathcal{A}, \mathcal{S}) = \mathcal{E}$. In [**90**], Thm 3.9, it is shown that the composition $\mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{X \mapsto \mathrm{Hom}(-,X)} \mathrm{AE}(\mathcal{E})$ is left exact and factors (uniquely up to isomorphism) over an exact functor $\mathrm{AE(f)}\colon \mathrm{AE}(\mathcal{F}) \to \mathrm{AE}(\mathcal{E})$.

Lemma 4.10. *Then the following are equivalent:*

(1) f *is fully faithful.*
(2) $\mathrm{AE}(f)$ *is homologically exact.*

Furthermore, if f is inclusion of an additively closed subcategory, we can identify $\mathrm{AE}(\mathcal{F})$ *with the essential image of* $\mathrm{AE(f)}$ *which is a partial resolving subcategory in* $\mathrm{AE}(\mathcal{E})$*. In this subcategory all objects have either grade* 2 *ore* 0*. The grade* 2*-objects are precisely the effaceable functors* $\mathrm{eff}(\mathcal{F})$*, i.e. we have*
$$\mathrm{AE}(\mathcal{F}) \cap \mathcal{C}^2(\mathcal{A}) = \mathrm{eff}(\mathcal{F})$$

Proof. The derived functor of $\mathrm{AE}(f)$ identifies with $K^b(f)\colon K^b(\mathcal{B}) \to K^b(\mathcal{A})$. Therefore f is fully faithful iff $K^b(f)$ is fully faithful iff $\mathrm{AE}(f)$ is homologically exact. The second claim follows from $\mathrm{AE}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}) = \mathrm{eff}(\mathcal{E})$ observed in [**90**], then $\mathrm{AE}(f)$ maps the torsion pair $(\mathrm{eff}(\mathcal{F}) = {}^{\perp}\mathcal{Q}, \mathrm{H}(\mathcal{F}) = \mathrm{copres}\, \mathcal{Q})$ where $\mathcal{Q} = \mathrm{AE}(\mathcal{F})$ to $(\mathrm{eff}(\mathcal{E}) = {}^{\perp}\mathcal{P}, \mathrm{H}(\mathcal{E}) = \mathrm{copres}\, \mathcal{P})$ where $\mathcal{P} = \mathrm{AE}(\mathcal{E})$ because it preserves projectives and is exact. Therefore the last claim follows. □

Remark 4.11. With the same proof we also show for an inflation-preserving $f\colon \mathcal{E} \to \mathcal{F}$: The functor f is fully faithful if and only if $\mathrm{H}(f)$ is homologically exact.

Lemma 4.12. *Let* $\mathcal{F} \in \mathrm{EX}(\mathcal{E})$ *and consider* $\mathrm{eff}(\mathcal{F})$ *as a full subcategory of* $\mathrm{eff}(\mathcal{E})$ *(by applying* $\Phi^{-1}_{\mathcal{A}|\mathcal{B}}$ *to it), then we have*
$$\mathrm{eff}(\mathcal{F}) \subseteq \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$$

is a Serre subcategory.
Furthermore if $\mathcal{E}$ restricts to $\mathcal{F}$ on $\mathcal{B}$ (i.e. every ses in $\mathcal{E}$ which has all three objects in $\mathcal{B}$ is exact in $\mathcal{F}$), then we have $\mathrm{eff}(\mathcal{F}) = \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$.

PROOF. Clearly $\mathrm{eff}(\mathcal{F}) \subseteq \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$ extension-closed. As $\mathcal{F}$ is an exact structure, $\mathrm{eff}(\mathcal{F})$ is already Serre subcategory in $\mathcal{C}^2(\mathcal{B}) \cong \mathcal{C}^2(\mathcal{A}|\mathcal{B})$, this implies it also is a Serre subcategory in $\mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$.
Assume now that $\mathcal{E}$ restricts to $\mathcal{F}$ on $\mathcal{B}$, then we have to show the other inclusion. Let $F \in \mathrm{eff}(\mathcal{E}) \cap \mathcal{C}^2(\mathcal{A}|\mathcal{B})$, then we find two $\mathcal{A}$-kernel-cokernel pairs representing F and one is an $\mathcal{E}$-ses and the other one is an $(i,p) \in \mathrm{KC}(\mathcal{A}|\mathcal{B})$. As $\mathcal{E}$-short exact sequences are closed under *homotopy* (Lemma 4.9) it follows that (i,p) is also an $\mathcal{E}$-short exact sequence. By our assumption, the $\mathcal{E}$-short exact sequences with all three terms in $\mathcal{B}$ are just the $\mathcal{F}$-short exact sequences, it follows that $F \in \mathrm{eff}(\mathcal{F})$. □

We will also need the following definition.

Definition 4.13. Let $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A})$ an additive subcategory, we define $\mathcal{B} := \mathcal{B}_{\mathcal{X}} \subseteq \mathcal{A}$ to be the full (additive) subcategory of objects $B \in \mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(-, B) \in \mathcal{X}$.
We consider the composition $\mathcal{P}^2(\mathcal{A}^{op}|\mathcal{B}^{op}) \to \mathrm{mod}_1 \mathcal{A} \to \underline{\mathrm{mod}}_1\mathcal{A}^{op}$ as the identity on objects. In this case we define the **relative transposed category** $\mathrm{Tr}_{\mathrm{rel}}(\mathcal{X})$ to be the full subcategory of objects X in $\mathcal{P}^2(\mathcal{A}^{op}|\mathcal{B}^{op})$ such that $X \cong \mathrm{Tr}(X')$ in $\underline{\mathrm{mod}}_1\mathcal{A}^{op}$ for some $X' \in \mathcal{X}$.

Remark 4.14. By definition $\mathcal{B}_{\mathcal{G}^2(\mathcal{A}|\mathcal{B})} = \mathcal{B} = \mathcal{B}_{\mathcal{J}^1(\mathcal{A}|\mathcal{B})}$ and

$$\mathrm{Tr}_{\mathrm{rel}}(\mathcal{G}^2(\mathcal{A}|\mathcal{B})) = \mathcal{G}^2(\mathcal{A}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}}), \quad \Omega\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}^1(\mathcal{A}|\mathcal{B})) = \mathcal{J}^1(\mathcal{A}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}})$$

We also remark that $\mathcal{B}_{\mathcal{C}^2(\mathcal{A}|\mathcal{B})} = \{0\}$.

We fix an exact structure $\mathcal{E}$ on $\mathcal{A}$. For every $\mathcal{B} \subseteq \mathcal{A}$ full additively closed subcategory let $\mathcal{C}_{\mathcal{B},max} \subseteq \mathrm{mod}_1 \mathcal{B}$ be the Serre subcategory corresponding to the maximal exact structure $\mathcal{F}$ on $\mathcal{B}$ such that the inclusion $\mathcal{F} \to \mathcal{E}$ is an exact functor. Recall that we have an equivalence $\Phi_{\mathcal{A}|\mathcal{B}} \colon \mathcal{C}^2(\mathcal{A}|\mathcal{B}) \to \mathcal{C}^2(\mathcal{B}), F \mapsto F|_{\mathcal{B}}$ of exact categories.

THEOREM 4.15. *Let $\mathcal{A}$ be an idempotent complete, small additive category and $\mathcal{E} = \mathcal{E}_{max}$ the maximal exact structure on it. Then the assignments*

$$\mathcal{F} \mapsto \mathrm{AE}(\mathcal{F}), \quad \mathcal{F} \mapsto \inf(\mathcal{F}), \quad \mathcal{F} = (\mathcal{B}, \mathcal{S}) \mapsto (\mathcal{B}, \mathrm{eff}(\mathcal{F}))$$

give bijections between $\mathrm{EX}(\mathcal{E})$ *and (1), (2) and (3) respectively.*

(1) *Partially resolving subcategories $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A})$ such that $\mathcal{X} \subseteq \mathcal{G}^2(\mathcal{A}|\mathcal{B})$ for $\mathcal{B} = \mathcal{B}_{\mathcal{X}}$ and $\mathrm{Tr}_{\mathrm{rel}}(\mathcal{X}) \subseteq \mathcal{P}^2(\mathcal{A}^{\mathrm{op}})$ is also partially resolving.*
(2) *Partially resolving subcategories $\mathcal{J} \subseteq \mathcal{P}^1(\mathcal{A})$ such that $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B})$ for $\mathcal{B} = \mathcal{B}_{\mathcal{J}}$ and $\Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}) \subseteq \mathcal{P}^1(\mathcal{A}^{\mathrm{op}})$ is also partially resolving.*
(3) *pairs of categories $(\mathcal{B}, \mathcal{C})$ with*
   (*) *$\mathcal{B} \subseteq \mathcal{A}$ a full additively closed subcategory and*
   (*) *$\mathcal{C} \subseteq \mathcal{C}^2(\mathcal{A}|\mathcal{B})$ a full additively closed subcategory such that $\Phi_{\mathcal{A}|\mathcal{B}}(\mathcal{C})$ is a Serre subcategory in $\mathcal{C}_{\mathcal{B},max}$.*

PROOF. We observe that (3) is just a trivial consequence of Enomoto's bijection, Theorem 6.4, we just state it here for completeness sake.
Let us turn to (1) and (2) and show the assignments are well-defined. Let $\mathcal{F} = (\mathcal{B}, \mathcal{S}) \in \mathrm{EX}(\mathcal{E})$.

(1) We already observed that $\mathcal{X} := \mathrm{AE}(\mathcal{F})$ is partially resolving in $\mathcal{P}^2(\mathcal{A})$. We want to see $\mathrm{AE}(\mathcal{F}^{op}) = \mathrm{Tr}_{\mathrm{rel}}(\mathcal{X})$. We denote by $\underline{\mathcal{G}^2(\mathcal{A})}$ the essential image of $\mathcal{G}^2(\mathcal{A}) \to \underline{\mathrm{mod}_1}\mathcal{A}$. When

we restrict the functor $\mathrm{Tr}_{\mathcal{A}}$ we a commutative diagram (*)

$$\begin{array}{ccc} \underline{\mathcal{G}^2(\mathcal{A}|\mathcal{B})} & \xrightarrow{\text{incl}} & \underline{\mathcal{G}^2(\mathcal{A})} \\ \downarrow{\scriptstyle \mathrm{Tr}_{\mathcal{B}}} & & \downarrow{\scriptstyle \mathrm{Tr}_{\mathcal{A}}} \\ \underline{\mathcal{G}^2(\mathcal{A}^{op}|\mathcal{B}^{op})} & \xrightarrow{\text{incl}} & \underline{\mathcal{G}^2(\mathcal{A}^{op})} \end{array}$$

The underline on the right hand side can either be seen as the essential image in $\underline{\mathrm{mod}_1}\mathcal{A}^{(op)}$ or the essential image in $\underline{\mathrm{mod}_1}\mathcal{B}^{(op)}$, the functors $\mathrm{Tr}_{\mathcal{A}}$ and $\mathrm{Tr}_{\mathcal{B}}$ coincide on this subcategory. This means that the relative trace category $\mathrm{Tr}_{rel}(\mathcal{X}) = \mathrm{Tr}_{\mathcal{B}}(\mathcal{X}) = \mathrm{AE}(\mathcal{F}^{op})$ and this is partially resolving in $\mathcal{P}^2(\mathcal{A}^{op})$.

(2) We already know that $\mathcal{J} := \mathrm{H}(\mathcal{F})$ is partially resolving in $\mathcal{P}^1(\mathcal{A})$ and we want to see $\mathrm{H}(\mathcal{F}^{op}) = \Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J})$. We look at the diagram (*). Not just trace also $\Omega_{\mathcal{A}}$ and $\Omega_{\mathcal{B}}$ identify on these subcategories, so $\Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}) = \Omega_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\mathcal{J}) = \mathrm{H}(\mathcal{F}^{\mathrm{op}})$ and this is partially resolving in $\mathcal{P}^1(\mathcal{A}^{op})$.

Now, we define the inverse assignments:

(1) Assume we have $\mathcal{X}$ as in (1) and define $\mathcal{B} = \mathcal{B}_{\mathcal{X}}$, then we see that $\mathcal{X} \subseteq \mathcal{P}^2(\mathcal{A}|\mathcal{B}) \cong \mathcal{P}^2(\mathcal{B})$ is resolving and as we have $\mathcal{X} \subseteq \mathcal{G}^2(\mathcal{A}|\mathcal{B})$, it follows as above $\mathrm{Tr}_{rel}(\mathcal{X}) = \mathrm{Tr}_{\mathcal{B}}(\mathcal{X}) \subseteq \mathcal{G}^2(\mathcal{A}^{op}|\mathcal{B}^{op})$ is resolving in $\mathcal{P}^2(\mathcal{A}^{op}|\mathcal{B}^{op}) \cong \mathcal{P}^2(\mathcal{B}^{op})$. By Theorem 6.8 it follows that $\mathcal{X} = \mathrm{AE}(\mathcal{F})$ for an exact structure $\mathcal{F}$ on $\mathcal{B}$. The inclusion $\mathcal{X} \subseteq \mathrm{AE}(\mathcal{E})$ apriori corresponds to a fully faithful left exact functor $f\colon \mathcal{F} \to \mathcal{E}$. But the inclusion $\mathrm{Tr}_{rel}(\mathcal{X}) \to \mathrm{AE}(\mathcal{E}^{op})$ corresponds to f^{op} also being left exact, we conclude that $f\colon \mathcal{E} \to \mathcal{F}$ is exact and so $\mathcal{F} \in \mathrm{EX}(\mathcal{E})$. This gives the inverse map.

(2) Assume we have $\mathcal{J}$ as in (2) and define $\mathcal{B} = \mathcal{B}_{\mathcal{J}}$, then $\mathcal{J}$ is resolving in $\mathcal{P}^1(\mathcal{A}|\mathcal{B}) \cong \mathcal{P}^1(\mathcal{B})$. As $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B})$ we conclude $\Omega_{\mathcal{A}}\mathrm{Tr}_{\mathrm{rel}}(\mathcal{J}) = \Omega_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\mathcal{J})$ is resolving in $\mathcal{P}^1(\mathcal{A}^{op}|\mathcal{B}^{op})$. By Theorem 6.10 it follows that $\mathcal{J} = \mathrm{H}(\mathcal{F})$ for an exact structure on $\mathcal{B}$ and as we have $\mathcal{J} \subseteq \mathcal{J}^1(\mathcal{A}|\mathcal{B})$ we conclude (using Lemma 4.9) hat all short exact sequences in $\mathcal{F}$ are mapped to kernel-cokernel pairs in $\mathcal{A}$. Then we look at the $H(\mathcal{F}) \subseteq H(\mathcal{E})$, by Appendix B, chapter 2, it corresponds to a fully faithful inflation-preserving functor $\mathcal{E} \to \mathcal{F}$, but as this functor also maps short exact sequences to kernel-cokernel pairs it is exact.

□

5. Meet-preserving of homological conditions

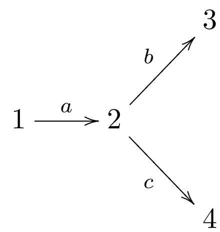
5.1. Examples for negative answers.

We collected these negative answers:

(1) homologically exactness and also homologically faithfulness
(2) gldim $= n$
(3) having enough projectives (or injectives)
(4) hereditary exact substructures in an hereditary exact category

Also, the subposet of hereditary exact substructures may not have a unique maximal element. The following is an example for (1).

Example 5.1. Intersections of homologically exact subcategories may not be homologically exact subcategories. We give an example of a resolving and a coresolving subcategory in an abelian category whose intersection is a semi-simple subcategory which is not homologically exact in the abelian category. Take $\mathcal{E} = \Lambda\,\mathrm{mod}$, with $\Lambda = KQ/I$ where (Q, I) is the following bound quiver



$I = (ba)$

Then this has nine indecomposable representations (the projectives, the injectives S_2 and $\mathrm{rad}(\mathrm{P}_1)$). Then $\mathcal{R} = \mathrm{add}(\Lambda \oplus \mathrm{rad}(\mathrm{P}_1) \oplus \mathrm{I}_1)$ is resolving and $\mathcal{C} = \mathrm{add}(\mathrm{D}\Lambda \oplus \mathrm{S}_2 \oplus \mathrm{P}_3)$ is coresolving. Their intersection is $\mathrm{add}(P_1 \oplus I_1 \oplus P_3)$. This is a semi-simple extension-closed subcategory (observe $P_1 = I_4$). Since $\mathrm{Ext}^2_\Lambda(I_1, P_3) \neq 0$ it is not homologically exact in $\mathcal{E}$.

For (3), one just has to observe that given infinitely many contravariantly finite generators $\mathcal{P}_i$ in an exact category, it is generally not true that $\bigvee \mathcal{P}_i$ (i.e. the smallest generator containing all $\mathcal{P}_i$) is contravariantly finite.

Example 5.2. Let $\mathcal{E} = \Lambda\,\mathrm{mod}$ with Λ the path algebra of the Kronecker quiver. We find finite dimensional preprojective generators $G_n = \Lambda \oplus \tau^-\Lambda \oplus \cdots \oplus \tau^{-n}\Lambda$, $n \in \mathbb{N}$ such that $\bigvee_{n\in\mathbb{N}} \mathrm{add}(G_n)$ is the preprojective component. Then we observe that the preprojective component is not contravariantly finite.

The following gives an example for (2), (4) and shows that homologically faithfulness is not preserved under forming meet.

Example 5.3. We give an example of an hereditary exact category with two hereditary exact substructures such that the intersection is no longer hereditary exact. Let $\mathcal{E} = \Lambda\,\mathrm{mod}$ where Λ is the path algebra of $1 \to 2 \to 3 \to 4$. We choose two generators $G_1 = \Lambda \oplus I_3 \oplus S_3 \oplus S_1$ and $G_2 = \Lambda \oplus S_2$. Let $\mathcal{E}_i = (\Lambda\,\mathrm{mod}, F_{G_i})$ be the exact substructure such that $\mathrm{Hom}(G_i, -)$ is exact on short exact sequences, then $\mathcal{E}_1 \cap \mathcal{E}_2 = (\Lambda\,\mathrm{mod}, F_{G_1 \oplus G_2})$. By looking at projective resolutions of non-projective indecomposables one easily see $\mathrm{gldim}(\Lambda\,\mathrm{mod}, F_{G_i}) = 1$ for $i = 1, 2$ and $\mathrm{gldim}(\Lambda\,\mathrm{mod}, F_{G_1 \oplus G_2}) = 2$.

5.2. Positive answers. Are already discussed in the introduction, we just prove here this missing Lemma:

Lemma 5.4. *Let $\mathcal{E}$ be an exact category and $n \in \mathbb{N}$. Then the following are equivalent*

(1) $\mathrm{gldim}\,\mathcal{E} \leq n$
(2) $\mathrm{Ext}^n(X, \sigma)$ *is right exact for all objects X and $\mathcal{E}$-short exact sequences σ*
(3) $\mathrm{Ext}^n(\sigma, X)$ *is right exact for all objects X and $\mathcal{E}$-short exact sequences σ*

Proof. We only show the equivalence of (1) and (2) (the other equivalence follows from passing to the opposite exact category). Clearly (1) implies (2) follows from the long exact sequence on the Ext-groups. So assume (2) and take $\sigma \in \mathrm{Ext}^{n+1}_{\mathcal{E}}(X, Y)$. Write σ as a concatenation $\sigma_1\sigma_2$ with $\sigma_1 \colon Y \rightarrowtail V \twoheadrightarrow W$ and apply $\mathrm{Hom}(X, -)$ to σ_1. We look at the connecting morphism

$$\mathrm{Ext}^n_{\mathcal{E}}(X, W) \to \mathrm{Ext}^{n+1}_{\mathcal{E}}(X, Y)$$

By [**126**, Cor. 4.2.12] this is given by concatenation with σ_1. In particular $\sigma_2 \mapsto \sigma$ and so σ is in the image. But as $\mathrm{Ext}^n_{\mathcal{E}}(X, \sigma_1)$ is right exact, this is zero. □

We say a bifunctor which is middle exact and fulfills the (corresponding) condition (2) and (3) from the previous lemma is **right exact**. So we can identify hereditary exact substructures with right exact subfunctors of $\mathrm{Ext}^1_{\mathcal{E}_{max}}$. These are usually not closed under intersection (see: negative answers). There also can not exist a structure as a complete lattice on hereditary exact substructures.

Example 5.5. This is an example with two maximal hereditary substructures. Let Λ be the path algebra of $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ bound by the relation $ba = 0$. We consider the abelian category $\mathcal{E} = \Lambda \operatorname{mod}$. Let $\mathcal{E}_i$ be the exact substructure with $\mathcal{P}(\mathcal{E}_i) = \operatorname{add}(\Lambda \oplus S_i)$, $i = 1, 2$. Both are hereditary exact and maximal wrt being hereditary.

6. Representation-finiteness

Let K be a field. Now, we assume that $\mathcal{A}$ is Krull-Schmidt K-linear category of finite representation-type (i.e. only finitely many indecomposable objects in $\mathcal{A}$). Given a full additively closed subcategory $\mathcal{M}$, then it is covariantly finite in a category $\mathcal{A}$ and we denote for any object A in $\mathcal{A}$ by $f_A \colon A \to M_A$ with M_A in $\mathcal{M}$ a left $\mathcal{M}$-approximation of A.

Lemma 6.1. *Let K be a field. Let $\mathcal{A}$ be a small K-linear additive, Hom-finite Kull-Schmidt category of finite type and $\mathcal{M} = \operatorname{add}(M)$ an additively closed subcategory. Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$ with enough projectives with $\mathcal{P}(\mathcal{E}) = \operatorname{add}(G)$ and let $\mathcal{F}$ be the maximal exact structure on $\mathcal{M}$ making the inclusion an exact functor. Then we have $\mathcal{P}(\mathcal{F}) = \operatorname{add}(M_G \oplus M_{\operatorname{coker} f_C})$ where we define $\operatorname{copres}_{\mathcal{E}}(\mathcal{M}) := \{X \in \mathcal{E} \mid \exists\, \mathcal{E}\text{-ses } X \rightarrowtail M' \twoheadrightarrow Y\}$ and since this is of finite type, we assume it is $\operatorname{add}(C)$ for an object C in $\mathcal{A}$.*

Proof. We first show that (1) M_G and (2) $M_{\operatorname{coker} f_C}$ as in the lemma lie in $\mathcal{P}(\mathcal{F})$:
(1) We show that f_G is an $\mathcal{E}$-deflation: Take $g \colon G_0 \twoheadrightarrow M_G$ be a $\mathcal{E}$-deflation with $\operatorname{add} G_0 \in \operatorname{add} G$. We may assume $G = G_0$. Then g factors over f, i.e. there exists an endomorphism $h \in \operatorname{End}(M_G)$ such that $g = fh$. By the obscure axiom h is an $\mathcal{E}$-deflation. It is an easy observation that then $\dim \operatorname{Hom}(G, \ker h) = 0$ and therefore h an isomorphism. This implies f is also an $\mathcal{E}$-deflation. In particular $\operatorname{Hom}(f, M) \colon \operatorname{Hom}(M_G, M) \to \operatorname{Hom}(G, M)$ is an isomorphism. This shows $M_G \in \mathcal{P}(\mathcal{F})$.
(2) The left $\operatorname{add}(M)$-approximation $f_C \colon C \to M_C$ is an $\mathcal{E}$-inflation, let $D := \operatorname{coker}(f_C)$. This implies we have a a left exact sequence $0 \to \operatorname{Ext}^1_{\mathcal{E}}(D, M) \to \operatorname{Ext}^1_{\mathcal{E}}(M_C, M) \to \operatorname{Ext}^1_{\mathcal{E}}(C, M)$. Now $f_D \colon D \to M_D$ together with the composition $M_C \to M_D$ and passing looking at the inclusion of the subfunctor $\operatorname{Ext}^1_{\mathcal{F}} \subseteq \operatorname{Ext}^1_{\mathcal{E}}|_{\mathcal{M}}$ we look at the following commutative diagramm

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ext}^1_{\mathcal{E}}(D, M) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{E}}(M_C, M) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{E}}(C, M) \\
 & & \uparrow & & \uparrow & & \\
 & & \operatorname{Ext}^1_{\mathcal{E}}(M_D, M) & & & & \\
 & & \uparrow & & & & \\
 & & \operatorname{Ext}^1_{\mathcal{F}}(M_D, M) & \longrightarrow & \operatorname{Ext}^1_{\mathcal{F}}(M_C, M) & &
\end{array}$$

Now, we claim $\operatorname{Ext}^1_{\mathcal{F}}(M_A, M) \to \operatorname{Ext}^1_{\mathcal{E}}(A, M)$ is injective for all objects A (then applied in the diagramm for $A = D$ and $A = C$ implies first that $\operatorname{Ext}^1_{\mathcal{F}}(M_D, M) \to \operatorname{Ext}^1_{\mathcal{F}}(M_C, M)$ in a monomorphism and then that $\operatorname{Ext}^1_{\mathcal{F}}(M_D, M) = 0$)).
Let us prove the claim: The map is given by pull-back $\mathcal{F}$-short exact sequences ending in M_A along the map f_A, assume the lower line pulls back to a split exact sequence. Then we find the dashed morphism $f \colon A \to M_1$ such that the triangle commutes

$$\begin{array}{ccccc}
 & & & & A \\
 & & & \swarrow & \downarrow f_A \\
M & \rightarrowtail & M_1 & \twoheadrightarrow & M_A
\end{array}$$

As $M_1 \in \mathcal{M}$ using that f_A is a left $\mathcal{M}$-approximation we find splitting $M_A \to M_1$ of the lower exact sequence. This show the injectivity.
Secondly, we need to see that $\mathcal{P}(\mathcal{F}) \subseteq \operatorname{add}(M_G \oplus M_D)$.
We claim (call this (*)): For every $\mathcal{E}$-exact sequence $(i, d) \colon X \rightarrowtail M_{G_1} \twoheadrightarrow Y$ with $G_2' \in \operatorname{add}(G)$ and $Y \in \operatorname{add}(M)$ we have i is a left $\operatorname{add}(M)$ approximation, i.e. we may assume $i = f_X$.

Before, we proof the claim, let us explain its consequence. Let $Q \in \mathcal{P}(\mathcal{F})$ and take an $\mathcal{E}$-deflation $q\colon G' \twoheadrightarrow Q$ with $G' \in \operatorname{add}(G)$. It factors over an $\mathcal{E}$-deflation $q'\colon M_{G'} \to Q$. Let $X = \ker q'$, then by claim (*), we have $Q = \operatorname{coker} f_X = M_{\operatorname{coker} f_X} \in \operatorname{add}(M_G \oplus M_D)$.
Proof of claim (*): Let $G = \bigoplus_{i\in I} G^i$ be a direct sum decomposition into indecomposables. We assume wlog $M_G = \bigoplus_{i\in I} M_{G^i}$. We use the horse-shoe lemma to produce split exact sequence $H_1 \rightarrowtail H_2 \twoheadrightarrow H_3$, $H_i \in \operatorname{add}(G)$ such that there is a morphism (p_1, p_2, p_3) to the short exact sequence (i, d) with all p_j $\mathcal{E}$-deflations. Then p_2, p_3 have to factor over f_{H_2}, f_{H_3} respectively and therefore also p_1 has to factor over f_{H_1} (all of these are $\mathcal{E}$-deflations). We find another split exact sequence $M_{H_1} \rightarrowtail M_{H_2} \twoheadrightarrow M_{H_3}$ such that there exists a morphism (h_1, h_2, h_3) of ses to (i, d) with all h_i are $\mathcal{E}$-deflations. As $M_{G_2'} \in \operatorname{add}(M_G)$ it follows that h_2 is a split epimorphism, i.e. we find another split exact sequence $M_{G_2} \rightarrowtail M_{H_2} \twoheadrightarrow M_{G_2'}$. Now, we define G_1 to be the largest common summand of the two summands H_1 and G_2 in H_2 and we set $G_1' = H_1/G_1$. As M_{G_1} is mapped under q_1 to zero, we get a commutative diagram

$$\begin{array}{ccc} M_{G_1'} & \xhookrightarrow[split]{} & M_{G_2'} \\ \big\downarrow q_1' & & \big\downarrow = \\ X & \xrightarrow{i} & M_{G_2'} \end{array}$$

with q_1' an $\mathcal{E}$-deflation. Right, now after all this affords we produced a split monomorphism $j\colon G_1' \to G_2'$ with $r\colon G_2' \to G_1'$, $rj = \mathrm{id}_{G_1'}$ such that we find a commutative diagram

$$\begin{array}{ccc} G_1' & \xhookrightarrow[j]{} & G_2' \\ \big\downarrow g_1 & & \big\downarrow f_{G_2'} \\ X & \xrightarrow{i} & M_{G_2'} \end{array}$$

with g_1 also an $\mathcal{E}$-deflation. Then we look at a morphism $t\colon X \to M$ and we want to see it factors over i. We have there exists $m\colon M_{G_2'} \to M$ such that

$$tg_1 r = m f_{G_2'} \quad \Rightarrow \quad tg_1 = m f_{G_2'} j = m i g_1$$

and since g_1 is an epimorphism it follows $t = mi$. □

Remark 6.2. Even in the representation-finite case: Exact subcategories can have more, less or equal number of indecomposable projectives to the exact category in which they are embedded into.

By a result of Enomoto, [**72**, Prop. 3.14, Cor. 3.15], every exact structure on $\mathcal{A}$ has enough projectives and enough injectives and is an Auslander-Reiten category.
Exact structures on $\mathcal{A}$ is the boolean lattice of generators. The lattice of all exact subcategories has for every additively closed subcategory $\mathcal{B}$ a boolean sublattice of all generators containing the generator constructed in the previous lemma (for $\mathcal{E} = \mathcal{E}_{max}$ the maximal exact structure on $\mathcal{A}$). The disjoint union of all these sublattices contains all exact subcategories in $\mathcal{A}$.
We also easily find:

Lemma 6.3. *Let $\mathcal{A}$ as through-out in this subsection. Then $\mathrm{EX}(\mathcal{A})$ is a finite poset, let $\mathcal{F}_i$ be an exact structure on $\mathcal{B}_i \subseteq \mathcal{A}$, $i = 1, 2$. Let $\mathcal{F}_1 \to \mathcal{F}_2$ be an arrow in the Hasse diagramm, then:*
(1) If the underlying additive categories are equal, then it is an arrow in the Boolean lattice corresponding to this subcategory.
(2) If they are not equal we have $|\mathcal{B}_1| < |\mathcal{B}_2|$ and $\mathcal{F}_1$ is the maximal exact structure making the inclusion $\mathcal{B}_1 \to \mathcal{F}_2$ exact and for all proper intermediate $\mathcal{B}_1 \subsetneq \mathcal{B} \subsetneq \mathcal{B}_2$, if $\mathcal{F}_{\mathcal{B}}$ is the maximal exact structure making the inclusion $\mathcal{B} \to \mathcal{F}_2$ exact then the inclusion $\mathcal{F}_1 \to \mathcal{F}_{\mathcal{B}}$ is not exact.

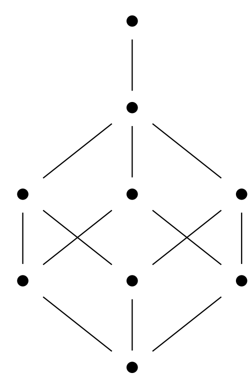
The proof is obvious.
With the previous two Lemmata we can theoretically compute these lattices. Instead we just look at the easiest non-trivial case and count how many objects we have (that already takes some time).

Exact structures on $\mathcal{A}$ is a graded poset by the map: $\mathcal{E} \mapsto |\mathcal{P}(\mathcal{E})| \in \mathbb{N}$ where the last one is the number of indecomposable projectives (up to isomorphism). Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$ and $E(\mathcal{E}) \in \{\mathrm{EX}(\mathcal{E}), \mathrm{Ex}(\mathcal{E}), \mathrm{ext}(\mathcal{E}), \mathrm{ex}(\mathcal{E})\}$. As these lattices are notoriously big, we look instead at the following simple generating function

$$\mu_{E(\mathcal{E})}(X,Y,T) := \sum_{\mathcal{E}' \in E(\mathcal{E}), \mathrm{gldim}\, \mathcal{E}' < \infty} X^{|\mathcal{P}(\mathcal{E}')|} Y^{|\mathcal{E}'|} T^{\mathrm{gldim}\, \mathcal{E}'}$$

e.g. if we set a_{ijk}^{E} to be the multiplicity of $X^i Y^j T^k$. For example, let Q be a Dynkin quiver, then $a_{nn0}^{\mathrm{ext}(KQ\,\mathrm{mod})}$ with $n = |Q_0|$ is the number of basic tilting KQ-modules.
Now, let us look at the simplest cases.
Let $\mathcal{E} = \Lambda$-mod with $\Lambda = K(1 \to 2 \to 3 \to \cdots \to n)$ and let G be basic module given by the direct sum of all indecomposable non-projectives. As a lattice $\mathrm{ex}(\mathcal{E})$ is a cube given by all summands of G, the meet of $\mathrm{add}(G')$ and $\mathrm{add}(G'')$ is given by $\mathrm{add}(G' \oplus G'')$, and the join is given by $\mathrm{add}(G) \cap \mathrm{add}(G')$.

Example 6.4. $n = 2$, then $\mathrm{EX}(\mathcal{E})$ has 9 elements, the cube on the bottom is $\mathrm{ADD}(\mathcal{A})$ (i.e. all split exact structures) and the maximal element is the abelian structure on $\mathcal{E}$, i.e. the Hasse diagramm looks like



Example 6.5. $n = 3$, we have 8 generators, so $\mathrm{ex}(\mathcal{E})$ has 8 elements, we have $2^6 = 64$ additively closed subcategories in $\mathrm{ADD}(\mathcal{A})$ of whom 34 are extension-closed, then $\mathrm{Ex}(\mathcal{E})$ has 56 elements and $\mathrm{EX}(\mathcal{E})$ has 95.

$$\begin{aligned}
\mu_{\mathrm{ex}(\mathbb{A}_3)}(X,Y,T) &= Y^6[X^6 + (X^3 + 2X^4 + 3X^5)T + X^4T^2] \\
\mu_{\mathrm{ext}(\mathbb{A}_3)}(X,Y,T) &= 1 + 6(XY) + 10(XY)^2 + 5(XY)^3 \\
&\quad + [4Y^3X^2 + (5Y^4 + 2Y^5 + Y^6)X^3]T \\
\mu_{\mathrm{Ex}(\mathbb{A}_3)}(X,Y,T) &= 1 + 6(XY) + 10(XY)^2 + 9(XY)^3 + 5(XY)^4 + 2(XY)^5 + (XY)^6 \\
&\quad + [4Y^3X^2 + (5Y^4 + 2Y^5 + Y^6)X^3 + (4Y^5 + 2Y^6)X^4 + 3Y^6X^5]T \\
&\quad + Y^6X^4T^2 \\
\mu_{\mathrm{EX}(\mathbb{A}_3)}(X,Y,T) &= 1 + 6XY + 15(XY)^2 + 20(XY)^3 + 15(XY)^4 + 6(XY)^5 + (XY)^6 \\
&\quad + +[4Y^3X^2 + (9Y^4 + 2Y^5 + Y^6)X^3 + (8Y^5 + 2Y^6)X^4 + 3Y^6X^5]T \\
&\quad + (Y^4X^3 + Y^6X^4)T^2
\end{aligned}$$

Example 6.6. $n = 4$, $E(\mathcal{E}) = \mathrm{ex}(\mathcal{E})$ has $2^6 = 64$ elements. Then one can calculate

$$\begin{aligned}
\mu_{\mathrm{ex}(\mathbb{A}_4)}(X,Y,T) = Y^{10}[X^{10} &+ (X^4 + 3X^5 + 7X^6 + 14X^7 + 12X^8 + 6X^9)T \\
&+ (3X^5 + 7X^6 + 5X^7 + 3X^8)T^2 + (X^6 + X^7)T^3]
\end{aligned}$$

Observe, that we have 42 exact substructures of gldim = 1, this means the poset of hereditary exact substructures has 43 elements, which is substantially more then the 24 submodule-closed generators.

CHAPTER 4

Exact categories represented by morphisms

1. Synopsis

For every module over a ring there is the associated module category over the endomorphism ring, i.e. a construction of a new module category over a ring.
This has the following generalization to exact categories: For an exact category $\mathcal{E}$ and a full subcategory $\mathcal{M}$ we look at all the $\mathcal{E}$-admissible morphisms S in $\mathcal{M}$ and define the category of S-presented functors $F\colon \mathcal{M}^{op} \to (\mathrm{Ab})$, this is a fully exact category of all finitely presented functors. Interestingly, we can also choose other classes of morphisms (for example deflations or inflations) and still obtain an exact category. Of course this topic deserves a systematic study that I can not give (due to time constraints).
We have already seen in Chapter 2, the Auslander exact category as an instance of this construction, we explain its generalizations which lead to the generality of a contravariantly finite generator $\mathcal{M}$ in an arbitrary exact category. We give a short history of ideas in the next section.
Furthermore, we have an exact category with enough projectives always presented by its admissible morphism between the projectives. Tilting subcategories are a generalization of the subcategory of projectives and we start using this construction in tilting theory for exact categories (cf. Chapter 10).
What is new? Functor categories represented by (general) classes of morphisms (in the literature you find either admissible morphisms or deflations). The generator correspondence for exact categories.

2. A short history of ideas

Of course, one would like to have an *endo-dictionary* translating properties into each other. Related to this is the question: Can one *reconstruct* the module category/exact category from this endomorphism ring/admissibly presented functor category?
This question has been answered in many different situations, we quickly survey the history of ideas here: We start recalling two result of M. Auslander.

Theorem 2.1. ***(Auslander correspondence, 1971***, [**17**]) *There exists a bijection between the set of Morita-equivalence classes of representation-finite finite-dimensional algebras* Λ *and that of finite-dimensional algebras* Γ *with* $\operatorname{gldim}\Gamma \leq 2 \leq \operatorname{domdim}\Gamma$. *It is given by* $\Lambda \mapsto \Gamma = \operatorname{End}_\Lambda(M)$ *where* $\operatorname{mod}\Lambda = \operatorname{add}(M)$

This has further generalizations to (from special to more general, some predate Auslander's result)

(*) The higher Auslander correspondence [**102**]
(*) The Morita-Tachikawa correspondence [**139**], [**178**]
(*) The generator correspondence [**177**], [**27**]

They are all instances of faithfully balancedness which we explain by considering an assignment of Morita equivalence classes of pairs of rings and modules and the assignment

$$\mathbb{E}\colon [\Lambda, {}_\Lambda M] \mapsto [\Gamma = \operatorname{End}_\Lambda(M), {}_\Gamma M]$$

Then we call a module ${}_\Lambda M$ **faithfully balanced** if $\mathbb{E}^2[\Lambda, M] = [\Lambda, M]$. All *correspondences* (for rings and modules) using this assignment $\mathbb{E}$ are instances of faithfully balanced modules. As a

feature, $\operatorname{Hom}(-, M)$ then always gives a duality between certain subcategories of the module categories [**136**, Lem 2.9] and all dualities given by such a Hom-functor arise from faithfully balanced modules (e.g. Matlis duality - [**128**]). The best studied examples apart from (co)generators are (co)tilting (cf. [**137**]) in which the duality becomes the Theorem of Brenner and Butler ([**42**]). There are many more correspondences of faithfully balanced modules, we recommend to read the introduction in [**136**].
In loc. cit. we generalized faithfully balanced to Auslander-Solberg exact structures of finite type on f.d. module categories of f.d. algebras. In Chapter 5 we look at faithfully balancedness in functor categories (because we need some of the results in tilting theory for exact categories).
(It is unclear how general faithfully balancedness can be defined - but all ambient exact categories are with enough projectives and we think that having enough projectives will always be an assumption for the set-up.)
Then, secondly

THEOREM 2.2. ***(Auslander's formula, 1966***, [**16**]*) Let $\mathcal{C}$ be a small abelian category and* $\operatorname{mod}_1 \mathcal{C}$ *the category of finitely presented additive functors $\mathcal{C}^{op} \to (Ab)$. Then this is an abelian category and there exists a left adjoint exact functor $L\colon \operatorname{mod}_1 \mathcal{C} \to \mathcal{C}$ to the Yoneda embedding. Its kernel is a Serre subcategory, called the* effaceable functors*, $\ker L = \operatorname{eff}(\mathcal{C})$ and there is an induced equivalence*

$$\operatorname{mod}_1 \mathcal{C}/\operatorname{eff}(\mathcal{C}) \to \mathcal{C}$$

This suggest a different way of reconstructing the category $\mathcal{C}$ from its module category $\operatorname{mod}_1 \mathcal{C}$, namely as localization with respect to the subcategory of effaceable functors. We call this approach *reconstruction using Auslander's formula*. Using this the following has been generalized to arbitrary exact categories

(*) The Auslander correspondence [**90**]
(*) The higher Auslander correspondence [**68**]
(*) The Morita-Tachikawa correspondence [**83**]
(*) The generator correspondence, cf. Theorem 3.28

Nevertheless, the assignment considered in all cases is the following:
To an exact category $\mathcal{E}$ and an additively closed subcategory $\mathcal{M}$, we assign the category $\operatorname{mod}_S \mathcal{M}$ of additive functors $F\colon \mathcal{M}^{op} \to (Ab)$ such that there exists an $\mathcal{E}$-admissible morphism $s\colon M_1 \to M_0$, $M_i \in \mathcal{M}$ such that $F = \operatorname{coker} \operatorname{Hom}_{\mathcal{M}}(-, s)$. We write this as assignment of (exact equivalence classes of) pairs of exact categories together with a subcategory.

$$\mathbb{E}'\colon [\mathcal{E}, \mathcal{M}] \mapsto [\operatorname{mod}_S \mathcal{M}, \operatorname{eff}(\mathcal{M})]$$

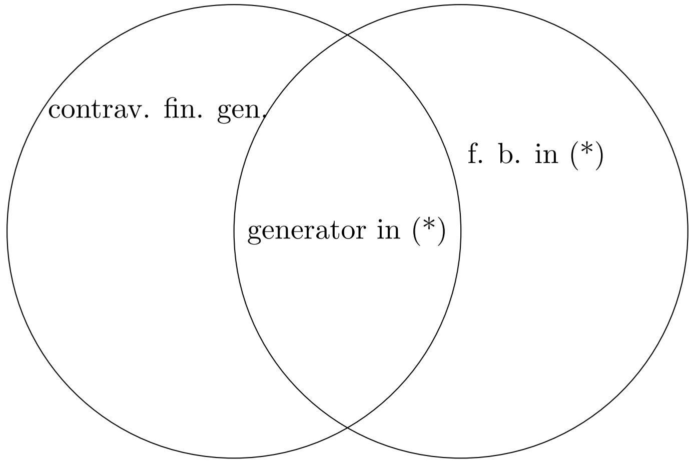
We will consider $\mathbb{E}'$ and $\mathbb{E}$ at least on the first entry as the same assignment.
The question is: "Can we find a localization sequence as in Auslanders formula which reconstructs $\mathcal{E}$ and $\mathcal{M}$ and therefore gives an inverse assignment to $\mathbb{E}'$?".

The obvious general open question

Open question 2.3. How does the first and the second type correspondence fit together? Both are crucially using adjoint pairs of functors, is there a joint generalization?

Let us look at an idempotent recollement on an endomorphism ring of a generator. In this situation we can study this generclosedator as a faithfully balanced module or we can look at the recollement and can recover the right hand side abelian category as a localization. Yet for other faithfully balanced modules this is not known to be true, so a description with an Auslander formula can not really be expected.

always small exact categories with subcategory $\mathcal{M}$:



where (*) is an exact category in which faithfully balanced (f.b.) is defined, so far only for Auslander-Solberg exact structures of finite type [**136**] and for categories of all additive functors $\operatorname{Mod}\mathcal{P}$ with $\mathcal{P}$ essentially small, cf. Chapter 5.

3. Presentations of exact categories

We start with a study of *exact* categories of the form $\operatorname{mod}_S \mathcal{M}$ (called exactly presented by $(\mathcal{M}, S)$) for some class of morphisms S in the first section. Then we translate properties of S into properties of $\operatorname{mod}_S \mathcal{M}$. The most important: S has weak kernels in S translates into $\operatorname{mod}_S \mathcal{M}$ is has enough projectives given by the presentables.

Definition 3.1. Given an additive category $\mathcal{M}$ and a class of morphisms S in $\mathcal{M}$. We say it is **closed under homotopy** if for two morphisms s, t in $\mathcal{M}$ with $\operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, s) \cong \operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, t)$ in $\operatorname{mod}_1 \mathcal{M}$ we have $s \in S$ if and only if $t \in S$.

Being homotopy-closed is often useful, such as:

Lemma 3.2. *If S is closed under homotopy and direct sums and summands of morphisms (i.e. $s, t \in S$ iff $s \oplus t \in S$) then* $\operatorname{mod}_S \mathcal{M}$ *is an additively closed subcategory in* $\operatorname{Mod}\mathcal{M}$.

The proof is obvious.

Remark 3.3. If S is homotopy closed then it contains all split epimorphisms. If every representable $\operatorname{Hom}(-, M)$ is of the form $\operatorname{coker}\operatorname{Hom}(-, s)$ for some $s \in S$ and if S is homotopy closed then all split admissible morphisms are contained in it.

Lemma 3.4. *If $S \subseteq \operatorname{Mor}\mathcal{M}$ is a class of morphisms which is closed under direct sums and summands. If S contains all split epimorphisms then S is homotopy-closed.*

Proof. Assume $F = \operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, s) = \operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, t)$. We look at the projective presentations of F in $\operatorname{Mod}\mathcal{M}$, say $s\colon M_1 \to M_0$, $t\colon N_1 \to N_0$. Then we have $s \oplus \operatorname{id}_{N_0} \oplus (N_1 \to 0) \cong t \oplus \operatorname{id}_{M_0} \oplus (M_1 \to 0)$ and by assumption $s \in S$ if and only if $t \in S$. □

Definition 3.5. Let $\mathcal{M}$ be an additive category and S a class of morphisms in $\mathcal{M}$. We define the **category of S-represented $\mathcal{M}$-modules** $\operatorname{mod}_S \mathcal{M}$ to be the full subcategory of $\operatorname{Mod}\mathcal{M}$ consisting of the functors $F\colon \mathcal{M}^{op} \to (\mathrm{Ab})$ such that there exists an exact sequence

$$\operatorname{Hom}(-, M_1) \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}(-, M_0) \to F \to 0$$

Definition 3.6. We say that a class of morphisms S on an additive category $\mathcal{M}$ (or the pair $(\mathcal{M}, S)$) is an **exact presentation** if $\mathcal{E} = \operatorname{mod}_S \mathcal{M}$ is an extension-closed subcategory in $\operatorname{Mod} \mathcal{M}$ (we will always equip it with this exact structure). In this case, we also say the exact category $\mathcal{E}$ is **exactly represented** by $(\mathcal{M}, S)$.

We will from now on assume that S is homotopy-closed and closed under direct sums.

Example 3.7. Exactly presented exact categories are precisely all fully exact subcategories of $\operatorname{mod}_1 \mathcal{M}$ for some additive category $\mathcal{M}$. If $\mathcal{E}$ is a fully exact subcategory of $\operatorname{mod}_1 \mathcal{M}$, take S to be the class of all morphisms s in $\mathcal{M}$ such that $\operatorname{coker} \operatorname{Hom}_{\mathcal{M}}(-, s)$ lies in $\mathcal{E}$, then obviously $\mathcal{E} = \operatorname{mod}_S \mathcal{M}$.

Definition 3.8. We say an exact category $\mathcal{E}$ is **projectively determined** (by $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{E})$) if a kernel-cokernel pair (i, p) is an exact sequence in $\mathcal{E}$ if and only if $(\operatorname{Hom}(Q, i), \operatorname{Hom}(Q, p))$ is an exact sequence of abelian groups for all $Q \in \mathcal{Q}$.
More generally, given an exact subcategory $i\colon \mathcal{E}' \subseteq \mathcal{E}$, we say that $\mathcal{E}'$ is **projectively determined** by $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{E}')$ **inside** $\mathcal{E}$ if: An $\mathcal{E}$-exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ with X, Y, Z in $\mathcal{E}'$ is $\mathcal{E}'$-exact if and only if $(\operatorname{Hom}(Q, i), \operatorname{Hom}(Q, p))$ is an exact sequence of abelian groups for all $Q \in \mathcal{Q}$.

In particular if $\mathcal{E}$ is projectively determined by $\mathcal{Q}$ than it is projectively determined by $\mathcal{Q}$ inside the maximal exact structure on the underlying additive category.

Example 3.9. Given an exact category $\mathcal{E}$ with enough projectives then $\mathcal{E}$ is projectively determined.

Example 3.10. Let $\mathcal{E}$ be an exact category. Every additive subcategory $\mathcal{M}$ of $\mathcal{E}$ gives an exact substructure $\mathcal{E}_{\mathcal{M}} \leq \mathcal{E}$ defined as follows: A $\mathcal{E}$-exact sequence (i, p) is an exact sequence in $\mathcal{E}_{\mathcal{M}}$ if and only if $(\operatorname{Hom}(Q, i), \operatorname{Hom}(Q, p))$ is an exact sequence of abelian groups for all $Q \in \mathcal{M}$.
These exact substructures are called Auslander-Solberg structures. By definition they are projectively determined by $\mathcal{M}$ inside $\mathcal{E}$. In particular, if $\mathcal{E}$ is projectively determined by $\mathcal{Q}$, then $\mathcal{E} = \mathcal{E}^{max}_{\mathcal{Q}}$ is the Auslander-Solberg structure given by $\mathcal{Q}$ of the maximal exact structure on the underlying additive category.

Example 3.11. Given an exact category $\mathcal{E}$ which is projectively determined by $\mathcal{Q}$. If $\mathcal{E}'$ is a fully exact subcategory which also contains $\mathcal{Q}$ then $\mathcal{E}'$ is also projectively declosedtermined by $\mathcal{Q}$ and also projectively determined by $\mathcal{Q}$ inside $\mathcal{E}$.

Definition 3.12. Let S be a class of morphisms in an additive category $\mathcal{M}$, then we say S has **weak kernels** in S if for every morphism $s\colon M \to N$ in S there exists another morphism $t\colon L \to M$ such that

$$\operatorname{Hom}_{\mathcal{M}}(-, L) \xrightarrow{\operatorname{Hom}(-,t)} \operatorname{Hom}_{\mathcal{M}}(-, M) \xrightarrow{\operatorname{Hom}(-,s)} \operatorname{Hom}_{\mathcal{M}}(-, N)$$

is exact in the middle (in $\operatorname{Mod} \mathcal{M}$).
Dually weak cokernels in S if S^{op} has weak kernels in S^{op}.

Lemma 3.13. *Let S be a class of morphisms closed under isomorphism, direct sums and summands and contains all split admissible morphisms. Let $\mathcal{E}$ be exactly presented by $(\mathcal{M}, S)$. Let $\widetilde{\mathcal{M}} = \{\operatorname{Hom}(-, M) \in \operatorname{mod}_1 \mathcal{M}\colon M \in \mathcal{M}\}$ and $\widetilde{S} := \{\operatorname{Hom}(-, s) \mid s \in S\}$.*
The following are equivalent

(a) *S has weak kernels in S*
(b) *$\widetilde{S}$ equals all $\mathcal{E}$-admissible morphisms in $\widetilde{\mathcal{M}}$*
(c) *$\mathcal{E}$ has enough projectives given by $\operatorname{add}(\widetilde{\mathcal{M}})$.*

Proof. Very easy. We leave it to the reader. □

Definition 3.14. We call a class of morphisms S in a category $\mathcal{M}$ **suitable** if $(\mathcal{M}, S)$ is an exact presentation and S has weak kernels in S and is closed under homotopy.
In this case we say that the exact category $\operatorname{mod}_S \mathcal{M}$ is **suitably presented** by $(\mathcal{M}, S)$.

We also observe that $\widetilde{\mathcal{M}} \subseteq \mathcal{E}$ implies $\mathcal{E}$ is projectively determined by $\widetilde{\mathcal{M}}$ (as $\operatorname{Mod}\mathcal{M}$ fulfills this and $\mathcal{E}$ is fully exact in it).

Lemma 3.15. *Let $\mathcal{E}$ be an exact category and $\mathcal{M}$ a full additively closed subcategory. Let S be either*

(1) *S_{adm} all $\mathcal{E}$-admissible morphisms in $\mathcal{M}$, or*
(2) *S_{infl} all $\mathcal{E}$-inflations in $\mathcal{M}$, or*
(3) *S_{defl} all $\mathcal{E}$-deflations in $\mathcal{M}$,*

then S is an exact presentation. If the ambient exact category $\mathcal{E}$ is clear, we will use the following notation:

$$(1)\ \operatorname{mod}_{\text{adm}}\mathcal{M} = \operatorname{mod}_{S_{\text{adm}}}\mathcal{M}, \quad (2)\ \mathrm{H}_{\mathcal{M}} := \operatorname{mod}_{S_{\text{infl}}}\mathcal{M}, \quad (3)\ \operatorname{eff}_{\mathcal{M}} = \operatorname{mod}_{\mathrm{S}_{\text{defl}}}\mathcal{M}$$

Proof. (1) The proof is an easy adaptation of [**90**, Prop. 3.5].
(2), (3) We just need to see that $H_{\mathcal{M}}$ and $\operatorname{eff}_{\mathcal{M}}$ are extension-closed in $\operatorname{mod}_{\text{adm}}\mathcal{M}$ but this follows from the horseshoe lemma and [**49**, Cor. 3.2]. □

Corollary 3.16. *If $\mathcal{E}$ is an exact category with enough projectives $\mathcal{P}$. Then*

$$\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_{\text{adm}}\mathcal{P}, \quad E \mapsto \operatorname{Hom}(-,E)|_{\mathcal{P}}$$

is an exact equivalence (i.e. equivalence which is an exact functor and its quasi-inverse is also an exact functor). If $\mathcal{P}$ is idempotent complete, then $\operatorname{mod}_{\text{adm}}\mathcal{P}$ is a resolving subcategory in $\operatorname{mod}_{\infty}\mathcal{P}$.

Now, we need the following observation:

Lemma 3.17. *(*[**50**, Lemma 21, 22]*) Let $\mathcal{E}$ be an exact category.*

(a) *$\mathcal{E}$-inflations is closed under direct summands iff*
$\mathcal{E}$ is weakly idempotent complete iff
$\mathcal{E}$-deflations are closed under direct summands.
(b) *$\mathcal{E}$-admissible morphisms are closed under direct summands if and only if $\mathcal{E}$ is idempotent complete.*

This can be used to:

Example 3.18. If $\mathcal{M} \subseteq \mathcal{E}$ is a contravariantly finite generating subcategory and S_{adm} the class of admissible morphisms on $\mathcal{M}$. Then S_{adm} has weak kernels in S_{adm} and $\operatorname{mod}_{\text{adm}}\mathcal{M}$ has enough projectives given by $\operatorname{add}(\widetilde{\mathcal{M}})$. We have that the functor $E \mapsto \operatorname{Hom}(-,E)|_{\mathcal{M}}$ restricts to a fully faithful functor $\Phi\colon \mathcal{E} \to \operatorname{mod}_{\text{adm}}\mathcal{M}$.
By the previous Lemma and Lemma 3.4: If $\mathcal{E}$ is idempotent complete then $(\mathcal{M}, S_{\text{adm}})$ is a suitable presentation.

Example 3.19. If the exact structure of $\mathcal{E}$ restricts to $\mathcal{M}$ to an abelian structure, then every $\mathcal{E}$-admissible has a kernel in $\mathcal{M}$ which is given by an $\mathcal{E}$-inflation and so the class of $\mathcal{E}$-admissible morphism on $\mathcal{M}$ coincides with all morphisms in $\mathcal{M}$ and this is suitable (this presents the abelian category $\operatorname{mod}_1\mathcal{M}$).

Here is a little warning.

Remark 3.20. Given a fully exact subcategory $\mathcal{F} \subseteq \mathcal{E}$, then there might be many more $\mathcal{E}$-admissible morphisms on $\mathcal{F}$ than there are $\mathcal{F}$-admissible ones. Even if $\mathcal{F}$ is homologically exact- just consider the case above: $\mathcal{F} = \mathcal{P}(\mathcal{E})$ is semi-simple, only projections onto summands are $\mathcal{F}$-admissible.

Remark 3.21. Let us summarize the discussion from before:
We have 1) ⇒ 2) ⇒ 3) ⇒ 4) with

1) Exact categories with enough projectives

2) Exactly presented categories, projectively determined by $\widetilde{\mathcal{M}}$
3) Exactly presented categories
4) Exact categories

We do not know a small exact category which is not exactly presented.
An example fulfilling 3) but not 2) is given by categories of effaceable functors on an exact category.
An example fulfilling 2) but not 1) is given by $\operatorname{mod}_1 \mathcal{M}$ (i.e. finitely presented additive functors in $\operatorname{Mod}\mathcal{M}$) where $\mathcal{M}$ does not have weak kernels (then this category has not enough projectives).

3.0.1. *Universal Property.* Now, we take $\mathcal{F}$ exactly presented by $(\mathcal{M}, S)$ and an additive functor $f\colon \mathcal{M} \to \mathcal{B}$ such that $f(s)$ has a cokernel for all $s \in S$. Then we can define a functor

$$\overline{f}\colon \mathcal{F} \to \mathcal{A}, \quad \overline{f}(\operatorname{coker}\operatorname{Hom}(-,s)) = \operatorname{coker} f(s)$$

Lemma 3.22. *In the above situation*

(1) *Assume that $\mathcal{A}$ is weakly idempotent complete. If $f\colon \mathcal{M} \to \mathcal{E}$ with $\mathcal{E}$ an exact structure on $\mathcal{A}$ and if $f(s)$ admissible for all $s \in S$, the $\overline{f}\colon \mathcal{F} \to \mathcal{E}$ is right exact.*
(2) *In the situation of (1); If all morphisms in $s \in S$ there exist a weak kernel $t \in S$ such that $(f(t), f(s))$ is exact in the middle then $\overline{f}\colon \mathcal{F} \to \mathcal{E}$ is exact.*

Proof. (1) Assume $F_1 \rightarrowtail F_2 \twoheadrightarrow F_3$ is exact in $\mathcal{F}$. We pick $s_i \in S$ such that $\operatorname{coker}\operatorname{Hom}_{\mathcal{M}}(-, s_i) = F_i$, $i = 1, 2, 3$. We now consider the $\operatorname{Hom}(-, s_i)$ as projective presentations of F_i in $\operatorname{Mod}\mathcal{M}$. We can assume by the horseshoe Lemma (using that we assume S is homotopy-closed) that these projective presentations are degree-wise split. As f is an additive functor, we obtain that $(f(s_1), f(s_2), f(s_3))$ is a morphism of two split exact sequences in $\mathcal{E}$. As $\mathcal{A}$ is weakly idempotent complete and $f(s_i)$ are $\mathcal{E}$-admissible we can apply the snake lemma and we obtain a right exact sequence $\overline{f}(F_1) \to \overline{f}(F_2) \twoheadrightarrow \overline{f}(F_3)$ on the cokernels.
(2) We repeat the same steps as in (1) but now with one longer projective presentations. The exactness of the outer sequences implies the exactness of the middle sequence in $\mathcal{F}$. Then apply the snake lemma.

□

Lemma 3.23. ***(Universal property)*** *Let $\mathcal{F}$ be suitably presented by $(\mathcal{M}, S)$. For every functor $f\colon \mathcal{M} \to \mathcal{E}$ which maps S to $\mathcal{E}$-admissible morphisms, the right exact functor $\overline{f}\colon \mathcal{F} \to \mathcal{E}$ with $\overline{f}(\operatorname{coker}(\operatorname{Hom}_{\mathcal{M}}(-,s))) = \operatorname{coker} f(s)$ for all $s \in S$ is (up to isomorphism of functors) the unique right exact functor F with $F \circ \mathbb{Y} = f$ where $\mathbb{Y}\colon \mathcal{M} \to \operatorname{mod}_{\mathrm{adm}}\mathcal{M}$, $M \mapsto \operatorname{Hom}_{\mathcal{M}}(-, M)$ is the Yoneda embedding.*

Proof. Let F be a right exact functor with $F \circ \mathbb{Y} = f$. Then as F is right exact and $\operatorname{Hom}(-,s)$ admissible for all $s \in S$ (because $\mathcal{F}$ is suitably presented). It follows that $F(\operatorname{coker}\operatorname{Hom}(-,s)) = \operatorname{coker} f(s) = \overline{f}(\operatorname{coker}\operatorname{Hom}(-,s))$. □

Remark 3.24. Observe that we do not need that $\mathcal{F}$ is suitably presented by $(\mathcal{M}, S)$ to show that $\overline{f}$ is right exact. But for the unique characterization we assume (even though it is a bit stronger than necessary).

3.1. Results for admissible morphisms. Let us come to the Yoneda embedding, recall from Lemma 3.13.

Remark 3.25. Let $\mathcal{M}$ be an additively closed subcategory in an idempotent complete exact category $\mathcal{E}$, then the Yoneda embedding $\mathbb{Y}\colon \mathcal{M} \to \operatorname{mod}_{\mathrm{adm}}\mathcal{M}$ reflects admissibility if and only if S_{adm} is suitable.

Definition 3.26. Let $\mathcal{E}$ be an exact category and $\mathcal{M} \subseteq \mathcal{E}$ be an additively closed subcategory and we denote by S the class of $\mathcal{E}$-admissible morphisms in $\mathcal{M}$. Then we define the following full subcategory of $\operatorname{mod}_S \mathcal{M}$

$$\operatorname{eff}_{\mathcal{M}} = \{F = \operatorname{coker} \operatorname{Hom}_{\mathcal{E}}(-, d)|_{\mathcal{M}} \mid d \text{ deflation } \}$$

and call this the subcategory of $\mathcal{M}$-effaceable functors.
We say that $\mathcal{M}$ **satisfies the Auslander formular** if $\operatorname{eff}_{\mathcal{M}}$ is a two-sided percolating subcategory (definition of [**90**]) and the quotient $\operatorname{mod}_S \mathcal{M}/\operatorname{eff}_{\mathcal{M}}$ is equivalent as an exact category to $\mathcal{E}$.

Lemma 3.27. *Let $\mathcal{M}$ be a contravariantly finite generator in $\mathcal{E}$, let $\underline{S}_{\text{adm}}$ be all $\mathcal{E}$-admissible morphisms in $\mathcal{M}$. Let $\operatorname{inc}\colon \mathcal{M} \subseteq \mathcal{E}$ be the inclusion functor and $L = \overline{\operatorname{inc}}\colon \operatorname{mod}_{\text{adm}} \mathcal{M} \to \mathcal{E}$ defined by $L(\operatorname{coker} \operatorname{Hom}_{\mathcal{M}}(-, s)) = \operatorname{coker} s$. We denote by $\Phi\colon \mathcal{E} \to \operatorname{mod}_{\text{adm}} \mathcal{M}$ the functor $\Phi(E) = \operatorname{Hom}_{\mathcal{A}}(-, E)|_{\mathcal{M}}$.*

(i) *L is exact, Φ is left exact and $\ker L = \operatorname{eff}_{\mathcal{M}}$*
(ii) *(L, Φ) are an adjoint pair, Φ is fully faithful, L is essentially surjective and $L \circ \Phi \cong \operatorname{id}_{\mathcal{E}}$*
(iii) *The subcategory $\operatorname{eff}_{\mathcal{M}}$ is percolating in $\operatorname{mod}_{\text{adm}} \mathcal{M}$ and the functor L factors over an equivalence of exact categories*

$$L'\colon \operatorname{mod}_{\text{adm}} \mathcal{M}/\operatorname{eff}_{\mathcal{M}} \to \mathcal{E}$$

Proof. (i) To see that L is exact we observe that we can find weak kernel of morphisms in $\mathcal{M}$ which give middle exact sequences and so Lemma 3.22 (2) applies. Clearly, for an $s \in S_{\text{adm}}$, we have: $L(\operatorname{coker} \operatorname{Hom}(-, s)) = \operatorname{coker}(s) = 0$ if and only if s is an $\mathcal{E}$-deflation.

We need to see that Φ maps deflations to admissible morphisms. So given an exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ we have a left exact sequence of functors $0 \to \Phi(X) \to \Phi(Y) \to \Phi(Z)$. So we need to see that if a is an inflation or a deflation then $\operatorname{coker} \Phi(a)$ in $\operatorname{Mod} \mathcal{M}$ is already in $\operatorname{mod}_{\text{adm}} \mathcal{M}$. Let $a\colon X \to Y$, let $p\colon M_Y \twoheadrightarrow Y$ be a right $\mathcal{M}$-approximation, we pull back p along a, i.e. we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{b} & M_Y \\ \downarrow & & \downarrow p \\ X & \xrightarrow[a]{} & Y \end{array}$$

We claim in both cases (a inflation or deflation) we have a commutative diagram with right exact rows

$$\begin{array}{ccccc} \Phi(R) & \xrightarrow{\Phi(b)} & \Phi(M_Y) & \twoheadrightarrow & F \\ \downarrow & & \downarrow p & & \downarrow = \\ \Phi(X) & \xrightarrow[\Phi(a)]{} & \Phi(Y) & \twoheadrightarrow & F \end{array}$$

Then let $r\colon M_R \twoheadrightarrow R$ be a right $\mathcal{M}$-approximation, it follows $br \in S_{\text{adm}}$ and $F = \operatorname{coker} \Phi(br)$.

(ii) As $\mathcal{M}$ is a contravariantly finite generator, L is surjective and Φ is well-defined and fully faithful. Furthermore, we have $L \circ \mathbb{Y} = \operatorname{inc}\colon \mathcal{M} \to \mathcal{E}$ and $\Phi|_{\mathcal{M}} = \mathbb{Y}$. For E in $\mathcal{E}$ we find an $\mathcal{E}$-admissible $s\colon M_1 \to M_0$ in S_{adm} such that $E = \operatorname{coker}(s)$ and $M_0 \to E$, $M_1 \to \operatorname{Im} s$ are $\mathcal{M}$-approximations, this implies that we have a right exact sequence $\Phi(M_1) \xrightarrow{\Phi(s)} \Phi(M_0) \twoheadrightarrow \Phi(E)$, the apply L to conclude $L\Phi(E) \cong E$.

For the adjunction, let $F \in \operatorname{mod}_{\text{adm}} \mathcal{M}$ and $E \in \mathcal{E}$, we claim $\operatorname{Hom}_{\operatorname{mod}_{\text{adm}} \mathcal{M}}(F, \Phi(E)) \cong \operatorname{Hom}_{\mathcal{E}}(L(F), E)$.

Choose $s \in S_{\text{adm}}$ such that $\Phi(M_1) \xrightarrow{\Phi(s)} \Phi(M_0) \twoheadrightarrow F$ is exact in $\operatorname{mod}_{\text{adm}} \mathcal{M}$. Now given a morphism $F \to \Phi(E)$, then the composition $\Phi(M_0) \to \Phi(M_1) \to \Phi(E)$ is zero. As Φ is fully faithful, there is a unique morphism $L(F) = \operatorname{coker}(s) \to E$. Conversely, as F is the cokernel of $\Phi(s)$, we have a unique morphism $c\colon F \to \Phi(\operatorname{coker}(s)) = \Phi L(F)$, so given an $a\colon L(F) \to E$ we just map it to $\Phi(a)c$.

(iii) The argument that $\mathrm{eff}_{\mathcal{M}}$ is percolating is just the observation that [**90**, Prop 3.6, 3.17] generalize to this set-up. By [**90**, Thm 2.12] we have the induced functor L' such that $L' \circ Q = L$. As Q, L are exact, the same is true for L'. By the same argument as in [**90**, Thm3.11], L' is an additive equivalence. As $L\Phi = \mathrm{id}_{\mathcal{E}}$ and $L = L'Q$, we see that $Q\Phi$ is the quasi-inverse of L'. We want to see that $Q\Phi$ is exact, as it is already left exact, it is enough to show that it preserves deflations. Let $f\colon X \twoheadrightarrow Y$ be an $\mathcal{E}$-deflation. We take a right $\mathcal{M}$-approximation $M_Y \twoheadrightarrow Y$ and pull back f along it, to an object Z. Then we take the $\mathcal{M}$-approximation of $M_Z \twoheadrightarrow Z$. Now, we have a commutative diagramm

$$\begin{array}{ccc} M_Z & \twoheadrightarrow & M_Y \\ \downarrow & & \downarrow \\ X & \twoheadrightarrow & Y \end{array}$$

with all arrows are $\mathcal{E}$-deflations. Now, Φ maps the right $\mathcal{M}$-approximations to deflations and $Q\Phi$ maps the $\mathcal{E}$-deflation $M_Z \to M_Y$ to one in $\mathrm{mod}_{\mathrm{adm}}\,\mathcal{M}/\,\mathrm{eff}_{\mathcal{M}}$. This implies by the diagonal $M_Z \to Y$ is mapped under $Q\Phi$ to a deflation and then by the obscure axiom it follows that $Q\Phi(f)$ is a deflation.

□

We look at the assignment of (exact equivalence classes of) pairs of exact categories together with a subcategory.

$$\mathbb{E}'\colon [\mathcal{E}, \mathcal{M}] \mapsto [\mathrm{mod}_S\,\mathcal{M}, \mathrm{eff}(\mathcal{M})]$$

As a reformulation we of (3) and (4) one gets:

THEOREM 3.28. ***(Generator correspondence for exact categories)*** *The assignment $\mathbb{E}'$ gives a bijection between*

(1) *Pairs of an exact category $\mathcal{E}$ together with a contravariantly finite generator $\mathcal{M}$*
(2) *Pairs of an exact category $\mathcal{F}$ with a percolating subcategory* eff *satisfying*
  (i) *$\mathcal{F}$ has enough projectives $\mathcal{P}$*
  (ii) $\mathrm{eff} \subseteq {}^{\perp}\mathcal{P} \quad (= \{X \in \mathcal{F} \mid \mathrm{Hom}(X, P) = 0\ \forall P \in \mathcal{P}\})$ *and* eff *is a torsion class*
  (iii) $\mathrm{Ext}^1(\mathrm{eff}, \mathcal{P}) = 0$

PROOF. First of all, we need to see that $\mathbb{E}'$ is well-defined, by Lemma 3.27, (3) we have that $\mathrm{eff}_{\mathcal{M}}$ is percolating and by Ex. 3.18 we have that $\mathcal{F} = \mathrm{mod}_S\,\mathcal{M}$ has enough projectives. For the properties (ii) and (iii), we leave the reader to check that the proofs of [**90**, Prop. 3.6, Prop. 3.17 (3)] generalize to this more general situation.
Define $\mathbb{F}[\mathcal{F}, \mathrm{eff}] := [\mathcal{E} = \mathcal{F}/\,\mathrm{eff}, Q(\mathcal{P}(\mathcal{F}))]$ where $Q\colon \mathcal{F} \to \mathcal{F}/\,\mathrm{eff}$ is the localization with respect to the percolating subcategory and $\mathcal{P} := \mathcal{P}(\mathcal{F})$ are the projectives in $\mathcal{F}$. We first show: Condition (ii) and (iii) ensure that we always have that $\mathcal{P} \subseteq \mathcal{F} \xrightarrow{Q} \mathcal{E}$ is fully faithful.
A morphism $P' \to P$ in $\mathcal{E}$ with $P, P' \in \mathcal{P}$ is given by an equivalence class of pairs $[f, s]$ with $f\colon X \to P$ and $s\colon X \to P'$ is $\mathcal{F}$-admissible such that $\ker(s), \mathrm{coker}(s) \in \mathrm{eff}$.
First we use that we have a torsion pair $(\mathrm{eff}, \mathcal{G})$ and therefore we find an exact sequence $E \rightarrowtail X \twoheadrightarrow G$ with $E \in \mathrm{eff}$, $G \in \mathcal{G}$. As $\mathrm{eff} \subseteq {}^{\perp}\mathcal{P}$, we find a morphism $g\colon G \to P$ such that $f\colon X \twoheadrightarrow G \to P$. By definition $[f, s] = [g, i]$ where $i\colon G \rightarrowtail P'$ is the induced inflation, observe that $\mathrm{coker}(i) = \mathrm{coker}(s) =: E' \in \mathrm{eff}$. So we look at the short exact sequence $G \rightarrowtail P' \twoheadrightarrow E'$ and apply $\mathrm{Hom}_{\mathcal{A}}(-, P)$. Using also (iii) we conclude that $\mathrm{Hom}_{\mathcal{A}}(P', P) \cong \mathrm{Hom}_{\mathcal{A}}(G, P)$, this means that $g\colon G \to P$ factors over $i\colon P' \to P$ uniquely. This implies the functor is full and also faithful because assume that a morphism $p\colon P' \to P$ fulfills $Q(p) = 0$, then there exists some $s\colon X \to P'$ with $\ker(s), \mathrm{coker}(s)$ in eff such that $ps = 0$, now, as s is admissible we have $pi = 0$ with $i\colon X/\,\mathrm{coker}(s) =: G \rightarrowtail P'$. But now $pi\colon G \to P$ is in the image of the isomorphism from before and therefore $p = 0$.

As localization with respect to percolating subcategory reflects admissibility (cf. [**90**, Thm 2.16] we get: $\mathcal{F}$-admissible morphism in $\mathcal{P}$ are precisely $\mathcal{E}$-admissible morphisms in $Q(\mathcal{P}) =: \mathcal{M}$. This bijection restricts to the following two classes:

(a) $\mathcal{F}$-admissible morphisms $p\colon P' \to P$ such that $\operatorname{coker}(p) \in \operatorname{eff}$
(b) $\mathcal{E}$-deflations $P' \to P$ with P, P' in $\mathcal{M}$

That is clear by definition as the bijection follows from applying Q and Q is an exact functor. Therefore we conclude that $\mathcal{F} = \operatorname{mod}_{\mathcal{F}-adm} \mathcal{P} \cong \operatorname{mod}_S \mathcal{M}$ where S are the $\mathcal{E}$-admissible morphisms in $\mathcal{M}$. Under this equivalence, we have the objects in eff, i.e. the objects represented by morphisms in class (a) are mapped to the objects represented by morphisms in $\mathcal{M}$ in class (b) which is the category $\operatorname{eff}_{\mathcal{M}}$. This shows $\mathbb{E}' \circ \mathbb{F}$ is the identity.
Now, we look at $\mathbb{F} \circ \mathbb{E}'$. By the previous Lemma we have $\mathcal{E} \cong \operatorname{mod}_S \mathcal{A}/\operatorname{eff}_{\mathcal{M}}$. We just need to see that $\mathcal{M} \cong Q(\mathcal{P}(\operatorname{mod}_S \mathcal{A}/\operatorname{eff}_{\mathcal{M}}))$. We define $\mathcal{F} := \operatorname{mod}_S \mathcal{A}$ and $\mathcal{P} = \mathcal{P}(\mathcal{F})$. As the S are suitable morphisms for $\mathcal{M}$ (cf. Example 3.18) we have that the Yoneda embedding $\mathbb{Y}\colon \mathcal{M} \to \mathcal{F}$ identifies $\mathcal{M}$ with the projectives $\mathcal{P}$ and S with the $\mathcal{F}$-admissible morphisms between the projectives. But as the composition $\mathcal{P} \subseteq \mathcal{F} \xrightarrow{Q} \mathcal{E}$ is fully faithful (see before), the claim follows. □

All other instances of $\mathbb{E}'$ in the history of idea section are specializations of the generator correspondence.

Open question 3.29. If $\mathcal{M}$ is a contravariantly finite generator and $\mathcal{E}$ has enough projectives, do we have an adjoint triple $(j_!, L, \Phi)$ defining the right half of a recollement of exact categories (cf. in [**90**], this exists for $\mathcal{M} = \mathcal{E}$). Then this should be used to find the right definition of faithfully balancedness in this situation.

3.2. Using other morphisms.

In joint work in progress (with Janina Letz, Marianne Lawson) we conjecture the following:
Let $\mathcal{E} = (\mathcal{A}, S)$ be an exact category. Let S_m be one of the following

(1) Let S_m be the $\mathcal{F}$-inflations for a supstructure $\mathcal{E} \leq \mathcal{F}$ on $\mathcal{A}$
(2) Let S_m be the class of all $\mathcal{A}$-monomorphisms.

Then we define $S \subseteq \operatorname{Mor}(\mathcal{A})$ to be the class of all morphisms s which factor as $s = ip$ with i in S_m and p an $\mathcal{E}$-deflation.
As all morphisms in S have a kernel in $\mathcal{A}$ which is an $\mathcal{E}$-inflation and therefore in S again: In particular S has weak kernels. As S contains all split admissible morphisms, by Lemma 3.4, we conclude that S is closed under homotopy. It is also straightforward to see that the proof in [**90**, Prop. 3.5] generalizes to show that $\operatorname{mod}_S \mathcal{A}$ is extension-closed in $\operatorname{mod}_1 \mathcal{A}$. This means we have S is suitable.

Conjecture 3.30. *Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category and S be a class of morphisms just described.*

(1) $\operatorname{eff}(\mathcal{E})$ *is a percolating subcategory in* $\operatorname{mod}_S \mathcal{A}$
(2) $\mathcal{E} \cong \operatorname{mod}_{\mathrm{adm}} \mathcal{A}/\operatorname{eff}(\mathcal{E}) \to \operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E})$ *is a fully exact subcategory.*

We look at the commutative diagramm

$$\begin{array}{ccc} \operatorname{mod}_S \mathcal{A} & \xrightarrow{\ i\ } & K^b(\mathcal{A}) \\ \downarrow & & \downarrow \\ \operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E}) & \longrightarrow & D^b(\mathcal{E}) \end{array}$$

where the functor i is maps $F = \operatorname{coker} \operatorname{Hom}(-, s)$ to the 3-term complex with $X_{-2} \to X_{-1} \to X_0$ defined as $\ker s \to X \xrightarrow{s} Y$.

We claim that $\operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E})$ *is an admissible exact subcategory in* $D^b(\mathcal{E})$ *(i.e. the exact structure of the localization coincides with all composable morphisms which are part of a triangle). Then we look at* $\mathcal{E} \subseteq \operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E}) \subseteq D^b(\mathcal{E})$ *and conjecture that* $D^b(\mathcal{E}) \to D^b(\operatorname{mod}_S \mathcal{A}/\operatorname{eff}(\mathcal{E}))$ *is a triangle equivalence.*

This would provide a useful tool to embed an exact category within its derived equivalence class into another exact category.

CHAPTER 5

On faithfully balancedness in functor categories

1. Synopsis

This is a generalization of some results of Ma-Sauter [**136**] from module categories over artin algebras to more general functor categories (and partly to exact categories). In particular, we generalize the definition of a faithfully balanced module to a *faithfully balanced subcategory* and find the generalizations of dualities and characterizations from Ma-Sauter.

2. Introduction

For an exact category $\mathcal{E}$ in the sense of Quillen and a full subcategory $\mathcal{M}$ we define categories $\mathrm{gen}_k^{\mathcal{E}}(\mathcal{M})$ (and $\mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M})$) of $\mathcal{E}$ (consisting of objects admitting a certain k-presentation in $\mathcal{M}$). We also consider the two functors $\Phi(X) := \mathrm{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{M}}, \Psi(X) := \mathrm{Hom}_{\mathcal{E}}(X, -)|_{\mathcal{M}}$.

We give the relatively obvious but technical generalizations of results in [**136**] related to these categories and functors. If $\mathcal{E}$ is a functor category (of some sort) these functors have adjoints and therefore stronger results can be found. We state here two of these:
Let $\mathcal{P}$ be an essentially small additive category. We denote by $\mathrm{Mod}\,\mathcal{P}$ the category of contravariant additive functors $\mathcal{P} \to (Ab)$ (and we set $\mathcal{P}\,\mathrm{Mod} := \mathrm{Mod}\,\mathcal{P}^{op}$). We write $\mathrm{mod}_k\,\mathcal{P}$ for the full subcategory which admit a k-presentation by finitely generated projectives. We denote by $h\colon \mathcal{P} \to \mathrm{Mod}\,\mathcal{P}$, $P \mapsto h_P = \mathrm{Hom}_{\mathcal{P}}(-, P)$ the Yoneda embedding.
Cogen^1*-duality:* Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and assume now $\mathcal{M} \subset \mathrm{mod}_k\,\mathcal{P}$. We shorten the notation $\mathrm{cogen}^k(\mathcal{M}) := \mathrm{cogen}^k_{\mathrm{mod}_k\,\mathcal{P}}(\mathcal{M}) \subset \mathrm{mod}_k\,\mathcal{P}$.
We say $\mathcal{M}$ is **faithfully balanced** if $h_P \in \mathrm{cogen}^1(\mathcal{M})$ for all $P \in \mathcal{P}$.

Lemma 2.1. *(cf. Lem. 4.11)* (cogen^1*-duality) If* $\mathcal{M}$ *is faithfully balanced, we denote by* $\tilde{\mathcal{M}} = \Psi(h_{\mathcal{P}}) \subset \mathcal{M}\,\mathrm{mod}_k$*, then* Ψ *defines a contravariant equivalence*

$$\mathrm{cogen}^1_{\mathrm{mod}_1\,\mathcal{P}}(\mathcal{M}) \longleftrightarrow \mathrm{cogen}^1_{\mathcal{M}\,\mathrm{mod}_1}(\tilde{\mathcal{M}})$$

The symmetry principle states as follows:

THEOREM 2.2. *(cf. Thm. 4.16, Symmetry principle). Let* $\mathcal{E}$ *be an exact category with enough projectives* $\mathcal{P}$ *and enough injectives* $\mathcal{I}$ *and* $k \geq 1$*. The following two statements are equivalent:*

(1) $\mathcal{P} \subset \mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M})$ *and* $\Phi(I) = \mathrm{Hom}_{\mathcal{E}}(-, I)|_{\mathcal{M}} \in \mathrm{mod}_k\,\mathcal{M}$ *for every* $I \in \mathcal{I}$
(2) $\mathcal{I} \subset \mathrm{gen}_k^{\mathcal{E}}(\mathcal{M})$ *and* $\Psi(P) = \mathrm{Hom}_{\mathcal{E}}(P, -)|_{\mathcal{M}} \in \mathcal{M}\,\mathrm{mod}_k$ *for every* $P \in \mathcal{P}$

A nice special case: Assume additionally that $\mathcal{E}$ is a Hom-finite K-category for a field K and $\mathcal{M} = \mathrm{add}(M)$ for an object $M \in \mathcal{E}$. Then the following two statements are equivalent:

(1) $\mathcal{P} \subset \mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M})$
(2) $\mathcal{I} \subset \mathrm{gen}_k^{\mathcal{E}}(\mathcal{M})$

Since: If we set $\Lambda = \mathrm{End}_{\mathcal{E}}(M)$, then $\mathrm{mod}_k\,\mathcal{M}$, $\mathcal{M}\,\mathrm{mod}_k$ can be identified with finite-dimensional (left and right) modules over Λ and $\Phi(I) = \mathrm{Hom}_{\mathcal{E}}(M, I), \Psi(P) = \mathrm{Hom}_{\mathcal{E}}(P, M)$ are by assumption finite-dimensional Λ-modules.

3. In additive categories

Here we want to extend Yoneda's embedding to a bigger subcategory: Let $\mathcal{C}$ be an additive category and $\mathcal{M}$ an essentially small full additive subcategory. A right $\mathcal{M}$-module is a contravariant additive functor from $\mathcal{M}$ into abelian groups. We denote by $\operatorname{Mod}\mathcal{M}$ the category of all right $\mathcal{M}$-modules. This is an abelian category. We have the fully faithful (covariant) Yoneda embedding $\mathcal{M} \to \operatorname{Mod}\mathcal{M}$ defined by $M \mapsto \operatorname{Hom}_{\mathcal{M}}(-, M)$. Clearly, we can extend this functor to a functor $\Phi\colon \mathcal{C} \to \operatorname{Mod}\mathcal{M},\ \Phi(X) := \operatorname{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{M}} = (-, X)|_{\mathcal{M}}$ where the last notation is our shortage for the Hom functor. The aim of this section is to define a subcategory $\mathcal{M} \subset \mathcal{G} \subset \mathcal{C}$ such that $\Phi|_{\mathcal{G}}$ is fully faithful.

We define a full subcategory of $\mathcal{C}$ as follows

$$\operatorname{gen}_1^{\mathrm{add}}(\mathcal{M}) := \left\{ Z \in \mathcal{C} \mid \begin{array}{c} \exists M_1 \xrightarrow{f} M_0 \xrightarrow{g} Z,\ M_i \in \mathcal{M},\ g = \operatorname{coker}(f) \text{ is an epim.} \\ (M, M_1) \to (M, M_0) \to (M, Z) \to 0 \\ \text{exact sequence of abelian groups } \forall M \in \mathcal{M} \end{array} \right\}$$

We observe that $g = \operatorname{coker}(f)$ and g an epimorphism is equivalent to that we have an exact sequence of $\mathcal{C}^{op}$-modules

$$0 \to (Z, -) \to (M_0, -) \to (M_1, -)$$

Furthermore the second line in the definition is equivalent to an exact sequence in $\operatorname{Mod}\mathcal{M}$

$$(-, M_1) \to (-, M_0) \to (-, Z)|_{\mathcal{M}} \to 0.$$

Dually, we define $\operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M}) := (\operatorname{gen}_1^{\mathrm{add}}(\mathcal{M}^{op}))^{op}$ where $\mathcal{M}^{op}$ is considered as a full additive subcategory of $\mathcal{C}^{op}$.

Lemma 3.1. (1) *The functor* $\operatorname{gen}_1^{\mathrm{add}}(\mathcal{M}) \to \operatorname{Mod}\mathcal{M}$ *defined by* $Z \mapsto (-, Z)|_{\mathcal{M}}$ *is fully faithful. We even have for every* $Z \in \operatorname{gen}_1^{\mathrm{add}}(\mathcal{M}), C \in \mathcal{C}$ *a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{C}}(Z, C) \to \operatorname{Hom}_{\operatorname{Mod}\mathcal{M}}((-, Z)|_{\mathcal{M}}, (-, C)|_{\mathcal{M}})$$

(2) *The functor* $\operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M}) \to \operatorname{Mod}\mathcal{M}^{op}$ *defined by* $Z \mapsto (Z, -)|_{\mathcal{M}}$ *is fully faithful. We even have for every* $Z \in \operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M}), C \in \mathcal{C}$ *a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{C}}(C, Z) \to \operatorname{Hom}_{\operatorname{Mod}\mathcal{M}^{op}}((Z, -)|_{\mathcal{M}}, (C, -)|_{\mathcal{M}})$$

Proof. We only prove (1), the second statement follows by passing to opposite categories. We consider the functor $\Phi\colon \mathcal{C} \to \operatorname{Mod}\mathcal{M}$ defined by $\Phi(X) := (-, X)|_{\mathcal{M}}$. Since $Z \in \operatorname{gen}_1^{\mathrm{add}}(\mathcal{M})$ we an exact sequences

$$0 \to (Z, C) \to (M_0, C) \to (M_1, C) \quad \text{of ab. groups}$$

and $\Phi(M_1) \to \Phi(M_0) \to \Phi(Z) \to 0$ in $\operatorname{Mod}\mathcal{M}$. By applying $(-, \Phi(C))$ to the second exact sequence we obtain an exact sequence

$$0 \to (\Phi(Z), \Phi(C)) \to (\Phi(M_0), \Phi(C)) \to (\Phi(M_1), \Phi(C)) \quad \text{of ab. groups.}$$

Since Φ is a functor, we find a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (Z, C) & \longrightarrow & (M_0, C) & \longrightarrow & (M_1, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\Phi(Z), \Phi(C)) & \longrightarrow & (\Phi(M_0), \Phi(C)) & \longrightarrow & (\Phi(M_1), \Phi(C)) \end{array}$$

By the Lemma of Yoneda, we have for every $F \in \operatorname{Mod}\mathcal{M}$ and $M \in \mathcal{M}$ that $\operatorname{Hom}_{\operatorname{Mod}\mathcal{M}}(\Phi(M), F) = F(M)$. This implies that the maps $(M_i, C) \to (\Phi(M_i), \Phi(C))$ are isomorphisms of groups. and therefore, the induced map on the kernels is an isomorphism. □

Remark 3.2. If $\mathcal{M}$ is not essentially small, $\operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(F, G)$ is not necessarily a set. But if one passes to the full subcategory of finitely presented $\mathcal{M}$-modules $\operatorname{mod}_1\mathcal{M}$, this set-theoretic issue does not arise: Observe that $Z \mapsto (-, Z)|_{\mathcal{M}}$ defines by definition a covariant functor

$$\Phi\colon \operatorname{gen}_1^{\mathrm{add}}(\mathcal{M}) \to \operatorname{mod}_1\mathcal{M},$$

the same proof as before shows that this is fully faithful. Similarly, the functor $Z \mapsto (Z,-)|_{\mathcal{M}}$ defines a fully faithful contravariant functor

$$\Psi\colon \operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M}) \to \operatorname{mod}_1 \mathcal{M}^{op}.$$

4. In exact categories

This section is a generalization of results from [**136**]. For exact categories we have subcategories of $\operatorname{cogen}^1_{\mathrm{add}}$ such that Ψ induces isomorphisms on (some) extension groups (cf. Lemma 4.3).
Given an exact category $\mathcal{E}$ with a full additive subcategory $\mathcal{M}$, we define $\operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M}) \subset \mathcal{E}$ to be the full subcategory of all objects X such that there is an exact sequence

$$0 \to X \to M_0 \to \cdots \to M_k \to Z \to 0$$

with $M_i \in \mathcal{M}, 0 \le i \le k$ such that for every $M \in \mathcal{M}$ the sequence

$$\operatorname{Hom}_{\mathcal{E}}(M_k, M) \to \cdots \to \operatorname{Hom}_{\mathcal{E}}(M_0, M) \to \operatorname{Hom}_{\mathcal{E}}(X, M) \to 0$$

is an exact sequence of abelian groups.
We define $\operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$ to be the full additive category of $\mathcal{E}$ given by all X such that there is an exact sequence

$$0 \to Z \to M_k \to \cdots \to M_0 \to X \to 0$$

with $M_i \in \mathcal{M}, 0 \le i \le k$ such that for every $M \in \mathcal{M}$ we have an exact sequence

$$\operatorname{Hom}_{\mathcal{E}}(M, M_k) \to \cdots \to \operatorname{Hom}_{\mathcal{E}}(M, M_0) \to \operatorname{Hom}_{\mathcal{E}}(M, X) \to 0$$

of abelian groups.
If it is clear from the context in which exact category we are working, then we leave out the index $\mathcal{E}$ and just write $\operatorname{cogen}^k(\mathcal{M})$ and $\operatorname{gen}_k(\mathcal{M})$.

Remark 4.1. Observe that $\operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M}) \subset \operatorname{cogen}^1_{\mathrm{add}}(\mathcal{M})$, $\operatorname{gen}^{\mathcal{E}}_k(\mathcal{M}) \subset \operatorname{gen}^{\mathrm{add}}_1(\mathcal{M})$ for $k \ge 1$ and therefore the functor $\Psi\colon X \mapsto (X,-)|_{\mathcal{M}}$ (resp. $\Phi\colon X \mapsto (-,X)|_{\mathcal{M}}$) is fully faithful on $\operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$ (resp. on $\operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$) by Lemma 3.1 and Remark 3.2.

Remark 4.2. Let $k \ge 1$. We denote by $\operatorname{mod}_k \mathcal{M}$ the category of $\mathcal{M}$-modules which admit a k-presentation (indexed from 0 to k) by finitely presented projectives. For $F \in \operatorname{mod}_k \mathcal{M}$ the Ext-groups $\operatorname{Ext}^i_{\mathcal{M}\operatorname{Mod}}(F,G)$ with $0 \le i < k$ are sets.
If $X \in \operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$, then we have $\Psi(X) = (X,-)|_{\mathcal{M}} \in \operatorname{mod}_k \mathcal{M}^{op} (=: \mathcal{M}\operatorname{mod}_k)$.
If $Y \in \operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$, then we have $\Phi(Y) = (-,Y)|_{\mathcal{M}} \in \operatorname{mod}_k \mathcal{M}$.

Since we are now working in exact categories, we observe the following isomorphisms on extension groups:

Lemma 4.3. *Let $k \ge 1$.*

(a) *If $X \in \operatorname{cogen}^k_{\mathcal{E}}(\mathcal{M})$, then the functor $Z \mapsto \Psi(Z) = (Z,-)|_{\mathcal{M}}$ induces a well-defined natural isomorphism of abelian groups*

$$\operatorname{Ext}^i_{\mathcal{E}}(Y,X) \to \operatorname{Ext}^i_{\mathcal{M}\operatorname{Mod}}(\Psi(X),\Psi(Y)), \quad 0 \le i < k$$

for all $Y \in \bigcap_{1 \le i < k} \ker \operatorname{Ext}^i_{\mathcal{E}}(-,\mathcal{M})$.

(b) *If $Y \in \operatorname{gen}^{\mathcal{E}}_k(\mathcal{M})$, then the functor $Z \mapsto \Phi(Z) = (-,Z)|_{\mathcal{M}}$ induces a well-defined natural isomorphism of abelian groups*

$$\operatorname{Ext}^i_{\mathcal{E}}(Y,X) \to \operatorname{Ext}^i_{\operatorname{Mod}\mathcal{M}}(\Phi(Y),\Phi(X)), \quad 0 \le i < k$$

for all $X \in \bigcap_{1 \le i < k} \ker \operatorname{Ext}^i_{\mathcal{E}}(\mathcal{M},-)$.

Proof. (a) the proof is a straight forward generalization of [**136**, Lemma 2.4, (2)](using Rem. 4.1) and (b) follows from (a) by passing to the opposite exact category $\mathcal{E}^{op}$. □

We will later use the following simple observation:

Remark 4.4. Let $\mathcal{E}$ be an exact category, $\mathcal{X}$ be a fully exact category and $\mathcal{M} \subset \mathcal{X}$ an additive subcategory. We say $\mathcal{X}$ is *deflation-closed* if for any deflation $d\colon X \to X'$ in $\mathcal{E}$ with X, X' in $\mathcal{X}$ it follows $\ker d \in \mathcal{X}$. The dual notion is *inflation-closed*.
If $\mathcal{X}$ is deflation-closed then $\mathrm{gen}_k^{\mathcal{X}}(\mathcal{M}) = \mathrm{gen}_k^{\mathcal{E}}(\mathcal{M}) \cap \mathcal{X}$. If $\mathcal{X}$ is inflation-closed then $\mathrm{cogen}^k_{\mathcal{X}}(\mathcal{M}) = \mathrm{cogen}^k_{\mathcal{E}}(\mathcal{M}) \cap \mathcal{X}$.

4.1. Inside functor categories. Let $\mathcal{P}$ be an essentially small additive category. We denote by $h\colon \mathcal{P} \to \operatorname{Mod}\mathcal{P}$, $P \mapsto h_P = \operatorname{Hom}_{\mathcal{P}}(-, P)$ the Yoneda embedding, we write $h_{\mathcal{P}}$ for the essential image of h.

4.1.1. *Adjoint functors.* Let now $\mathcal{M}$ be an essentially small full additive subcategory of $\operatorname{Mod}\mathcal{P}$. We consider the contravariant functor

$$\begin{aligned} \Psi\colon \operatorname{Mod}\mathcal{P} &\to \mathcal{M}\operatorname{Mod}, \\ X &\mapsto \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}} = (X,-)|_{\mathcal{M}} \end{aligned}$$

We also consider the contravariant functor

$$\begin{aligned} \Psi'\colon \mathcal{M}\operatorname{Mod} &\to \operatorname{Mod}\mathcal{P} \\ Z &\mapsto (P \mapsto \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(Z, \Psi(h_P))) \end{aligned}$$

We generalize [13, Lemma 3.3].

Lemma 4.5. *The functors Ψ and Ψ' are contravariant adjoint functors, i.e. the following is a (bi)natural isomorphim*

$$\chi\colon \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X, \Psi'(Z)) \to \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(Z, \Psi(X))$$

defined as follows: A natural transformation $f \in \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X, \Psi'(Z))$, is determined by for every $P \in \mathcal{P}, x \in X(P), M \in \mathcal{M}$ a group homomorphism

$$f_{P,x}(M)\colon Z(M) \mapsto \Psi(h_P)(M) = M(P)$$

then, we define a natural transformation $\chi(f)\colon Z \to \Psi(X) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}}$ for $M \in \mathcal{M}$ as follows

$$\begin{aligned} \chi(f)(M)\colon Z(M) &\to \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X, M), \\ z &\mapsto (X(P) \xrightarrow{f_{P,-}(z)} M(P), x \mapsto f_{P,x}(M)(z))_{P\in\mathcal{P}} \end{aligned}$$

Proof. We define $\chi'\colon \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(Z, \Psi(X)) \to \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X, \Psi'(Z))$ as follows: For $g\colon Z \to \Psi(X) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}}$ we have for every $M \in \mathcal{M}, z \in Z(M)$ a natural transformation $g_{M,z}\colon X \to M$, i.e. for every $P \in \mathcal{P}$ a group homomorphism

$$g_{M,z}(P)\colon X(P) \to M(P), x \mapsto g_{M,z}(P)(x),$$

then we define $\chi'(g)(P)\colon X(P) \to \Psi'(Z)(P) = \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(Z, (h_P,-)|_{\mathcal{M}})$ as follows

$$x \mapsto (Z(M) \to M(P), z \mapsto g_{M,z}(P)(x))_{M\in\mathcal{M}}.$$

Then χ' is the inverse map to χ. □

Remark 4.6. Given an adjoint pair of contravariant functors Ψ and Ψ', the natural isomorphisms

$$\operatorname{Hom}(X, \Psi(Z)) \to \operatorname{Hom}(Z, \Psi'(X))$$

induce natural transformations $\alpha\colon \mathrm{id} \to \Psi'\Psi$ (and $\alpha'\colon \mathrm{id} \to \Psi\Psi'$) as follows

$$\operatorname{Hom}(X,X) \xrightarrow{\Psi(-)} \operatorname{Hom}(\Psi(X), \Psi(X)) \cong \operatorname{Hom}(X, \Psi'\Psi(X)), \quad \mathrm{id}_X \mapsto \alpha_X$$

in this case we have triangle identities

$$\begin{aligned} \mathrm{id}_{\Psi(X)} &= (\Psi(X) \xrightarrow{\alpha'_{\Psi(X)}} \Psi\Psi'\Psi(X) \xrightarrow{\Psi(\alpha_X)} \Psi(X)) \\ \mathrm{id}_{\Psi'(Z)} &= (\Psi'(Z) \xrightarrow{\alpha_{\Psi'(Z)}} \Psi'\Psi\Psi'(Z) \xrightarrow{\Psi'(\alpha'_Z)} \Psi'(Z)) \end{aligned}$$

In [**183**, section4] a tensor bifunctor is introduced

$$-\otimes_{\mathcal{M}} -\colon \operatorname{Mod}\mathcal{M} \times \mathcal{M}\operatorname{Mod} \to (Ab), (F,G) \mapsto F \otimes_{\mathcal{M}} G$$

Now, we consider the covariant funtor

$$\Phi\colon \operatorname{Mod}\mathcal{P} \to \operatorname{Mod}\mathcal{M}, \quad X \mapsto \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(-,X)|_{\mathcal{M}} =: (-,X)|_{\mathcal{M}}$$

and the following covariant functor

$$\Phi'\colon \operatorname{Mod}\mathcal{M} \to \operatorname{Mod}\mathcal{P}, \quad Z \mapsto (P \mapsto Z \otimes_{\mathcal{M}} \Psi(h_P))$$

Lemma 4.7. *The functor Φ is right adjoint to Φ', i.e. we have a (bi)natural maps*

$$\operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(\Phi'(Z), X) \to \operatorname{Hom}_{\operatorname{Mod}\mathcal{M}}(Z, \Phi(X))$$

Remark 4.8. If $F\colon \mathcal{C} \leftrightarrow \mathcal{D}\colon G$ is an adjoint pair of functors (with F left adjoint to G), then we have a unit $u\colon 1_{\mathcal{C}} \to GF$ and a counit, $c\colon FG \to 1_{\mathcal{D}}$. Let $\mathcal{C}_u$ be the full subcategory of objects in X in $\mathcal{C}$ such that $u(X)$ is an isomorphism. Let $\mathcal{D}_c$ be the full subcategory of objects Y in $\mathcal{D}$ such that $c(Y)$ is an isomorphism. Then, the triangle identities show directly that F, G restrict to quasi-inverse equivalences $F\colon \mathcal{C}_u \leftrightarrow \mathcal{D}_c\colon G$.

4.1.2. $\boxed{\operatorname{cogen}^k}$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and assume now $\mathcal{M} \subset \operatorname{mod}_k \mathcal{P}$. In this subsection we study $\operatorname{cogen}^k(\mathcal{M}) := \operatorname{cogen}^k_{\operatorname{mod}_k \mathcal{P}}(\mathcal{M}) \subset \operatorname{mod}_k \mathcal{P}$.
Our aim is to give a different description of the categories $\operatorname{cogen}^k(\mathcal{M})$ (cf. Lemma 4.9) and to introduce *faithfully balancedness* which leads to the cogen^1 duality (cf. Lemma 4.11).

We have the contravariant functor

$$\Psi\colon \operatorname{Mod}\mathcal{P} \to \mathcal{M}\operatorname{Mod}, \quad X \mapsto \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}}$$

and $\Psi|_{\operatorname{cogen}^k(\mathcal{M})}\colon \operatorname{cogen}^k(\mathcal{M}) \to \mathcal{M}\operatorname{mod}_k$ is fully faithful for $1 \le k < \infty$.
The natural transformation $\alpha\colon \operatorname{id}_{\operatorname{Mod}\mathcal{P}} \to \Psi'\Psi$, for $X \in \operatorname{Mod}\mathcal{P}$ is given by a morphism in $\operatorname{Mod}\mathcal{P}$, $\alpha_X\colon X \to \Psi'\Psi(X) = \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(\Psi(X), \Psi(h_-))$ which is defined at $P \in \mathcal{P}$ via

$$X(P) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P, X) \to \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(\operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-)|_{\mathcal{M}}, \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P,-)|_{\mathcal{M}})$$
$$f \mapsto [\operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(X,-) \xrightarrow{-\circ f} \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P,-)]|_{\mathcal{M}}$$

We observe that α_M is an isomorphism for every $M \in \mathcal{M}$ (since

$$\begin{aligned}(\Psi'\Psi(M))(P) &= \operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(\operatorname{Hom}_{\mathcal{M}}(M,-), \Psi(h_P)) \\ &= \Psi(h_P)(M) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{P}}(h_P, M) = M(P)\end{aligned}$$

using Yoneda's Lemma twice).

Lemma 4.9. *For $1 \le k \le \infty$ we have $\operatorname{cogen}^k_{\operatorname{mod}_k \mathcal{P}}(\mathcal{M})$ equals*

$$\begin{aligned}\{X \in \operatorname{mod}_k -\mathcal{P} \mid \alpha_X \text{ isom. }, \Psi(X) \in \mathcal{M}\operatorname{mod}_k,\\ \operatorname{Ext}^i_{\mathcal{M}\operatorname{Mod}}(\Psi(X), \Psi(h_P)) = 0, 1 \le i < k, \forall P \in \mathcal{P}\}\end{aligned}$$

Proof. The proof is a straight forward generalization of [**136**, Lemma 2.2,(1)] (the functor $\operatorname{Hom}_\Gamma(-,M)$ has to be replaced by applying $\operatorname{Hom}_{\mathcal{M}\operatorname{Mod}}(-, \Psi(h_P))$ for all $P \in \mathcal{P}$). □

Definition 4.10. We say $\mathcal{M}$ is **faithfully balanced** if $h_{\mathcal{P}} \subset \operatorname{cogen}^1(\mathcal{M})$.

Lemma 4.11. *(cogen^1 duality) If $\mathcal{M}$ is faithfully balanced, we denote by $\tilde{\mathcal{M}} = \Psi(h_{\mathcal{P}}) \subset \mathcal{M}\operatorname{mod}_k$, then Ψ defines a contravariant equivalence*

$$\operatorname{cogen}^1_{\operatorname{mod}_1 \mathcal{P}}(\mathcal{M}) \longleftrightarrow \operatorname{cogen}^1_{\mathcal{M}\operatorname{mod}_1}(\tilde{\mathcal{M}})$$

and contravariant equivalences

$$\operatorname{cogen}^k_{\operatorname{mod}_k \mathcal{P}}(\mathcal{M}) \longleftrightarrow \operatorname{cogen}^1_{\mathcal{M}\operatorname{mod}_1}(\tilde{\mathcal{M}}) \cap \bigcap_{1 \le i < k} \ker(\operatorname{Ext}^i_{\mathcal{M}\operatorname{mod}_k}(-, \tilde{\mathcal{M}}))$$

Proof. Let $k=1$. Since we have an adjoint pair of contravariant functors Ψ, Ψ' it follows from the triangle identities (cf. Remark 4.6): If α_X is an isomorphism then also $\alpha'_{\Psi(X)}$ and if α'_Z is an isomorphism then also $\alpha_{\Psi'(Z)}$. Now, since $\mathcal{M}$ is faithfully balanced we have that Ψ induces an equivalence $\mathcal{P}^{op} \cong \tilde{\mathcal{M}} = \Psi(h_{\mathcal{P}})$ by Lemma 3.1. It follows from the definition of Ψ' and a right module version of Lemma 4.9 that $\mathrm{cogen}^1(\tilde{\mathcal{M}}) = \{Z \in \mathcal{M}\,\mathrm{mod}_1 \mid \alpha'_Z \text{ isom}\}$.
The rest is a straightforward generalization of the proof of [**136**, Lemma 2.9]. $\square$

4.1.3. $\boxed{\mathrm{gen}_k}$. We study $\mathrm{gen}_k(\mathcal{M}) = \mathrm{gen}_k^{\mathrm{Mod}\,\mathcal{P}}(\mathcal{M}) \subset \mathrm{Mod}\,\mathcal{P}$. We again give a different description of these categories using tensor products of $\mathcal{M}$-modules (cf. Lemma 4.13). This is the main ingredient in the proof of the symmetry principle in the next subsection.

We have the covariant functor

$$\Phi\colon\ \mathrm{Mod}\,\mathcal{P} \to \mathrm{Mod}\,\mathcal{M}, \quad X \mapsto \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(-,X)|_{\mathcal{M}}$$

and $\Phi|_{\mathrm{gen}_k(\mathcal{M})}\colon\ \mathrm{gen}_k(\mathcal{M}) \to \mathrm{mod}_k\,\mathcal{M}$ is fully faithful. We have an induced covariant functor

$$\varepsilon = \Phi' \circ \Phi\colon\ \mathrm{Mod}\,\mathcal{P} \to \mathrm{Mod}\,\mathcal{P}, \quad X \mapsto \varepsilon_X$$

defined for $P \in \mathcal{P}$ as

$$\varepsilon_X(P) := \Phi(X) \otimes_{\mathcal{M}} \Psi(h_P)$$

and a natural transformation $\varphi\colon \varepsilon \to \mathrm{id}_{\mathrm{Mod}\,\mathcal{P}}$, for $X \in \mathrm{Mod}\,\mathcal{P}$ this is given by a morphism $\varphi_X\colon \varepsilon_X \to X$ which is defined at $P \in \mathcal{P}$ via

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(-,X)|_{\mathcal{M}} \otimes_{\mathcal{M}} (\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(h_P,-)|_{\mathcal{M}}) &\to \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(h_P,X) = X(P)\\ \underbrace{g \otimes f}_{\in \mathrm{Hom}(M,X)\otimes_{\mathbb{Z}}\mathrm{Hom}(h_P,M)} &\mapsto g \circ f\end{aligned}$$

Remark 4.12. Φ and is right adjoint functor of Φ' between abelian categories therefore Φ is left exact and Φ' is right exact, φ is the counit of this adjunction. If $M \in \mathcal{M}$, then φ_M is an isomorphism.

Lemma 4.13. *For* $1 \le k \le \infty$ *we have*

$$\begin{aligned}&\mathrm{gen}_k^{\mathrm{Mod}\,\mathcal{P}}(\mathcal{M}) =\\ &\{X \in \mathrm{Mod}\,\mathcal{P} \mid \varphi_X \textit{ isom. }, \Phi(X) \in \mathrm{mod}_k\,\mathcal{M},\ \mathrm{Tor}^i_{\mathcal{M}}(\Phi(X),\Psi(h_P)) = 0, 1 \le i < k, \forall P \in \mathcal{P}\}\end{aligned}$$

Proof. Let $X \in \mathrm{gen}_k(\mathcal{M})$, then there exists an exact sequence $M_k \to \cdots \to M_0 \to X \to 0$ such that Φ preserves its exactness, this implies $\Phi(X) \in \mathrm{mod}_k\,\mathcal{M}$. Now, we apply $\varepsilon = \Phi'\Phi$ and consider the commutative diagram

$$\begin{array}{ccccccccc} M_k & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & X & \longrightarrow & 0\\ \uparrow{\scriptstyle \varphi_{M_k}} & & & & \uparrow{\scriptstyle \varphi_{M_0}} & & \uparrow{\scriptstyle \varphi_X} & & \\ \varepsilon_{M_k} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_{M_0} & \longrightarrow & \varepsilon_X & \longrightarrow & 0 \end{array}$$

Now, since Φ' is right exact and φ_{M_i} is an isomorphism for $0 \le i \le k$, we conclude that φ_X is an isomorphism and the lower row is exact. This implies $\mathrm{Tor}^i_{\mathcal{M}}(\Phi(X),\Psi(h_P)) = 0, 1 \le i < k$.
Conversely, if we take $X \in \mathrm{Mod}\,\mathcal{P}$ fulfilling the assumptions in the set bracket of the lemma. We can apply Φ' to the projective k-presentation of $\Phi(X)$, then we can find a diagram as before but this time we know from the assumptions that the bottom row is exact. Furthermore, since φ_* is an isomorphism in all places of the diagram, we have that also the top row is exact. This implies $X \in \mathrm{gen}_k^{\mathrm{Mod}\,\mathcal{P}}(\mathcal{M})$. $\square$

4.2. The symmetry principle. Now, we study these subcategories in more general exact categories. For an exact category $\mathcal{E}$ with enough projectives $\mathcal{P}$ and an exact category $\mathcal{F}$ with enough injectives $\mathcal{I}$, we consider the covariant, exact, fully faithful functors

$$\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_\infty \mathcal{P}, \quad X \mapsto \operatorname{Hom}_{\mathcal{E}}(-,X)|_{\mathcal{P}}$$
$$\mathbb{I}\colon \mathcal{F}^{\mathrm{op}} \to \operatorname{mod}_\infty \mathcal{I}^{op}, \quad X \mapsto \operatorname{Hom}_{\mathcal{F}}(X,-)|_{\mathcal{I}^{op}}$$

cf. [**73**, Prop. 2.2.1, Prop.2.2.8]

Remark 4.14. For an additive category $\mathcal{M}$ of $\mathcal{E}$ (resp. of $\mathcal{F}$) we have:

$$\mathbb{P}(\operatorname{gen}_k^{\mathcal{E}}(\mathcal{M})) = \operatorname{Im}\mathbb{P} \cap \operatorname{gen}_k^{\operatorname{Mod}\mathcal{P}}(\mathbb{P}(\mathcal{M})),$$
$$\mathbb{I}((\operatorname{cogen}_{\mathcal{F}}^k(\mathcal{M}))^{op}) = \mathbb{I}(\operatorname{gen}_k^{\mathcal{F}^{op}}(\mathcal{M}^{op})) = \operatorname{Im}\mathbb{I} \cap \operatorname{gen}_k^{\operatorname{Mod}\mathcal{I}^{op}}(\mathbb{I}(\mathcal{M}^{op}))$$

This follows from remark 4.4 since $\mathbb{P}\colon \mathcal{E} \to \operatorname{Im}\mathbb{P}$ is an equivalence of exact categories and $\operatorname{Im}\mathbb{P}$ is deflation-closed in $\operatorname{mod}_\infty \mathcal{P}$ and $\operatorname{mod}_\infty \mathcal{P}$ is deflation-closed in $\operatorname{Mod}\mathcal{P}$. The second statement follows by passing to the opposite category.

As before, let $\Phi\colon \mathcal{E} \to \operatorname{Mod}\mathcal{M}, \Phi(X) = \operatorname{Hom}_{\mathcal{E}}(-,X)|_{\mathcal{M}}$, $\Psi\colon \mathcal{E} \to \mathcal{M}\operatorname{Mod}, \Psi(X) = \operatorname{Hom}_{\mathcal{E}}(X,-)|_{\mathcal{M}}$. We have the immediate corollary:

Corollary 4.15. *(of Lem. 4.13 and Rem. 4.14) (1) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and $\mathcal{M}$ a full additive subcategory. Then the following are equivalent:*

(1) $X \in \operatorname{gen}_k^{\mathcal{E}}(\mathcal{M})$
(2) $\Phi(X) \in \operatorname{mod}_k \mathcal{M}$ *and for every* $P \in \mathcal{P}$*:*

$$\Phi(X) \otimes_{\mathcal{M}} \Psi(P) \to \operatorname{Hom}_{\mathcal{E}}(P,X), \quad g \otimes f \mapsto g \circ f$$

is an isomorphism, $\operatorname{Tor}^i_{\mathcal{M}}(\Phi(X),\Psi(P)) = 0$, $1 \le i < k$.

(2) If $\mathcal{E}$ is an exact category with enough injectives $\mathcal{I}$ and $\mathcal{M}$ a full additive subcategory. Then the following are equivalent:

(1) $X \in \operatorname{cogen}_{\mathcal{E}}^k(\mathcal{M})$
(2) $\Psi(X) \in \mathcal{M}\operatorname{mod}_k$ *and for every* $I \in \mathcal{I}$*:*

$$\Phi(I) \otimes_{\mathcal{M}} \Psi(X) \to \operatorname{Hom}_{\mathcal{F}}(X,I), \quad g \otimes f \mapsto g \circ f$$

is an isomorphism, $\operatorname{Tor}^i_{\mathcal{M}}(\Phi(I),\Psi(X)) = 0$, $1 \le i < k$.

Theorem 4.16. *(Symmetry principle). Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ and $k \ge 1$. The following two statements are equivalent:*

(1) $\mathcal{P} \subset \operatorname{cogen}_{\mathcal{E}}^k(\mathcal{M})$ *and* $\Phi(I) = \operatorname{Hom}_{\mathcal{E}}(-,I)|_{\mathcal{M}} \in \operatorname{mod}_k \mathcal{M}$ *for every* $I \in \mathcal{I}$
(2) $\mathcal{I} \subset \operatorname{gen}_k^{\mathcal{E}}(\mathcal{M})$ *and* $\Psi(P) = \operatorname{Hom}_{\mathcal{E}}(P,-)|_{\mathcal{M}} \in \mathcal{M}\operatorname{mod}_k$ *for every* $P \in \mathcal{P}$

Proof. We consider $\mathbb{P}, \mathbb{I}$ as before defined for the category $\mathcal{E}$. Then, it is straight forward from the previous Lemma to see that (1) and (2) are both equivalent to for all $P \in \mathcal{P}, I \in \mathcal{I}$, $\Psi(P) \in \mathcal{M}\operatorname{mod}_k, \Phi(I) \in \operatorname{mod}_k \mathcal{M}$ and

$$\Phi(I) \otimes_{\mathcal{M}} \Psi(P) \to \operatorname{Hom}_{\mathcal{E}}(P,I), \quad g \otimes f \mapsto g \circ f$$

is an isomorphism, $\operatorname{Tor}^i_{\mathcal{M}}(\Phi(I),\Psi(P)) = 0$, $1 \le i < k$. Therefore (1) and (2) are equivalent. □

Part 2

Derived methods

CHAPTER 6

Derived categories and functors for exact categories

This includes a joint result with Juan Omar Gomez.

1. Synopsis

By now, derived categories and derived functors for abelian categories are standard topics in a course on homological algebra. This is a an introduction to derived categories of exact categories assuming that the reader is familiar with the theory for abelian categories. Other sources which include exact categories are [**119**], [**126**], [**49**].
Our treatment of derived functors is only shortly summarizing the results in [**119**].
What is new? We characterize when derived categories of exact categories are locally small (i.e. Hom-classes are sets).

2. Why derived categories?

This is an attempt in trying to explain in a nutshell why derived categories have homological algebra as their heart.
An exact category (in the sense of Quillen) is an additive category together with a collection of kernel-cokernel pairs called short exact sequences fulfilling axioms such that $\mathrm{Ext}^1_{\mathcal{E}}$ = (taking equivalence classes of short exact sequences) becomes an additive bifunctor. Using longer exact sequences one can find higher Ext-functors Ext^n - for the moment we assume that these are all set-valued functors (cf. next section).
Homological algebra for exact categories is the study of the bifunctors $\mathrm{Ext}^n_{\mathcal{E}}$ and in particular the conversion of short exact sequences into long exact sequences using higher Ext-groups.
Philosophically: Can we enlarge $\mathcal{E}$ to a category $\mathcal{D}$ such that these bifunctors become restrictions of the Hom-functor and are connected by an auto-equivalence $\Sigma\colon \mathcal{D} \to \mathcal{D}$ as
$\mathrm{Ext}^n_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{D}}(X, \Sigma^n Y)$?
For every short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in $\mathcal{E}$ representing $\sigma \in \mathrm{Ext}^1_{\mathcal{E}}(Z,X) = \mathrm{Hom}_{\mathcal{D}}(Z, \Sigma X)$ we look at the sequences in $\mathcal{D}$

$$X \to Y \to Z \xrightarrow{\sigma} \Sigma X$$

and call these 'distinguished triangles with three objects in $\mathcal{E}$'. This *wish list* on such a category $\mathcal{D}$ has been formalized in the notion of a triangulated category with initial data an additive category $\mathcal{D}$ with an auto-equivalence Σ, called *suspension*, and a collection of distingished triangles such that a list of axioms is fulfilled (cf. TR0-TR5 in [**115**, Def. 10.1.6]). As the long exact sequences have no negative parts, we find another condition which the bounded derived category has to fulfill

$$\mathrm{Ext}^{-n}_{\mathcal{E}}(X,Y) = \mathrm{Hom}(X, \Sigma^{-n} Y) = 0 \quad \forall X, Y \in \mathcal{E},\ n \geq 1.$$

The structure preserving functors between triangulated categories are called *triangle functors.*
'Structure preserving functors' from exact categories into triangulated categories are called δ-functors (in the sense of Keller [**117**] or one can view them as extriangulated functors in the sense of [**37**]):

Definition 2.1. ([**117**]) Let $\mathcal{E}$ be an exact category and $\mathcal{D}$ be a triangulated category. A δ-functor $\mathcal{E} \to \mathcal{D}$ consists of a pair (F, δ) consisting of an additive functor $F\colon \mathcal{E} \to \mathcal{D}$ and an assignment δ

mapping short exact sequences $\sigma = (i,d)\colon X \rightarrowtail Y \twoheadrightarrow Z$ to a morphism $\delta_\sigma\colon F(Z) \to \Sigma F(X)$ fitting into a distinguished triangle

$$F(X) \xrightarrow{F(i)} F(Y) \xrightarrow{F(d)} F(Z) \xrightarrow{\delta_\sigma} \Sigma F(X).$$

We call a δ-functor $\mathcal{E} \to \mathcal{D}$ *homological* if δ induces natural isomorphisms $\mathrm{Ext}^n_{\mathcal{E}}(Z,X) \to \mathrm{Hom}_{\mathcal{D}}(F(Z), \Sigma^n F(X))$ for every $n \in \mathbb{Z}$.

So, we hope for the following naive definition:
The bounded derived category should be the *universal homological δ-functor into triangulated categories*. This means we have an homological δ-functor $\mathcal{E} \to \mathrm{D}^b(\mathcal{E})$ such that every homological δ-functor $\mathcal{E} \to \mathcal{D}$ factors as $\mathcal{E} \to \mathrm{D}^b(\mathcal{E}) \xrightarrow{R} \mathcal{D}$. We also want R to be unique up to natural isomorphism. Unfortunately, uniqueness is in general unknown. We call it a **realization functor** for $\mathcal{E}$ (if it exists).
The existence of R can be proven when restricting to suitably *enhanced* triangulated categories (filtered derived [**40**], Neeman enhanced [**142**] or algebraic due to [**123**], proven in [**133**]). Uniqueness is usually not discussed, it requires restriction to triangle functors which are preserving the fixed enhancement (for algebraic triangulated categories it follows from the construction, cf. [**133**]).

Remark 2.2. Why would one also look at $\mathrm{D}^+(\mathcal{E})$, $\mathrm{D}^-(\mathcal{E})$, $\mathrm{D}(\mathcal{E})$?
The reason is that we can usually not define right and left derived functors on $\mathrm{D}^b(\mathcal{E})$ but if we extend our derived category we (often) can.
At the level of positive (resp. negative) derived categories we have an explicit method to calculate (at least partially) right (resp. left) derived functor using Deligne's right (resp. left) acyclic objects. For example, injective objects are always right acyclic and injective coresolutions (if they exist) can then be used to calculate right derived functors (as in the abelian case). We come back to this in detail in the last section of this chapter.
Our motivation to look at $\mathrm{D}(\mathcal{E})$ is not so strong but if you want for example a triangulated category with arbitrary set-valued coproducts then you would look at $\mathrm{D}(\mathcal{E})$ where $\mathcal{E}$ has arbitrary coproducts.

2.1. Explicit construction(s). Let us start with an exact category $\mathcal{E}$ with underlying additive category $\mathcal{A}$ and $* \in \{\emptyset, +, -, b\}$.

2.1.1. Variant 1*: As Verdier quotient.* Given a full triangulated subcategory in a triangulated category $\mathcal{U} \subseteq \mathcal{T}$ there exists a triangle functor $Q_{\mathcal{U}}\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$, called the **Verdier localization**, which fulfills the following universal property: Every triangle functor $\mathcal{T} \to \mathcal{R}$ which annihilates the objects of $\mathcal{U}$ factors uniquely over a triangle functor $\mathcal{T}/\mathcal{U} \to \mathcal{R}$.
There are two well-known *issues*

(1) $\mathcal{T}/\mathcal{U}$ is defined by a localization and the Hom-classes may not always be sets.
(2) Let $\mathcal{U} \subset \overline{\mathcal{U}} \subseteq \mathcal{U}^{\oplus}$ with $\overline{\mathcal{U}}$ the saturation (i.e. the closure of $\mathcal{U}$ under isomorphism in $\mathcal{T}$) and $\mathcal{U}^{\oplus}$ the thick closure (i.e. closure under direct summands and isomorphism in $\mathcal{T}$). Then clearly these larger categories fulfill the same universal property and $\mathcal{T}/\mathcal{U} = \mathcal{T}/\overline{\mathcal{U}} = \mathcal{T}/\mathcal{U}^{\oplus}$. Therefore, some authors consider Verdier quotients with respect to thick subcategories.

Then take the homotopy category of the additive category $\mathrm{K}^*(\mathcal{A})$. This is a triangulated category (cf. [**115**, Thm 11.3.8]).
The subcategory of $\mathcal{E}$-acyclic complexes $\underline{\mathrm{Ac}}^*(\mathcal{E})$ is a full triangulated subcategory (cf. [**141**], 1.1).

Definition 2.3. The ($*$-)**derived category** of $\mathcal{E}$ is defined as the Verdier quotient

$$\mathrm{D}^*(\mathcal{E}) := \mathrm{K}^*(\mathcal{A})/\underline{\mathrm{Ac}}^*(\mathcal{E})$$

Then problem (2) can be fully answered when looking at properties of the underlying additive category. We say an additive category $\mathcal{A}$ is **weakly idempotent complete (wic) (resp. impotent complete (ic))** if every idempotent endomorphism $e\colon A \to A$ has a kernel (resp. has kernel and image and gives rise to a split exact sequence $\ker e \to A \to \mathrm{Im}\, e$).

In [**49**], exact structure on an exact category is extended functorially to an exact structure on an idempotent completion of the underlying additive category. This gives an exact functor

$$\mathcal{E} \to \mathcal{E}^{ic}$$

Similarly a weakly idempotent completion can be constructed. This gives exact functors

$$\mathcal{E} \to \mathcal{E}^{wic} \to \mathcal{E}^{ic}$$

In [**141**], the following (2c) has been proven (and (2a,b) is partly attributed to Thomason):

(2a) For $* = b$.
 $\underline{\mathrm{Ac}}^b(\mathcal{E})$ is saturated if and only it is thick if an only if $\mathcal{E}$ is weakly idempotent complete.
 For every exact category $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}^{wic})$ is a triangle equivalence.

(2b) For $* = \pm$.
 In this case we have both triangle equivalences $\mathrm{D}^{\pm}(\mathcal{E}) \to \mathrm{D}^{\pm}(\mathcal{E}^{wic}) \to \mathrm{D}^{\pm}(\mathcal{E}^{ic})$.

(2c) For $* = \emptyset$.
 $\underline{\mathrm{Ac}}(\mathcal{E})$ is saturated if and only it is thick if an only if $\mathcal{E}$ is idempotent complete.
 Then $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\mathcal{E}^{ic})$ is a triangle equivalence.

Remark 2.4. Then $\mathrm{D}^b(\mathcal{E})$ may not be idempotent complete: Balmer-Schlichting [**31**] showed that the idempotent completion of a triangulated category has a natural triangulated structure and $\mathrm{D}^b(\mathcal{E}^{ic})$ is triangle equivalent to $(\mathrm{D}^b(\mathcal{E}))^{ic}$.

2.1.2. Variant 2*: As Localization of exact category of complexes with respect to a biresolving subcategory.* This is a recent construction of Rump [**168**, Thm 5], cf. also [**170**].
A full additive subcategory $\mathcal{C}$ in an exact category $\mathcal{F}$ is a **biresolving subcategory** if it is thick (i.e. closed under direct summands and every $\mathcal{C}$ satisfies the 2-out of 3-property for short exact sequences (i.e. for every $\mathcal{F}$-short exact sequence $X \to Y \to Z$ with two out of X, Y, Z in $\mathcal{C}$ the third is also in $\mathcal{C}$) and it is generating-cogenerating (i.e. for every X in $\mathcal{F}$ there is a $\mathcal{F}$-deflation $d\colon C^0 \to X$ and an $\mathcal{F}$-inflation $i\colon X \to C^1$ with $C^0, C^1 \in \mathcal{C}$). (We slightly differ with this definition from loc. cit, as there it is not assumed that $\mathcal{C}$ is closed under direct summands.)
Then the localization $\mathcal{F}/\mathcal{C}$ is defined as follows: First consider $[\mathcal{C}]$ to be the ideal (i.e. subfunctor of Hom) given by all morphisms factoring through $\mathcal{C}$. We take $\Sigma(\mathcal{C}) \subseteq (\mathcal{F},)$ the class of all morphisms which become in the ideal quotient category $\mathcal{F}/[\mathcal{C}]$ a monomorphism and also an epimorphism. Then in loc. cit. it is shown that there exists a left and right calculus of fractions and

$$\mathcal{F}/\mathcal{C} := \Sigma(\mathcal{C})^{-1}\mathcal{F}$$

admits a structure of triangulated category, loc. cit Theorem 5.
Now we apply this as follows: We assume that $\mathcal{E}$ is weakly idempotent complete (in loc. cit this is not assumed).
We take *-chain complexes $\mathrm{Ch}^*(\mathcal{E})$ in $\mathcal{A}$. We see this as an exact category with short exact sequences are degree-wise $\mathcal{E}$-short exact sequences (cf. [**49**, Lem. 9.1]). We look at the full subcategory $\mathrm{Ac}^*(\mathcal{E})$ of $\mathcal{E}$-acyclic-complexes. This is a biresolving subcategory of the exact category $\mathrm{Ch}^*(\mathcal{E})$, cf. [**168**, Example 2]. Then this gives the second definition of the derived category (cf. [**170**])

$$\mathrm{D}^*(\mathcal{E}) := \mathrm{Ch}^*(\mathcal{E})/\mathrm{Ac}^*(\mathcal{E})$$

Then one can show the following Lemma as a corollary, the canonical functor of the localization is also the composition $L\colon \mathrm{Ch}^*(\mathcal{E}) \to \mathrm{K}^*(\mathcal{A}) \to \mathrm{D}^*(\mathcal{E})$. In [**175**, Tag 014Z], Lemma 13.12.1, it has been explained how to construct a δ such that (L, δ) is a δ-functor (even though in loc. cit. the category is abelian, the same arguments work for general exact categories). The construction goes as follows: Given a short exact sequence $\sigma\colon A^\bullet \xrightarrow{a} B^\bullet \xrightarrow{b} C^\bullet$ in $\mathrm{Ch}^*(\mathcal{E})$, one constructs a quasi-isomorphism (i.e. morphism with acyclic cone) $q\colon C(a) \to C$ where $C(a)$ is the cone of a, then one has a standard triangle in $\mathrm{K}^*(\mathcal{A})$ for the morphism a, and this gives a distinguished triangle in $\mathrm{D}^*(\mathcal{E})$. In particular, we have in this triangle a morphism $p\colon C(a) \to \Sigma A^\bullet$, so we have a well-defined

$$\delta_\sigma := p \circ q^{-1}\colon C^\bullet \to \Sigma A^\bullet \quad \in \mathrm{D}^*(\mathcal{E})$$

Recall that a complex is numbered as follows by the integers $\cdots \to X^n \to X^{n+1} \to \cdots$. We also have a shift on $T\colon \mathrm{Ch}(\mathcal{E}) \to \mathrm{Ch}(\mathcal{E})$, where $T(X)$ is the shift of complexes to the right, i.e. $T(X^\bullet)^n := X^{n+1}$.
For a complex X concentrated in degree 0 we have the short exact sequence $\eta_X\colon TX \rightarrowtail (X \xrightarrow{1_X} X) \twoheadrightarrow X$ with $(X \xrightarrow{1_X} X)$ is the 2-term complex concentrated in degrees $0, 1$. We have $L \circ T \cong \Sigma \circ L$ (we say L is **shift invariant**). Also, by definition $\delta_{\eta_X} = 1_{L(X)}$ (we say that L is **normed**).

Lemma 2.5. *Let $\mathcal{E}$ be weakly idempotent complete exact category. Then*

(1) *the canonical functor $L\colon \mathrm{Ch}^*(\mathcal{E}) \to \mathrm{D}^*(\mathcal{E})$ is a δ-functor which is shift invariant and normed.*
(2) *Every shift invariant, normed δ-functor $G\colon \mathrm{Ch}^{\mathrm{b}}(\mathcal{E}) \to \mathcal{D}$ factors uniquely as $\overline{G} \circ L$ with $\overline{G}\colon \mathrm{D}^b(\mathcal{E}) \to \mathcal{D}$ a triangle functor.*

Proof. (1) has already been explained before the lemma.
(2) Let $G = (G, \delta^G)\colon \mathrm{Ch}^{\mathrm{b}}(\mathcal{E}) \to \mathcal{D}$ be a shift invariant δ-functor.
The property being *normed* can be replace by mapping split acyclic complexes to zero, this means that G factors over a triangle functor $\mathrm{K}^b(\mathcal{A}) \to \mathcal{D}$. As it maps $\mathcal{E}$-short exact sequences to triangles, the triangle functor maps acyclic complexes to zero (cf. same argument in [**133**, Lem 3.5]) and therfore factors over a triangle functor $\overline{G}\colon \mathrm{D}^b(\mathcal{E}) \to \mathcal{D}$. Assume now, we have a second triangle functor $H\colon \mathrm{D}^b(\mathcal{E}) \to \mathcal{D}$ with $H \circ L = \overline{G} \circ L$. Now, L factors as $L_V \circ L_K$ with $L_K\colon \mathrm{Ch}^b(\mathcal{E}) \to \mathrm{K}^b(\mathcal{A})$ is just the ideal quotient and $L_V\colon \mathrm{K}^b(\mathcal{A}) \to \mathrm{D}^b(\mathcal{E})$ is the Verdier quotient. By the universal property of the Verdier quotient it is enough to see that $H \circ L_V = \overline{G} \circ L_V$. But L_K is full, so every morphism is of the form $L_{\mathrm{K}}(f)$ for some morphism and we see that $HL_V = \overline{G}L_V$. □

Remark 2.6. One can also see the derived categories of exact categories as homotopy categories associated to certain model categories (different choices might lead to the same homotopy categories). Also derived functors can be constructed in this more general set-up. For example look into [**81**] and [**96**].

2.2. Completions.

2.2.1. *Countable envelope.* Exact categories $\mathcal{E}$ always have a countable envelope $\tilde{\mathcal{E}}$ constructed in [**116**], Appendix B:

First construct $\mathcal{FE}$ with objects are sequences $X = (X^0 \xrightarrow{i_X^0} X^1 \xrightarrow{i_X^1} X^2 \to \cdots)$ of consecutively composable inflations, and morphisms $X \to Y$ are sequences of $(f^p\colon X^p \to Y^p)_{p\in\mathbb{N}_0}$ such that $i_Y^p f^p = f^{p+1} i_X^p$ for all $p \geq 0$. Then $\mathcal{FE}$ is an exact category with a sequence (j, e) is a short exact sequence if and only if (j^p, e^p) are short exact sequences in $\mathcal{E}$ for all $p \geq 0$. Then we define $\tilde{\mathcal{E}}$ as the category with the same objects and morphisms $\mathrm{Hom}(X, Y) = \lim_p \operatorname{colim}_q \mathrm{Hom}(X^p, Y^q)$. By construction we have a functor $\mathcal{FE} \to \tilde{\mathcal{E}}$, we call a sequence $(\tilde{j}, \tilde{e})$ an exact sequence if there exists an exact sequence (j, e) in $\mathcal{FE}$ which maps via the natural functor to it. Observe that the underlying additive category of $\tilde{\mathcal{E}}$ has countable coproducts (by taking only split inflations in a sequence of inflations as before). We recall Keller's results.

Theorem 2.7. *([**116**, Appendix B])*

(a) *$\tilde{\mathcal{E}}$ is an exact category wrt the exact sequences described before. The constant functor $E\colon \mathcal{E} \to \tilde{\mathcal{E}}$, $X \to (X = X = X \cdots)$ is a homologically exact functor (i.e. exact and inducing isomorphisms on all Ext-groups).*
(b) *The following are equivalent:*
  (b1) *$\mathcal{E}$ is locally small (i.e. has Hom-sets and not classes) and $\mathrm{Ext}^n_{\mathcal{E}}$ are set-valued for all $n \geq 1$*
  (b2) *$\tilde{\mathcal{E}}$ is locally small and $\mathrm{Ext}^n_{\tilde{\mathcal{E}}}$ are set-valued for all $n \geq 1$*

Proof. or where to find it: For (b) observe that (a) already implies (b2) ⇒ (b1). The other implication is using that (b1) implies that $\mathcal{F}\mathcal{E}$ has set-valued Ext^n for all $n \geq 1$ and then use [**116**, Lemma in B.3]. □

Remark 2.8. By the previous Lemma we have an induced fully faithful triangle functor $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\tilde{\mathcal{E}})$ but we do not know if $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\tilde{\mathcal{E}})$ is faithful.

2.2.2. *Completion of small exact categories.* Small exact categories $\mathcal{E}$ have a completion $\overrightarrow{\mathcal{E}}$ (called a locally coherent exact category) with respect to filtered colimits, cf. [**152**] and Appendix in Chapter 2.

$$\mathcal{E} \to \overrightarrow{\mathcal{E}}$$

Then $\mathcal{E}$ is a full homologically exact category of $\overrightarrow{\mathcal{E}}$. Also, in loc. cit, the author shows that $\overrightarrow{\mathcal{E}}$ is a so-called exact category of Grothendieck type which implies that it has enough injectives.

Remark 2.9. $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\overrightarrow{\mathcal{E}})$ might be not full (examples are given by [**153**]), a characterization when this triangle functor is faithful is unknown.

The difference to the countable envelope is that we assume here that $\mathcal{E}$ is (essentially) small. If we drop this assumption then we do not know much about the Ind-completion (but it can have not enough injectives).

2.3. Passing between different boundedness levels.

Lemma 2.10. *([**119**]) By construction we have triangle functors* $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^{+/-}(\mathcal{E}) \to D(\mathcal{E})$. *They are all fully faithful.*

Let us consider two fixed exact categories $\mathcal{E}, \mathcal{E}'$ and the following three statements

(D^b) $\mathrm{D}^b(\mathcal{E})$ and $\mathrm{D}^b(\mathcal{E}')$ are triangle equivalent.
(D^+) $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^+(\mathcal{E}')$ are triangle equivalent.
(D) $\mathrm{D}(\mathcal{E})$ and $\mathrm{D}(\mathcal{E}')$ are triangle equivalent.

We are now looking at situations where one holds and another one not.

(a) If $\mathcal{E} \neq \mathcal{E}^{ic} = \mathcal{E}'$ then (D), (D^+) and not (D^b) holds (cp. [**141**]).
(b) If $\mathcal{E} \subseteq \mathcal{E}'$ is a coresolving subcategory of $\mathcal{E}'$, then we have (D^+). It is finitely coresolving if and only if (D^b) holds. It is n-coresolving for some $n \geq 0$ if and only if (D) holds.
    This way, we find an instance where (D^+) and not (D^b) and also not (D) holds. Reference [**92**]. Also: an instance where (D^b) and not (D) holds.

This leaves only the following open: Does (D^b) imply (D^+)? I do not know.

Remark 2.11. When restricting to exact categories with certain similar properties, there are still interdependences. For module categories of rings (cf. main theorem in [**158**]). This has been generalized to functor categories [**10**]. The best explanation for this is given by Neeman's theory of approximable triangulated categories, cf. [**143**].

3. When is it a category?

In this text, we will work in the framework of ZFCU (Zermelo-Frankel, the axiom of choice and the axiom of the universe). We fix an infinite universe $\mathcal{U}$.

Let X be a set, we will say that X is: $\mathcal{U}$*-small* if $X \in \mathcal{U}$, $\mathcal{U}$*-class* if $X \subset \mathcal{U}$ or $\mathcal{U}$*-large set* if $X \not\subseteq \mathcal{U}$. In general, we will drop $\mathcal{U}$ from the notation.

We allow a category to have a class of objects and to have a class as morphisms between any two objects. We say that a category is *locally small* if $\mathrm{Hom}(X,Y)$ is a small set for every two objects X,Y - usually in the literature: locally small categories are called categories.

As usual, we let Set denote the category of small sets with morphisms given by maps between them. We will need a *big version* of this category, namely $\widehat{\mathrm{Set}}$, where we allow the objects to be classes as well as the morphisms between them. We denote $\widehat{\mathrm{Gps}}, \widehat{\mathrm{Ab}}\ldots$ denote the 'big' versions the categories of groups, abelian groups, etc.

Let $\mathcal{E}$ be a (locally small Quillen-) exact category.

THEOREM 3.1. *The following are equivalent:*

(1) $\mathrm{Ext}^n_{\mathcal{E}}$ *is set-valued for all* $n \geq 0$
(2) $\mathrm{D}^b(\mathcal{E})$ *is a locally small.*

If $\mathcal{E}$ *has countable coproducts then they are also equivalent to the following conditions:*

(3) $\mathrm{D}^-(\mathcal{E})$ *is locally small*
(3') $\mathrm{D}^+(\mathcal{E})$ *is locally small.*
(4) $\mathrm{D}(\mathcal{E})$ *is locally small.*

Remark 3.2. If we can proof that the inclusion of $\mathcal{E}$ into its countable envelope $\tilde{\mathcal{E}}$ induces a faithful functor $\mathrm{D}(\mathcal{E}) \to \mathrm{D}(\tilde{\mathcal{E}})$, then we can drop the assumption that $\mathcal{E}$ has to have countable coproducts because we could just replace $\mathcal{E}$ by its countable envelop (cp. subsubsection 2.2.1) .

Let us state a very easy corollary

Corollary 3.3. *If* $\mathcal{E}$ *is an exact category. We denote by* $\mathcal{E}^{ic}$ *its idempotent completion. Then:* $\mathrm{D}^b(\mathcal{E})$ *is a locally small if and only if* $\mathrm{D}^b(\mathcal{E}^{ic})$ *is locally small.*

PROOF. As $\mathcal{E}$ is homologically exact in $\mathcal{E}^{ic}$ (cf. [**31**]), we conclude that every $\mathrm{Ext}^n_{\mathcal{E}^{ic}}((X,e_1),(Y,e_2))$ is a summand of $\mathrm{Ext}^n_{\mathcal{E}}(X,Y)$. Then use Theorem 3.1. ☐

Corollary 3.4. *For every locally coherent exact category* $\mathcal{E}$*, the derived categories* $\mathrm{D}^*(\mathcal{E})$ *with* $* \in \{b, \pm, \emptyset\}$ *are locally small.*

PROOF. Locally coherent exact categories have enough injectives therefore all Ext^n are set-valued. They have countable coproducts therefore the theorem directly applies. ☐

For the proof of Theorem 3.1 we observe the following:

Lemma 3.5. *Let* $\mathcal{G}$ *be a triangulated category together with a homological functor* $F\colon \mathcal{G} \to \widehat{\mathrm{Ab}}$. *Consider the full subcategory* $\mathcal{C}$ *of* $\mathcal{G}$ *on the objects* X *such that* $F(\Sigma^n X) \in \mathrm{Ab}$ *for all* $n \in \mathbb{Z}$. *Then* $\mathcal{C}$ *is a thick subcategory of* $\mathcal{G}$.

PROOF. This is an immediate consequence of the fact that Ab is a closed under extensions. ☐

Then we have this easy corollary:

Corollary 3.6. *Let* $\mathcal{T}$ *be a triangulated category, and* $\mathcal{C} \subseteq \mathcal{T}$ *be a full locally small subcategory and assume that* $\mathcal{C}$ *is closed under all shifts. Then* $\mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$ *is also locally small.*

PROOF. Let $\mathcal{D}$ be the full subcategory of $\mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$ on the objects X such that $\mathrm{Hom}_{\mathcal{T}}(X,C)$ and $\mathrm{Hom}_{\mathcal{T}}(C,X)$ are small sets for any $C \in \mathcal{C}$. Note that $\mathcal{D}$ is closed under arbitrary shift by the hypothesis on $\mathcal{C}$. By Lemma 3.5, it follows that $\mathcal{D}$ is thick, and since it contains $\mathcal{C}$ we deduce that $\mathcal{D} = \mathrm{Thick}_{\mathcal{T}}(\mathcal{C})$. ☐

Proof of Theorem 3.1. (n) implies (1) for $n \in \{2,3,3',4\}$: Follows directly since for all X, Y in $\mathcal{E}$ we have $\mathrm{Ext}^n_{\mathcal{E}}(X,Y) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(X, \Sigma^n Y)$ for all $n \in \mathbb{Z}$ where $\mathrm{Ext}^0_{\mathcal{E}}(X,Y) = \mathrm{Hom}_{\mathcal{E}}(X,Y)$ and $\mathrm{Ext}^{<0}_{\mathcal{E}}(X,Y) = 0$, cp [**126**, Prop. 4.2.11]. As $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^*(\mathcal{E})$ is fully faithful for $* = \pm, \emptyset$, the other implications also follow.
(1) implies (2): Just take $\mathcal{T} = \mathrm{D}^b(\mathcal{E})$ and $\mathcal{C} = \mathrm{add}(\mathcal{E}[n], n \in \mathbb{Z})$. Then, by definition $\mathcal{C}$ is locally small if and only if (1) in the Theorem 3.1 is fulfilled. Since $\mathrm{Thick}_{\mathrm{D}^b(\mathcal{E})}(\mathcal{C}) = \mathrm{D}^b(\mathcal{E})$, it follows (2) from the previous Corollary.
Now assume that $\mathcal{E}$ has countable coproducts, let $\mathcal{A}$ be the underlying additive category. Then, every object X in $\mathrm{K}^-(\mathcal{A})$ fits into a triangle

$$\bigoplus_{n\leq 0} \sigma_{\geq n} X \to \bigoplus_{n \leq 0} \sigma_{n \geq 0} X \to X \xrightarrow{+1}$$

where $\sigma_{\geq n}X$ is the (brutal) truncation of a complex X is defined as $\cdots 0 \to X^n \to X^{n+1} \to \cdots$ (see e.g. [**126**, Ex. 4.2.2]). By [**126**, Lemma 1.1.8], the Verdier quotient functor commutes with countable coproducts and maps this distinguished triangle to a distinguished triangle in $\mathrm{D}^-(\mathcal{E})$.
(2) implies ((3) and (3')): This means the extension-closure of $\mathrm{D}^b(\mathcal{E})$ in $\mathrm{D}^-(\mathcal{E})$ is the whole triangulated category and therefore Corollary 3.6 implies the claim. The argument for $\mathrm{D}^+(\mathcal{E})$ is analogue using brutal truncation in the other direction.
((3) and (3')) implies (4): Now, we look at the unbounded homotopy category $\mathrm{K}(\mathcal{A})$. Brutal truncation yields a distinguished triangle

$$\sigma_{\geq 0} X \to X \to \sigma_{<0} X \xrightarrow{+1}$$

Then passing to the Verdier quotient we can look at the smallest additive subcategory of $\mathrm{D}(\mathcal{E})$ containing $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^-(\mathcal{E})$ and call this category $\mathcal{C} = \mathrm{D}^-(\mathcal{E}) \vee \mathrm{D}^+(\mathcal{E})$. If $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^-(\mathcal{E})$ are locally small, then $\mathcal{C}$ is also locally small by Lemma 3.7. Clearly $\mathcal{C}$ is closed under all shifts. The previous triangle shows that $\mathcal{C}$ is a thick generator for $D(\mathcal{E})$ and therefore $\mathrm{D}(\mathcal{E})$ is also a locally small category.

□

Lemma 3.7. *Let $\mathcal{E}$ be an exact category with countable coproducts. Assume that $\mathrm{D}^+(\mathcal{E})$ and $\mathrm{D}^-(\mathcal{E})$ are locally small. Then for every $X \in \mathrm{D}^-(\mathcal{E}), Y \in \mathrm{D}^+(\mathcal{E})$ we have that $\mathrm{Hom}_{\mathrm{D}(\mathcal{E})}(X,Y)$ and $\mathrm{Hom}_{\mathrm{D}(\mathcal{E})}(Y,X)$ are sets.*

Proof. For X in $\mathrm{D}^-(\mathcal{E})$ we find a distinguished triangle (see above):

$$\bigoplus_{n\leq 0} \sigma_{\geq n} X \to \bigoplus_{n \leq 0} \sigma_{n \geq 0} X \to X \xrightarrow{+1}$$

Then apply $\mathrm{Hom}(-,Y)$ and apply $\mathrm{Hom}(Y,-)$ with $Y \in \mathrm{D}^+(\mathcal{E})$. The rest is obvious (use the five terms of the long exact sequences with $\mathrm{Hom}(X,Y)$ and resp $\mathrm{Hom}(Y,X)$ in the middle, $\mathrm{Hom}(\bigoplus X_i, Y) \cong \prod_i \mathrm{Hom}(X_i, Y)$ and $\mathrm{Hom}(Y, \bigoplus_i X_i) \cong \mathrm{Hom}(Y, X_i)$ implies that the other four terms are small abelian groups). □

Remark 3.8. As far as we know there is no characterization of all higher Ext-functors in an exact category being set-valued. Nevertheless there is the following list of examples where this is fulfilled.

(a) If $\mathcal{E}$ is essentially small. One reference for this [**126**], Lemma 4.2.17 together with Prop. 4.3.4.
(b) If $\mathcal{E}$ has enough projectives or enough injectives (more generally of $\mathrm{K}^b(\mathcal{E})$ has enough K-projectives or enough K-injectives). A reference for this [**126**], Cor. 4.3.2, p.123.
(c) If $\mathcal{E}$ has a small generator or a cogenerator.

We give some examples of categories to see that (1) can not be weakened.

Example 3.9. We will look at representations of *quivers* where we allow the vertices and arrows to be a proper classes and representations are to be understood as in vector spaces over some field K and of finite total dimension. Furthermore, we will always impose the relations that the composition of any two composable arrows is zero. These are abelian categories. Let's construct some examples. We fix a proper class M (i.e. this is not a set).

(1) We look at Q with two vertices 1 and 2 and arrows $a_m \colon 1 \to 2$ for each m in M. Then this gives an abelian category with Hom-sets but $\mathrm{Ext}^1(S_1, S_2)$ is not a set since we find for every m in M a short exact sequence

$$0 \to S_2 \to I_m \to S_1 \to 0$$

with I_m is the representation with $a_m = \mathrm{id_K}$ and $a_n = 0$ for all $n \neq m$. These are pairwise non-isomorphic.

(2) Now we look at a quiver with vertices $1, 2$ and $v_m, m \in M$ and arrows $b_m \colon 1 \to v_m, c_m \colon v_m \to 2$ for every $m \in M$ and the relations $c_m b_m = 0$ for all $m \in M$. Then this gives an abelian category with Hom and Ext^1 are set-valued. But for every $m \in M$ we have an exact sequence

$$0 \to S_2 \to J_m \to L_m \to S_1 \to 0$$

with J_m 2-dimensional given by $b_n = 0$ for all $n \in M$ and $c_m = \mathrm{id}_K$, $c_n = 0$ for $n \neq m$ and L_m 2-dimensional given by $b_m = \mathrm{id_K}$, $b_n = 0$ for $n \neq m$, $c_n = 0$ for all $n \in M$. Again these are pairwise non-isomorphic, so $\mathrm{Ext}^2(S_1, S_2)$ is not a set.

(3) Fix an integer $t \geq 1$. Now look at the quiver with vertices $1, 2$ and $v_{1,m}, \ldots, v_{t,m}$ for every $m \in M$ and arrows $a_{1,m} \colon 1 \to v_{1,m}, a_{2,m} \colon v_{1,m} \to v_{2,m}, \ldots, a_{t,m} \colon v_{t-1,m} \to v_{t,m}, a_{t+1,m} \colon v_{t,m} \to 2$. Again we impose $a_{i+1,m} a_{i,m} = 0$ for $1 \leq i \leq t$ and $m \in M$. With a similar argument as in the previous cases one shows that $\mathrm{Ext}^{t+1}(S_1, S_2)$ is not a set. But Hom and Ext^i are set-valued for $1 \leq i \leq t$.

4. Keller's approach to deriving additive functors between exact categories

Or, as much as we could verify of it. We summarize the construction of [119], in a simple language. If $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ and $\mathcal{F} = (\mathcal{B}, \mathcal{S}')$ we look at the class $\mathcal{S}_f := \{(i, d) \in \mathcal{S} \mid (f(i), f(\mathrm{D})) \in \mathcal{S}'\}$ which we call f-exact sequences. If $(i, d) \in \mathcal{S}_f$, we call i an f-inflation and d an f-deflation.

Definition 4.1. Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ an exact category. Let $\mathcal{X}$ be a class of kernel-cokernel pairs on $\mathcal{A}$. We call an object $X \in \mathcal{A}$ **right $\mathcal{X}-$acyclic** if every $X \xrightarrow{i} Y \xrightarrow{d} Z$ with $(i, d) \in \mathcal{S}$ fulfills $(i, d) \in \mathcal{X}$. We call $\mathcal{C}_{\mathcal{X}}$ the full subcategory of $\mathcal{A}$ of right $\mathcal{X}$-acyclic objects.

Definition 4.2. Let $\mathcal{S}_f^{pb}$ be the class of all $\sigma \in \mathcal{S}$ such that there exists a commutative diagram

$$\begin{array}{ccccc} X & \rightarrowtail & Y' & \twoheadrightarrow & Z' \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ X & \rightarrowtail & Y & \twoheadrightarrow & Z \end{array}$$

with $\sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0)$ in the upper row and an f-exact short exact sequence $\sigma_f \in \mathcal{S}_f$ in lower row.

We define the category of **right f-acyclics** as the full subcategory $\mathcal{C} := \mathcal{C}_{\mathcal{S}_f^{pb}}$ of $\mathcal{A}$.

This will later be used to show that this notion is the one mentioned in the literature.

Lemma 4.3. *The following are equivalent for $X \in \mathcal{E} = (\mathcal{A}, \mathcal{S})$.*

(1) *X is right f-acyclic*

(2) *Every morphism $M \to X$ in $\mathrm{K}^+(\mathcal{A})$ with M an $\mathcal{E}$-acyclic complex factors as $M \to M' \to X$ with M' an $\mathcal{E}$-acyclic and f-exact complex (i.e. the corresponding short exact sequences are all in $\mathcal{S}_f$).*

Proof. Assume $f\colon M \to X$ is a chain map with $M = (M^n, d^n)$ acyclic and X in degree 0. Then such a map is given by a morphism $h\colon N = \operatorname{Im} d^0 \to X$. Therefore, it always factors over the push-out of $\operatorname{Im} d^0 \rightarrowtail M^1 \twoheadrightarrow \operatorname{Im} d^1$ along h. This means we may assume wlog that M is a short exact sequence $\sigma\colon X \rightarrowtail Y \twoheadrightarrow Z$ (in degrees $0,1,2$) and the morphism $M \to X$ is the identity on X in degree 0 and zero in all other degrees.
Assume (2): If this now factors over an f-exact short exact sequence $\sigma\colon X' \rightarrowtail Y' \twoheadrightarrow Z'$ then we find that σ' pulls back to $\sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0)$.
Assume (1): If $\sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0)$ is a pull back of $\sigma_f \in \mathcal{S}_f$, then we have that $M \to X$ factors over $\sigma \to \sigma \oplus (X_1 \xrightarrow{1} X_1 \to 0) \to \sigma_f =: M'$. □

Remark 4.4. Recall in Chapter 1 we showed: If f is left or right exact or if f is fully faithful with extension-closed essential image, then $\mathcal{S}_f$ is already an exact structure.

Lemma 4.5. *Assume that $\mathcal{E}_f = (\mathcal{A}, \mathcal{S}_f)$ is an exact substructure of $\mathcal{E}$. Then an object X in $\mathcal{A}$ is right f-acyclic if and only if the natural morphism*

$$\operatorname{Ext}^1_{\mathcal{E}_f}(Y, X) \to \operatorname{Ext}^1_{\mathcal{E}}(Y, X)$$

is an isomorphism for all $Y \in \mathcal{A}$. In particular, $\mathcal{C}$ is extension- and inflation-closed in $\mathcal{E}$.
Furthermore, Given an $\mathcal{E}$-short exact sequence $X \xrightarrow{i} Y \xrightarrow{d} Z$ with X, Y, Z in $\mathcal{C}$ then it is f-exact (i.e $(i, d) \in \mathcal{S}_f$)

Proof. This is easy to see. □

Definition 4.6. If $\mathcal{E}_f = (\mathcal{A}, \mathcal{S}_f)$ is an exact substructure of $\mathcal{E}$, we say f has enough right f-acyclics if $\mathcal{C}$ is cogenerating in $\mathcal{E}$ (i.e. every object $X \in \mathcal{E}$ there exists an inflation $X \to C$ with $C \in \mathcal{C}$). Then it is a coresolving subcategory and $i\colon \mathrm{D}^+(\mathcal{C}) \to \mathrm{D}^+(\mathcal{E})$ is a triangle equivalence and we have a triangle functor

$$rf\colon \mathrm{D}^+(\mathcal{E}) \xrightarrow{i^{-1}} \mathrm{D}^+(\mathcal{C}) \xrightarrow{f} \mathrm{D}^+(\mathcal{F})$$

In general, we do not see why $\mathcal{C}$ should be extension-closed in $\mathcal{E}$ nor why it should satisfy condition (C2) from [**119**] (this is claimed in [**119**, Lemma 15.3]). As this is not proven in loc. cit. one should treat it as a conjecture.
There are two strategies how one can pass to an extension-closed subcategory $\mathcal{C}_2 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}$ $i = 1, 2$. Either $X \in \mathcal{C}_1$ are all objects such that every $\mathcal{E}$-short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ is also f-exact, or one looks at the maximal exact substructure $\mathcal{E}_{f,max} \leq \mathcal{E}$ making the functor f exact and then $X \in \mathcal{C}_2$ are all objects such that all $\mathcal{E}$-short exact sequences $X \rightarrowtail Y \twoheadrightarrow Z$ are already $\mathcal{E}_{f,max}$-short exact sequences. The advantage of $\mathcal{C}_2$ is that we can generalize the previous Lemma imediately and conclude that $\mathcal{C}_2$ is extension- and inflation-closed.

Example 4.7. Observe that the full subcategory on $\mathcal{E}$-injectives $\mathcal{I} = \mathcal{I}(\mathcal{E})$ is contained in $\mathcal{C}_2$. So if $\mathcal{E}$ has enough injectives then it also has enough right f-acyclics (using the subcategory $\mathcal{C}_2$).

4.1. $rf = Rf$. We explain now why this notion of right f-acyclic coincides with the one defined in [**119**] and then conclude rf is the right derived functor Rf of f.
Generally, given a triangle functor $F\colon \mathcal{T} \to \mathcal{T}'$ and a fixed Verdier quotient $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{M} =: \mathbb{D}$ (we think of this as a derived category). We will choose $F\colon \mathrm{K}^+(\mathcal{A}) \xrightarrow{\mathrm{K}^+(f)} \mathrm{K}^+(\mathcal{B}) \to \mathrm{D}^+(\mathcal{F})$ and $Q\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{E})$. Keller constructs (following roughly Deligne) a (possibly zero) triangulated subcategory $\mathcal{U}$ of $\mathcal{T}$ such that $\mathcal{U}/(\mathcal{U} \cap \mathcal{M})$ is a triangulated subcategory of $\mathcal{T}/\mathcal{M}$ such that $F|_{\mathcal{U}}$ factors over a triangle functor $\mathcal{U}/\mathcal{U} \cap \mathcal{M} \to \mathcal{T}'$ (and this triangle functor is isomorphic to the restriction of Deligne's Rf):

Definition 4.8. A triangulated subcategory $\mathcal{U}$ of $\mathcal{T}$ is called **right cofinal** (wrt $\mathcal{M}$) if every morphism $M \to X$ with $M \in \mathcal{M}$, $X \in \mathcal{U}$ factors as $M \to M' \to X$ with $M' \in \mathcal{U} \cap \mathcal{M}$.

In this case, the induced triangle functor $\mathcal{U}/(\mathcal{U} \cap \mathcal{M}) \to \mathbb{D}$ is fully faithful (cf. [**119**, 10.3]).
Now let $\ker F$ be the thick subcategory of $\mathcal{T}$ with objects $T \in \mathcal{T}$ such that $F(T) \cong 0$.
This definition differs from [**119**, Lem. 14.1], explanation see below:

Definition 4.9. We say $X \in \mathcal{T}$ is F**-split** if every morphism $M \to X$ in $\mathcal{T}$ with $M \in \mathcal{M}$ factors as $M \to M' \to X$ with $M' \in \mathcal{M} \cap \ker F$.
We call $\mathcal{U}$ be the full subcategory of $\mathcal{T}$ with F-split objects.

Remark 4.10. The characterization of F-split objects in [**119**, 14.1, (iii)] does not seem to give a triangulated subcategory. We use Lipman's stronger definition ([**135**], Def. (2.2.5), Ex. 2.2.8(D)) as in that case the F-split objects are shown to be a triangulated subcategory.

Lemma 4.11. *(cf.* [**135**, Lemma 2.2.5.1]*) $\mathcal{U}$ is a triangulated subcategory of $\mathcal{T}$.*

Observe, that $\mathcal{M} \cap \ker F \subseteq \mathcal{M} \cap \mathcal{U}$ because given $X \in \mathcal{M} \cap \ker F$ and $M \to X$ a morphism with $M \in \mathcal{M}$ we can consider the factorization over $M' = X \xrightarrow{1_X} X$, to see that $X \in \mathcal{U}$. In particular, $\mathcal{U}$ is right cofinal in $\mathcal{T}$.
But also by definition $X \in \mathcal{U} \cap \mathcal{M} \subseteq \ker F$ because now we can take $1_X \colon X \to X$ and it has to factor over $M' \in \ker F \cap \mathcal{M}$ as $X \in \mathcal{U}$. This means $1_{F(X)} = F(1_X) \cong 0$ and therefore $F(X) \cong 0$. By the universal property of the Verdier quotient, F factors over a triangle functor $\mathcal{U}/(\mathcal{U} \cap \mathcal{M}) \to \mathcal{T}'$.

Then we just cite the following result as we do not remind the reader of Deligne's definition of the derived functor.

Lemma 4.12. *(*[**119**, section 14]*) The triangle functor $\mathcal{U}/\mathcal{U} \cap \mathcal{M} \to \mathcal{T}'$ coincides with Deligne's $RF|_{\mathcal{U}/\mathcal{M} \cap \mathcal{U}}$.*

We call $\mathcal{C}^{Del} := \mathcal{U} \cap \mathcal{A} \subseteq \mathrm{K}^+(\mathcal{A})$ the category of **right F-acyclics** (i.e. these are the F-split stalk complexes in the homotopy category). Then:

Corollary 4.13. *(of Lemma 4.3)* $\mathcal{C}^{Del} = \mathcal{C}$

CHAPTER 7

Tilting theory in exact categories

1. Synopsis

We define tilting subcategories in arbitrary exact categories to achieve the following. Firstly: Unify existing definitions of tilting subcategories to arbitrary exact categories. We discuss standard results for tilting subcategories: Auslander correspondence, Bazzoni description of the perpendicular category.
Secondly: We treat the question of induced derived equivalences separately - given a tilting subcategory $\mathcal{T}$, we ask if a functor on the perpendicular category induces a derived equivalence to a (certain) functor category $\operatorname{mod}_\infty \mathcal{T}$ over $\mathcal{T}$. If this is the case, we call the tilting subcategory *ideq* tilting. We prove a generalization of Miyashita's theorem (which is itself a generalization of a well-known theorem of Brenner-Butler) and characterize exact categories with enough projectives allowing ideq tilting subcategories.
In particular, this is always fulfilled if the exact category is abelian with enough projectives.

2. Introduction

Tilting theory (= categories with tilting objects) is originally defined for categories of finitely generated modules over artin algebras (Brenner-Butler) and tilting modules were still assumed of pd ≤ 1. Then this was generalized to arbitrary projective dimension by [**87**], Chapter 3 - here you also find a detailed account of the beginning of tilting theory (starting with BGP reflection functors). Afterwards this was generalized in several directions ([**180**], [**137**], [**57**], [**6**], [**58**], [**88**], [**7**] Chapter 5, [**158**] - referred to (by us) as: *infinite* or *big* tilting or tilting in triangulated categories). The different developments at that time (2007) were captured in the handbook of tilting theory [**7**]. Since then, many more generalizations were found, e.g. the later discussed recent works [**185**], [**138**], [**169**]. For exact categories: The first occurrence in [**13**] is tilting objects in exact substructures of $\operatorname{mod} \Lambda$ for an artin algebra Λ. In [**126**], Chapter 7, a tilting object in an exact category is defined as a self-orthogonal object which generates the bounded derived category of the exact category as a thick subcategory. In [**185**], the authors define tilting subcategories for extriangulated categories with enough projectives and enough injectives. The literature on *infinite* or *big* tilting in exact categories will not be considered here (this includes [**169**]). An alternative definition of tilting in exact categories can be found in [**138**] which covers big and small tilting in exact categories (as far as we can see: Here is an additional axiom (T3) required in loc. cit. which we are not using).

Our motivation is to generalize and unify the following classical and recent definitions and results (which we only sketch as follows):

(A) Tilting modules over artin algebras induce derived equivalences ([**87**] Chapter 3, [**158**]).
(B) Relative tilting modules over artin algebras induce derived equivalences ([**13**], [**45**]).
(C) Tilting bundles over projective varieties over a field induce derived equivalences ([**28**], [**39**]).
(D) Zhang-Zhu [**185**] introduce tilting subcategories for extriangulated categories with enough projectives and enough injectives. They prove a generalization of a result called Auslander-correspondence.

To include (C),(D): We must drop the assumption that the exact category has enough projectives and we have to generalize from tilting modules to tilting subcategories. Our definition can be found as Definition 5.1. Furthermore, we shortly discuss standard results from tilting theory:
1. *Bazzoni's description of the perpendicular category* cf. Corollary 5.5. This gives a description of the perpendicular category of a tilting subcategory which is practical to find examples (see e.g. special tilting).
2. *Auslander correspondence* cf. subsection 5.1.1. This means a characterization of the subcategories which arise as perpendicular categories of tilting subcategories.
As this topic has so many precursors, we discuss compatibility with other definitions of tilting (this does not claim completeness due to the amount of literature on the subject) in section 6.
To generalize (A),(B): We need to find a *tilting functor* which induces in (A) and (B) a triangle equivalence as claimed.
We introduce in section 7 the notion ideq tilting, which means that the tilting functor induces a triangle equivalence on the bounded derived categories. We prove, if for a an n-tilting subcategory $\mathcal{T}$, the category $\operatorname{mod}_\infty \mathcal{T}$ has finite global dimension, then $\mathcal{T}$ is ideq tilting (cf. Prop. 7.8)- this generalizes (C). We prove the following results as generalizations of (A) and (B) respectively:

Theorem 2.1. *(cf. Thm 7.14) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then the following are equivalent:*

(1) *$\mathcal{E}$ is equivalent as an exact category to a finitely resolving subcategory $\overline{\mathcal{E}}$ of $\operatorname{mod}_\infty \mathcal{P}$, i.e. $\overline{\mathcal{E}}$ is resolving and for every $F \in \operatorname{mod}_\infty \mathcal{P}$ there is a finite exact sequence $0 \to E_n \to \cdots \to E_0 \to F \to 0$ with some $E_i \in \overline{\mathcal{E}}$ for some $n \geq 0$.*
(2) *There is an $n \in \mathbb{N}_0$ and an n-tilting subcategory of $\mathcal{E}$ which is ideq n-tilting.*
(3) *For every $n \geq 0$, every n-tilting subcategory of $\mathcal{E}$ is ideq n-tilting.*

Corollary 2.2. *(cf. Cor. 7.34) Let $\mathcal{P}$ be an idempotent complete, additive category. Let $\mathcal{E}$ be an exact substructure of $\operatorname{mod}_\infty \mathcal{P}$, with enough projectives $\mathcal{Q}$. Then for every $n \geq 0$, every n-tilting subcategory of $\mathcal{E}$ is ideq n-tilting.*

To prove the first theorem, we prove a *Miyashita Theorem* (generalization of Brenner Butler's theorem) cf. Theorem 7.13. This describes the image of the perpendicular category of an ideq tilting subcategory $\mathcal{T}$ under the tilting functor $(X \to \operatorname{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{T}})$.

As standing assumption: We will always assume that the exact category is idempotent complete.
To introduce definitions of subcategories and recall results from the literature, we start in section 2 with preliminaries on subcategories of exact categories and in section 3 we give a quick introduction to the bounded derived category.
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3. Some definitions of subcategories

Let $\mathcal{A}$ be an idempotent complete category. Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be throughout this section be an exact category in the sense of Quillen (it consists of an additive category $\mathcal{A}$ together with a class of kernel-cokernel pairs $\mathcal{S}$, referred to as **short exact sequences** which satisfy the axioms of [**49**], Def. 2.1).
For a kernel-cokernel pair $(i, d) \in \mathcal{S}$ we call i an **inflation** and d a **deflation**. We will denote by $\mathcal{P}(\mathcal{E})$ the projective objects in $\mathcal{E}$ and by $\mathcal{I}(\mathcal{E})$ the injectives. Since this is common practice, we will also often denote the underlying additive category $\mathcal{A}$ again by $\mathcal{E}$ - we think the reader can handle this level of ambiguity.

Definition 3.1. If $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ is an exact category and $\mathcal{X}$ a full subcategory which is closed under extensions. Then we call $\mathcal{X}$ **fully exact subcategory** if we consider it together with the exact structure $\mathcal{S}|_{\mathcal{X}}$ (i.e. the short exact sequences in $\mathcal{S}$ where all three terms lie in $\mathcal{X}$). We will write $\mathcal{P}(\mathcal{X})$ for the Ext-projectives in $\mathcal{X}$.

3.1. Subcategories generated by or orthogonal to a subcategory. We call a morphism $f\colon X \to Y$ in $\mathcal{E}$ **admissible** if it factors as $f = d \circ i$ for an inflation i and a deflation d. We say $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g admissible is exact at Y if $\operatorname{Im} f = \ker g$ and $\ker g \to Y \to \operatorname{Im} g$ is an exact sequence in $\mathcal{E}$. A sequence of composable morphisms is **exact** if every morphism is admissible and the sequence is exact at every intermediate object (for short exact sequences we sometimes leave out the zeros in the beginning and end). We call a sequence

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \to X_0 \xrightarrow{f_0} X_{-1} \xrightarrow{f_{-1}} 0$$

right exact if there is an exact sequence

$$0 \to Z \to X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \to X_0 \xrightarrow{f_0} X_{-1} \xrightarrow{f_{-1}} 0$$

We say a (co- or contravariant) functor $F\colon \mathcal{E} \to (Ab)$ into abelian groups **is exact on** the right exact sequence if F maps all short exact sequences

$$\ker f_i \to X_i \to \operatorname{Im} f_i, \quad -1 \leq i \leq n$$

to short exact sequences in abelian groups.
Let $\mathcal{X}$ be a full additive subcategory of $\mathcal{E}$ and an integer $n \geq 0$ and a subset $I \subset \mathbb{N}_0$, we define the following full subcategories of $\mathcal{E}$

$$\begin{aligned}
\operatorname{gen}_n(\mathcal{X}) &= \{M \in \mathcal{E} \mid \exists \text{ right exact } X_n \to \cdots \to X_0 \to M \to 0,\ X_i \in \mathcal{X} \\
&\qquad\qquad \operatorname{Hom}_{\mathcal{E}}(X, -) \text{ exact on it for every } X \in \mathcal{X}\} \\
\operatorname{pres}_n(\mathcal{X}) &= \{M \in \mathcal{E} \mid \exists \text{ right exact } X_n \to \cdots \to X_0 \to M \to 0,\ X_i \in \mathcal{X}\} \\
\operatorname{Res}_n(\mathcal{X}) &= \{M \in \mathcal{E} \mid \exists \text{ exact } 0 \to X_n \to \cdots \to X_0 \to M \to 0,\ X_i \in \mathcal{X}\} \\
\mathcal{X}^{\perp_I} &= \{M \in \mathcal{E} \mid \operatorname{Ext}^i_{\mathcal{E}}(X, M) = 0 \text{ for all } i \in I, X \in \mathcal{X}\}
\end{aligned}$$

We write $\mathcal{X}^{\perp_{\geq n}} := \mathcal{X}^{\perp_{[n,\infty)}}, \mathcal{X}^{\perp} := \mathcal{X}^{\perp_{\geq 1}}$. We define $\operatorname{gen}_\infty(\mathcal{X})$ (resp. $\operatorname{pres}_\infty(\mathcal{X})$) analogously to $\operatorname{gen}_n(\mathcal{X})$ (resp. $\operatorname{pres}_n(\mathcal{X})$) with infinite resolutions.
We define $\operatorname{gen}(\mathcal{X}) := \operatorname{gen}_0(\mathcal{X}), \operatorname{pres}(\mathcal{X}) := \operatorname{pres}_0(\mathcal{X})$ and observe $\operatorname{gen}(\mathcal{X}) = \operatorname{pres}(\mathcal{X})$ is equivalent to $\mathcal{X}$ is contravariantly finite in $\operatorname{pres}(\mathcal{X})$. Also, we set

$$\operatorname{Res}(\mathcal{X}) := \bigcup_{n \geq 1} \operatorname{Res}_n(\mathcal{X})$$

We have obvious inclusions

$$\operatorname{Res}_n(\mathcal{X}) \subset \operatorname{pres}_n(\mathcal{X}) \supset \operatorname{gen}_n(\mathcal{X}), \quad \operatorname{Res}(\mathcal{X}) \subset \operatorname{pres}_\infty(\mathcal{X}) \supset \operatorname{gen}_\infty(\mathcal{X})$$

We denote the dual notions with $\operatorname{cogen}^n(\mathcal{X}), \operatorname{copres}^n(\mathcal{X}), n \leq \infty$ and $\operatorname{Cores}_n(\mathcal{X}), \operatorname{Cores}(\mathcal{X}), {}^{\perp_I}\mathcal{X}$ respectively.
Observe that these categories depend on the exact structure $\mathcal{S}$, so if there is the possibility of confusion, we will endow them with an index $\mathcal{E}$ (or $\mathcal{S}$).

Definition 3.2. Let X be an object in $\mathcal{E}$. We say $\operatorname{pd}_{\mathcal{E}} X \leq n$ (resp. $\operatorname{id}_{\mathcal{E}} X \leq n$) if $\operatorname{Ext}^{n+1}_{\mathcal{E}}(X, -) = 0$ (resp. $\operatorname{Ext}^{n+1}_{\mathcal{E}}(-, X) = 0$).
For a subcategory $\mathcal{X}$ we define

$$\operatorname{pd}_{\mathcal{E}} \mathcal{X} := \sup\{\operatorname{pd}_{\mathcal{E}} X \mid X \in \mathcal{X}\} \quad \in \mathbb{N}_0 \cup \{\infty\}$$

and analogously $\operatorname{id}_{\mathcal{E}} \mathcal{X}$. We call $\mathcal{P}^{\leq n}$ the full subcategory of all objects X with $\operatorname{pd}_{\mathcal{E}} X \leq n$ and $\mathcal{P}^{<\infty}$ the subcategory of all objects X with $\operatorname{pd}_{\mathcal{E}} X < \infty$. The subcategories $\mathcal{I}^{\leq n}$, $\mathcal{I}^{<\infty}$ are defined dually.

Remark 3.3. We would like to know the common definition of an exact category $\mathcal{E}$ with "$\mathcal{E} = \mathcal{P}^{<\infty}$" since we use it so frequently.
If the exact category has the Jordan-Hölder property and only finitely many $\mathcal{E}$-simples, then $\operatorname{gldim} \mathcal{E} = \max_{S \text{ simple}} \operatorname{pd} S$. Therefore, in that case $\mathcal{E} = \mathcal{P}^{<\infty}$ is equivalent to $\operatorname{gldim} \mathcal{E} < \infty$.
The *finitistic dimension conjecture* holds for $\mathcal{E}$ if

$$\operatorname{gldim}(\mathcal{P}^{<\infty}) < \infty.$$

So, also in this case, $\mathcal{E} = \mathcal{P}^{<\infty}$ is equivalent to $\operatorname{gldim} \mathcal{E} < \infty$.

(As far, as we know this conjecture is open for $\mathcal{E} = \operatorname{mod}\Lambda$ where Λ is an artin algebra. But is known to fail in some other abelian categories.)

We observe the following lemma.

Lemma 3.4. *Let $\mathcal{E}$ be an exact category and $\mathcal{X}$ a full, self-orthogonal subcategory.*

(a) *Then we have* $\operatorname{Res}(\mathcal{X}) \subset \mathcal{X}^{\perp}$ *and* $\operatorname{Res}(\mathcal{X})$ *is an extension-closed subcategory.*
(b) *Let $n \geq 1$. If $\operatorname{pd}_{\mathcal{E}} \mathcal{X} \leq n$, then we have* $\operatorname{pres}_{n-1}(\mathcal{X}) \subset \mathcal{X}^{\perp}$ *and* $\operatorname{pres}_m(\mathcal{X})$ *is extension-closed for all $n-1 \leq m \leq \infty$.*

Proof. ad (a) Since $\mathcal{X}^{\perp}$ is inflation-closed it follows $\operatorname{Res}(\mathcal{X}) \subset \mathcal{X}^{\perp}$. To see that $\operatorname{Res}(\mathcal{X})$ is extension closed one can use literally proof of the horseshoe lemma (replacing the projectives with $\mathcal{X}$), cf. [**49**], Thm. 12.8.
ad (b) Given an exact sequence $0 \to Z \to X_{n-1} \to \cdots \to X_0 \to S \to 0$ with $X_i \in \mathcal{X}$, we see by dimension shift that $S \in \mathcal{X}^{\perp}$ using $\operatorname{pd}_{\mathcal{E}} \mathcal{X} \leq n$. This proves $\operatorname{pres}_{n-1}(\mathcal{X}) \subset \mathcal{X}^{\perp}$ and therefore $\operatorname{pres}_m(\mathcal{X}) \subset \mathcal{X}^{\perp}$, $n-1 \leq m \leq \infty$. Again by the horseshoe lemma as in loc. cit it follows that $\operatorname{pres}_m(\mathcal{X})$ is extension closed.

□

3.2. Thick subcategories.

Recall that a full subcategory $\mathcal{L}$ of $\mathcal{E}$ is called **thick**, if it is closed under summands and for every short exact sequence $A \to B \to C$ in $\mathcal{S}$ the following holds: If any two objects of A, B, C are in $\mathcal{L}$, then the third is also in $\mathcal{L}$.
We denote by $\operatorname{Thick}(\mathcal{X})$ the smallest thick subcategory of $\mathcal{E}$ that contains $\mathcal{X}$. By definition, we have $\operatorname{Res}(\mathcal{X}) \subset \operatorname{Thick}(\mathcal{X})$ and $\operatorname{Cores}(\mathcal{X}) \subset \operatorname{Thick}(\mathcal{X})$.

Example 3.5. The categories $\mathcal{P}^{<\infty}$ and $\mathcal{I}^{<\infty}$ are thick subcategories of $\mathcal{E}$.

Definition 3.6. Let $\mathcal{E}$ be an exact category and $\mathcal{X}$ a full subcategory. We call $\mathcal{X}$ **inflation-closed** if for every inflation $i\colon X \to Y$ with X, Y in $\mathcal{X}$ one also has $\operatorname{coker} i$ in $\mathcal{X}$. Dually, one defines **deflation-closed**.

Proposition 3.7. *([**19**], Prop. 3.5 and Prop. 3.6, Thm 1.1) Let $\mathcal{E}$ be an exact category and $\mathcal{X}$ a fully exact subcategory which has enough projectives $\mathcal{P} = \mathcal{P}(\mathcal{X})$.*
Assume $\mathcal{X}$ is an inflation-closed subcategory. Then we have

(1) $\operatorname{Thick}(\mathcal{X}) = \operatorname{Cores}(\mathcal{X})$,
(2) $\operatorname{Cores}_n(\mathcal{P}) = \operatorname{Cores}_n(\mathcal{X}) \cap {}^{\perp}\mathcal{X}$ *and therefore also* $\operatorname{Cores}(\mathcal{P}) = \operatorname{Cores}(\mathcal{X}) \cap {}^{\perp}\mathcal{X}$,
(3) *$\mathcal{X}$ is covariantly finite in* $\operatorname{Cores}(\mathcal{X})$

Example 3.8. Let $\mathcal{E}$ be an exact category. We denote by $\mathcal{P} = \mathcal{P}(\mathcal{E})$ the projectives. Then, $\mathcal{P}$ fulfills the dual conditions of Prop. 3.7 and therefore

$$\operatorname{Thick}(\mathcal{P}) = \operatorname{Res}(\mathcal{P})$$

are all objects which admit a finite projective resolution. If $\mathcal{E}$ has enough projectives, then $\mathcal{P}^{<\infty} = \operatorname{Res}(\mathcal{P})$.

3.2.1. *Thick subcategories in triangulated categories.* Let $\mathcal{C}$ be a triangulated category. We call a full subcategory **thick** if it is closed under summands and a triangulated subcategory.
Let $\mathcal{X}$ be an additive subcategory of $\mathcal{C}$, we write $\operatorname{Thick}_{\Delta}(\mathcal{X})$ for the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{X}$.

3.3. Resolving subcategories.

Definition 3.9. Let $\mathcal{E}$ be an exact category. Let $\mathcal{X}$ be a full subcategory.
We say $\mathcal{X}$ is **resolving** if it is extension closed, deflation-closed and $\operatorname{pres}(\mathcal{X}) = \mathcal{E}$.
We say $\mathcal{X}$ is **coresolving** if it is extension closed, inflation-closed and $\operatorname{copres}(\mathcal{X}) = \mathcal{E}$.
We say $\mathcal{X}$ is

(*) **finitely resolving** if it is resolving and $\operatorname{Res}(\mathcal{X}) = \mathcal{E}$,
(*) $n-$**resolving** if it is resolving and $\operatorname{Res}_n(\mathcal{X}) = \mathcal{E}$,
(*) **uniformly finitely resolving** if it is n-resolving for some $n \geq 0$.

Dually one defines finitely coresolving, n-coresolving and uniformly finitely coresolving.

Example 3.10. Let $\mathcal{E}$ be an exact category with enough projectives and $\mathcal{X}$ be any additive subcategory. Then ${}^{\perp}\mathcal{X}$ is a resolving subcategory. Dually, if $\mathcal{E}$ has enough injectives and $\mathcal{X}$ is an additive subcategory, then $\mathcal{X}^{\perp}$ is a coresolving subcategory.

3.4. Functor categories. Let $\mathcal{X}$ be an additive category. If $\mathcal{X}$ is essentially small we can define the category $\operatorname{Mod}\mathcal{X}$ of right $\mathcal{X}$-modules as the category of contravariant additive functors on $\mathcal{X}$ to abelian groups (if $\mathcal{X}$ is not essentially small the class of all natural transformations between two $\mathcal{X}$-modules is not necessarily a set). This is an abelian category with enough projectives and injectives. The projectives are summands of arbitrary direct sums of representable functors $\operatorname{Hom}_{\mathcal{X}}(-, X)$ for some $X \in \mathcal{X}$ which we call the *finitely generated projectives.* Similarly, one can define the category $\mathcal{X}\operatorname{Mod} := \operatorname{Mod}\mathcal{X}^{op}$ of left $\mathcal{X}$-modules. We define full subcategories

$$\operatorname{mod}_{\infty}\mathcal{X} \subset \operatorname{mod}_n -\mathcal{X} \subset \operatorname{Mod}\mathcal{X}, \quad n \in \mathbb{N}_0$$

via $F \in \operatorname{mod}_n -\mathcal{X}$ if there is a projective n-presentation (indexed by 0 up to n) by finitely generated projectives, i.e. there is an exact sequence

$$\operatorname{Hom}_{\mathcal{X}}(-, X_n) \to \cdots \to \operatorname{Hom}_{\mathcal{X}}(-, X_0) \to F \to 0,$$

we define $\operatorname{mod}_{\infty} -\mathcal{X}$ analogue with an infinite sequence as above and we call $\operatorname{mod}\mathcal{X} := \operatorname{mod}_0 -\mathcal{X}$ the *finitely generated* $\mathcal{X}$-modules, $\operatorname{mod}_1\mathcal{X}$ as the *finitely presented* $\mathcal{X}$-modules. All these are fully exact subcategories of $\operatorname{Mod}\mathcal{X}$.

Definition 3.11. Let $f\colon X_1 \to X_0$ be a morphism in $\mathcal{X}$, we say that f **admits a weak kernel** if there exists a morphism $g\colon X_2 \to X_1$ such that

$$\operatorname{Hom}_{\mathcal{X}}(-, X_2) \xrightarrow{g\circ} \operatorname{Hom}_{\mathcal{X}}(-, X_1) \xrightarrow{f\circ} \operatorname{Hom}_{\mathcal{X}}(-, X_0)$$

is exact in $\operatorname{Mod}\mathcal{X}$. We say that $\mathcal{X}$ **admits weak kernels** if every morphism in $\mathcal{X}$ admits one.

THEOREM 3.12. *([16], Prop. 2.1) The category* $\operatorname{mod}_1\mathcal{X}$ *equals* $\operatorname{mod}_{\infty}\mathcal{X}$ *if and only if* $\mathcal{X}$ *admits weak kernels. In this case* $\operatorname{mod}_{\infty}\mathcal{X}$ *is abelian.*

Remark 3.13. It is common practice to ignore the assumption $\mathcal{X}$ *essentially small* because as soon as one looks at $\operatorname{mod}_1\mathcal{X}$ the class of all natural transformations between two finitely presented $\mathcal{X}$-modules is a set.

3.5. Embedding exact categories with enough projectives into a functor category. If $\mathcal{E}$ is an exact category with enough projectives $\mathcal{P}$, then we the Yoneda embedding induces by composition with the restriction-on-$\mathcal{P}$ functor a functor

$$\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_{\infty}\mathcal{P}, \quad E \mapsto \operatorname{Hom}_{\mathcal{E}}(-, E)|_{\mathcal{P}}$$

We write $\operatorname{Im}\mathbb{P}$ for the essential image. For us, the main observation is the following:

Proposition 3.14. *([73] Prop. 2.2.1, Prop. 2.2.8) The functor* $\mathbb{P}$ *is fully faithful and induces isomorphisms on all higher extension groups,* $\operatorname{Im}\mathbb{P}$ *is extension closed.*
If $\mathcal{E}$ *is idempotent complete, then* $\operatorname{Im}\mathbb{P}$ *is a resolving subcategory of* $\operatorname{mod}_{\infty}\mathcal{P}$*.*

We observe also the following obvious lemma.

Lemma 3.15. *Given two exact categories* $\mathcal{E}$ *and* $\mathcal{E}'$ *with enough projectives* $\mathcal{P}$ *and* $\mathcal{P}'$ *respectively. Then the following are equivalent:*

(1) $\mathcal{E}$ *and* $\mathcal{E}'$ *are equivalent as exact categories.*

(2) *There is an equivalence* $\Psi\colon \mathcal{P} \to \mathcal{P}'$ *of additive categories with the property: A morphism* f *in* $\mathcal{P}$ *is admissible in* $\mathcal{E}$ *if and only if* $\Psi(f)$ *in* $\mathcal{P}'$ *is admissible in* $\mathcal{E}'$.
(3) *There is an additive equivalence* $\Phi\colon \mathcal{E} \to \mathcal{E}'$ *with* $\Phi(\mathcal{P}) = \mathcal{P}'$.

Proof. (2) implies (1): Given Ψ as in (2). We extend Ψ to an equivalence $\Psi\colon \mathcal{E} \to \mathcal{E}'$. For an (in $\mathcal{E}$) admissible $f\colon P_1 \to P_0$ we map $\operatorname{coker}(f)$ to $\operatorname{coker}\Psi(f)$ (and then check that this is well-defined and gives an equivalence of categories).
Conversely, if $\Psi\colon \mathcal{E} \to \mathcal{E}'$ is an equivalence of exact categories it maps admissible morphsims to admissible morphisms, projectives to projectives. Then it restricts to an equivalence as in (2).
(3) implies (1): An additive equivalence preserves kernel-cokernel pairs. Kernel-cokernel pairs are short exact sequences in $\mathcal{E}$ (resp. $\mathcal{E}'$) if and only if $\operatorname{Hom}(P,-)$ are exact on them for all $P \in \mathcal{P}$ (resp. $\mathcal{P}'$). Therefore Φ is an equivalence of exact categories. □

Corollary 3.16. *Let* $\mathcal{E}$ *be an exact category with enough projectives* $\mathcal{P}$. *The following are equivalent:*

(a) $\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_\infty \mathcal{P}$ *is an equivalence.*
(b) *The* $\mathcal{E}$*-admissible morphisms in* $\mathcal{P}$ *are precisely those such that* $\ker \operatorname{Hom}_{\mathcal{P}}(-,h)(\in \operatorname{Mod}-\mathcal{P})$ *lies in* $\operatorname{mod}_\infty \mathcal{P}$.

Proof. Clearly, we have (b) implies (a) and (a) implies (b) by the previous Lemma. □

Lemma 3.17. *Let* $\mathcal{E}$ *be an exact category with enough projectives* $\mathcal{P}$. *Then the following are equivalent:*

(1) $\operatorname{Im}\mathbb{P}$ *is finitely resolving in* $\operatorname{mod}_\infty \mathcal{P}$.
(2) *For every morphism* $f\colon P_1 \to P_0$ *in* $\mathcal{P}$ *admitting an infinite sequence of successive weak kernels in* $\mathcal{P}$ *(i.e.* $\operatorname{Hom}_{\mathcal{P}}(-,f)$ *is part of a projective resolution by finitely generated projectives of an object in* $\operatorname{mod}_\infty \mathcal{P}$*) there exists a complex in* $\mathcal{E}$

$$0 \to M_n \to M_{n-1} \to \cdots \to M_2 \to P_1 \xrightarrow{f} P_0$$

depending on f *such that*

$$0 \to \mathbb{P}(M_n) \to \mathbb{P}(M_{n-1}) \to \cdots \to \mathbb{P}(M_2) \to \mathbb{P}(P_1) \to \mathbb{P}(P_0)$$

is exact in $\operatorname{mod}_\infty \mathcal{P}$.

Proof. (2) implies (1): If $M \in \operatorname{mod}_\infty \mathcal{P}$, we choose $f\colon P_1 \to P_0$ admitting a weak kernel such that $M \cong \operatorname{coker}\mathbb{P}(f)$. Then choose the complex as in (2), to obtain that $M \in \operatorname{Res}(\operatorname{Im}\mathbb{P})$, and therefore $\operatorname{Im}\mathbb{P}$ is finitely resolving. Conversely, let $f\colon P_0 \to P_1$ be a morphism as in (2). This implies $Z = \ker \operatorname{Hom}_{\mathcal{P}}(-,f) \in \operatorname{mod}_\infty \mathcal{P}$. Assume (1), i.e. $\operatorname{Res}(\operatorname{Im}\mathbb{P}) = \operatorname{mod}_\infty \mathcal{P}$. This means there exists an exact sequence

$$0 \to \mathbb{P}(M_n) \to \cdots \to \mathbb{P}(M_2) \to Z \to 0$$

in $\operatorname{mod}_\infty \mathcal{P}$ with $M_i \in \mathcal{E}$. Since $\mathbb{P}$ is fully faithful, the claim follows. □

4. The derived category of an exact category

We recall from Buehler [49], section 9: Let $\mathcal{A}$ be an idempotent complete additive category. One can define the category of chain complexes $\operatorname{Ch}(\mathcal{A})$ and the homotopy category $\operatorname{K}(\mathcal{A})$, whose objects equal those if $\operatorname{Ch}(\mathcal{A})$ but the morphisms are the quotient group given by chain maps modulo chain maps homotopic to the zero map. $\operatorname{K}(\mathcal{A})$ has the structure of a triangulated category induced by strict triangles in $\operatorname{Ch}(\mathcal{A})$. A chain complex

$$A = (\cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} \cdots) \quad \in \operatorname{Ch}(\mathcal{A})$$

is called **left bounded** if $A^n = 0$ for $n << 0$, **right bounded** if $A^n = 0$ for $n >> 0$ and **bounded** if $A^n = 0$ for $|n| >> 0$. We denote by $\operatorname{K}^+(\mathcal{A}), \operatorname{K}^-(\mathcal{A})$ and $\operatorname{K}^b(\mathcal{A})$ the full subcategories of $\operatorname{K}(\mathcal{A})$ whose objects are the left bounded, right bounded and bounded chain complexes respectively. By definition $K^b(\mathcal{A}) = K^+(\mathcal{A}) \cap K^-(\mathcal{A})$. The subcategories $K^*(\mathcal{A})$ with $* \in \{+,-,b\}$ are triangulated subcategories but not closed under isomorphism in $K(\mathcal{A})$ unless $\mathcal{A} = 0$.

Now, let us assume that $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ is an exact category. A chain complex $A = (A^\bullet, d^\bullet)$ in $\mathrm{Ch}(\mathcal{A})$ is called **acyclic** if every differential factors as $d^n \colon A^n \to Z^{n+1}A \to A^{n+1}$ such that $Z^nA \to A^n \to Z^{n+1}A$ is an exact sequence (i.e. in $\mathcal{S}$).

Neeman proved that the mapping cone of a chain map between acyclic complexes is acyclic. This implies that the full subcategory $\mathrm{Ac}(\mathcal{E}) = \mathrm{Ac}_{\mathcal{S}}(\mathcal{A})$ of $K(\mathcal{A})$ given by acyclic complexes is a triangulated subcategory of $K(\mathcal{A})$. If $\mathcal{A}$ is idempotent complete then $\mathrm{Ac}(\mathcal{E})$ is closed under isomorphism, every null-homotopic chain complex is acyclic and $\mathrm{Ac}(\mathcal{E})$ is a thick subcategory of $K(\mathcal{A})$. We define $\mathrm{Ac}^*(\mathcal{E}) := \mathrm{Ac}(\mathcal{E}) \cap K^*(\mathcal{A})$ for $* \in \{+,-,b\}$. If $\mathcal{A}$ is weakly idempotent complete then the categories $\mathrm{Ac}^*(\mathcal{E})$ are thick subcategories of $K^*(\mathcal{A})$ for $* \in \{+,-,b\}$.

Given any triangulated category $\mathcal{C}$ and thick subcategory $\mathcal{T}$ the **Verdier quotient**

$$\mathcal{C}/\mathcal{T}$$

is defined via a localization (cf. [126], section 3.2, Prop. 3.2.2). It is again a triangulated category and the canonical functor $\mathcal{C} \to \mathcal{C}/\mathcal{T}$ is an exact functor.

As explained in [49], since $\mathcal{A}$ is idempotent complete, $\mathrm{Ac}^*(\mathcal{E})$ is a thick subcategory of $K^*(\mathcal{A})$ for $* \in \{\pm, b, \emptyset\}$. The **derived category** of an exact category $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ is defined as the Verdier quotient

$$D(\mathcal{E}) := K(\mathcal{A})/\mathrm{Ac}(\mathcal{E})$$

Similarly, we define the **bounded/left bounded derived category** as the Verdier quotients

$$D^*(\mathcal{E}) := K^*(\mathcal{A})/\mathrm{Ac}^*(\mathcal{E}), \quad * \in \{b, \pm\}$$

For more details and basic properties we refer to [126], Chapter 4 and [49], section 10.

Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. We define $K^{b,-}(\mathcal{P})$ as the full subcategory of $K^-(\mathcal{P})$ given by all complexes X such that there exists an $n \geq 1$ such that d^{n-1} and d^{-n} are admissible and the truncated complexes

$\tau_{\leq(-n)}X = (\cdots \to X^{(-n)-1} \to X^{(-n)} \to \ker d^{(-n)} \to 0 \cdots)$,

$\tau_{\geq n}X = (\cdots 0 \to \operatorname{coker} d^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots)$

are acyclic, i.e. in $\mathrm{Ac}(\mathcal{E}) \cap K^-(\mathcal{P})$. Observe, that this depend on the ambient exact category $\mathcal{E}$, even though the notation suggests otherwise.

Lemma 4.1. *([126], Cor. 4.2.9) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then, there are triangle equivalences*

(1) $K^-(\mathcal{P}) \to D^-(\mathcal{E})$
(2) $K^{b,-}(\mathcal{P}) \to D^b(\mathcal{E})$

Definition 4.2. Let $\mathcal{E}$ be an exact category and $\mathcal{X}$ a fully exact subcategory. We write $D^*(\mathcal{X})$, $* \in \{b,+,-\}$ for its derived categories. There always exists an exact functor $D^*(\mathcal{X}) \to D^*(\mathcal{E})$ induced by the inclusion $\mathcal{X} \to \mathcal{E}$ (because this is an exact functor).

THEOREM 4.3. [92] *If $\mathcal{E}$ is an exact category and $\mathcal{X}$ is a resolving or coresolving subcategory.*

(i) *If $\mathcal{X}$ is resolving, the inclusion $\mathcal{X} \to \mathcal{E}$ induces a triangle equivalence*

$$D^-(\mathcal{X}) \to D^-(\mathcal{E})$$

if $\mathcal{X}$ is coresolving on $D^+()$.

(ii) *If $\mathcal{X}$ is finitely resolving or finitely coresolving, the inclusion induces a triangle equivalence*

$$D^b(\mathcal{X}) \to D^b(\mathcal{E}).$$

(iii) *If $\mathcal{X}$ is uniformley finitely resolving or uniformly finitely coresolving, it induces a triangle equivalence*

$$D(\mathcal{X}) \to D(\mathcal{E}).$$

Remark 4.4. If $\mathcal{E}$ is an exact category with enough projectives and $\mathcal{X}$ a resolving subcategory, then the exact functor $\mathcal{X} \to \mathcal{E}$ induces a commuting diagram of exact functors

$$\begin{array}{ccc} \mathrm{D}^-(\mathcal{X}) & \longrightarrow & \mathrm{D}^-(\mathcal{E}) \\ \uparrow & & \uparrow \\ \mathrm{D}^b(\mathcal{X}) & \longrightarrow & \mathrm{D}^b(\mathcal{E}) \end{array}$$

where the vertical arrows are the induced triangle functors by construction. By Lemma 4.1 they are fully faithful. By Theorem 4.3,(i), the upper horizontal functor is a triangle equivalence. This implies that the lower horizontal functor is fully faithful, too.

5. Tilting theory in exact categories

We define tilting subcategories and go through the list of results (cf. introduction) that we demand for tilting theory (i.e. induced derived equivalence, Aulsander correspondence, Bazzoni-like result and Brenner-Butler theorem).

5.1. Tilting subcategories.

Definition 5.1. Let $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ be an exact category and $\mathcal{T} \subset \mathcal{E}$ a full subcategory. We call $\mathcal{T}$ a **tilting** subcategory if

(T1) $\mathcal{T}$ is self-orthogonal and $\mathcal{T}^\perp$ has enough projectives which are given by $\mathcal{P}(\mathcal{T}^\perp) = \mathcal{T}$ (equivalently: $\mathcal{T}$ is closed under taking summands in $\mathcal{E}$ and $\mathcal{T} \subset \mathcal{T}^\perp \subset \mathrm{gen}(\mathcal{T})$)
(T2) $\mathcal{T}^\perp$ is finitely coresolving, i.e. it is coresolving and $\mathrm{Cores}(\mathcal{T}^\perp) = \mathcal{E}$.

By Prop. 3.7, if we assume (T1), we can replace (T2) by an equivalent:

(T2') $\mathrm{Thick}(\mathcal{T}^\perp) = \mathcal{E}$.

We call $\mathcal{T}$ n-**tilting** if it is tilting with $\mathrm{Cores}_n(\mathcal{T}^\perp) = \mathcal{E}$. We call an object T in $\mathcal{E}$ **tilting** if $\mathrm{add}(T)$ is a tilting subcategory. Assumption (T1) implies $\mathcal{T}$ and $\mathcal{T}^\perp$ are closed under summands and we have a well-defined fully faithful exact functor into infinitely presented $\mathcal{T}$-modules $f_{\mathcal{T}} \colon \mathcal{T}^\perp \to \mathrm{mod}_\infty \mathcal{T}$, $X \mapsto \mathrm{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{T}}$ since $\mathcal{T} = \mathcal{P}(\mathcal{T}^\perp)$ and $\mathcal{T}^\perp$ has enough projectives. Assumption (T2) implies that the inclusion $\mathcal{T}^\perp \subset \mathcal{E}$ induces a triangle equivalence $\mathrm{D}^b(\mathcal{T}^\perp) \to \mathrm{D}^b(\mathcal{E})$. Assumption (T1) and (T2) imply that the inclusion $\mathcal{T} \to \mathcal{E}$ gives rise to a triangle equivalence

$$\mathrm{K}^{b,-}(\mathcal{T}) \cong \mathrm{D}^b(\mathcal{T}^\perp) \to \mathrm{D}^b(\mathcal{E}).$$

Here is another way to express (T2) but we will need the assumption that the exact category has enough injectives.

Lemma 5.2. *Let $\mathcal{T}$ be an additive subcategory in an exact category $\mathcal{E}$. Then:* $\mathrm{Cores}(\mathcal{T}^\perp) = \mathcal{E}$ *(resp.* $\mathrm{Cores}_n(\mathcal{T}^\perp) = \mathcal{E}$*) implies* $\mathcal{E} = \bigcup_{n \geq 0} \mathcal{T}^{\perp_{\geq n+1}}$ *(resp.* $\mathrm{pd}_{\mathcal{E}} \mathcal{T} \leq n$*).*
If $\mathcal{E}$ has enough injectives , then the converse is true.

Proof. Observe that $A \in \mathrm{Cores}_n(\mathcal{T}^\perp)$ implies $\mathrm{Ext}^i(T, A) = 0, i \geq n+1$ for every $T \in \mathcal{T}$, i.e. $A \in \mathcal{T}^{\perp_{\geq n+1}}$. If $\mathcal{E}$ has enough injectives and $A \in \mathcal{T}^{\perp_{\geq n+1}}$, then we have by dimension shift $\Omega^{-n} A \in \mathcal{T}^\perp$. So, the injective coresolution of A truncated at the n-th cosyzygy shows that $A \in \mathrm{Cores}_n(\mathcal{T}^\perp)$. □

Corollary 5.3. *Assume that $\mathcal{T}$ is a tilting subcategory of an exact category $\mathcal{E}$ and $n \geq 0$. Then the following are equivalent:*

(1) *$\mathcal{T}$ is n-tilting*

(2) $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} \leq n$

Proof. By Lemma 5.2, we have the implication (1) implies (2). Assume (2), we claim: $\mathrm{Cores}(\mathcal{T}^{\perp}) \subset \mathrm{Cores}_n(\mathcal{T}^{\perp})$. Let $A \in \mathrm{Cores}(\mathcal{T}^{\perp})$, then there exists an exact sequence

$$0 \to A \to X_0 \to \cdots \to X_s \to 0, \quad X_i \in \mathcal{T}^{\perp}, 0 \leq i \leq s$$

Wlog. assume that $s > n-1$. Let $Z_i = \mathrm{Im}(X_i \to X_{i+1}), Z_{-1} := A$. Applying $\mathrm{Hom}(T,-)$, $T \in \mathcal{T}$, gives by dimension shift $\mathrm{Ext}^j_{\mathcal{E}}(T, Z_{n-1}) \cong \mathrm{Ext}^{j+1}_{\mathcal{E}}(T, Z_{n-2}) \cong \cdots \cong \mathrm{Ext}^{j+n}_{\mathcal{E}}(T, A) = 0$ for all $j \geq 1$ since $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} \leq n$. So, $Z_{n-1} \in \mathcal{T}^{\perp}$. □

Lemma 5.4. *Let $\mathcal{T}$ be an additive category of an exact category $\mathcal{E}$. If $\mathcal{T}$ fulfills (T1) and $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} \leq n$, then we have*

$$\mathcal{T}^{\perp} = \mathrm{gen}_{n-1}(\mathcal{T}) = \mathrm{pres}_{n-1}(\mathcal{T}) = \mathrm{gen}_{\infty}(\mathcal{T}) = \mathrm{pres}_{\infty}(\mathcal{T}).$$

Proof. The assumption $\mathcal{T}^{\perp_{\geq n+1}} = \mathcal{E}$ together with $\mathcal{T}$ self-orthogonal implies by an easy dimension shift argument that $\mathrm{pres}_{n-1}(\mathcal{T}) \subset \mathcal{T}^{\perp}$. Now, by assumption (T1) we have $\mathcal{T}^{\perp} \subset \mathrm{gen}_{\infty}(\mathcal{T})$ since $\mathcal{T} = \mathcal{P}(\mathcal{T}^{\perp})$ and $\mathcal{T}^{\perp}$ has enough projectives. Since $\mathrm{gen}_{\infty}(\mathcal{T}) \subset \mathrm{pres}_{n-1}(\mathcal{T})$ trivially and $\mathrm{gen}_{n-1}(\mathcal{T})$ and $\mathrm{pres}_{\infty}(\mathcal{T})$ are intermediate between these two, the claim follows. □

For infinitely generated tilting modules, the perpendicular category of a tilting module of finite projective dimension has been described in [**35**], Theorem 3.11. In [**182**], Theorem 1.1, this description has been proven for relative tilting modules over artin algebras. Therefore, we call the generalization to tilting subcategories in exact categories *Bazzoni's description.*

Corollary 5.5. *(*Bazzoni's description*) Let $\mathcal{T}$ be an additive category of an exact category $\mathcal{E}$ which is closed under taking summands and assume $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} \leq n$, then (T1) is equivalent to the following:*

$$\text{(T1')} \quad \mathcal{T} \subset \mathcal{T}^{\perp} \subset \mathrm{pres}_n(\mathcal{T})$$

and also to

$$\text{(T1'')} \quad \mathcal{T} \subset \mathcal{T}^{\perp} \subset \mathrm{add}(\mathrm{pres}_n(\mathcal{T}))$$

where for a full subcategory $\mathcal{M}$ of $\mathcal{E}$: $\mathrm{add}(\mathcal{M})$ denotes the full subcategory of all summands of finite direct sums of objects in $\mathcal{M}$.
Furthermore, in this case we have $\mathrm{pres}_{n-1}(\mathcal{T}) = \mathrm{pres}_n(\mathcal{T}) = \mathcal{T}^{\perp}$.

Proof. The implication "(T1) implies (T1')" is given by Lemma 5.4 using the inclusions $\mathrm{pres}_{\infty}(\mathcal{T}) \subset \mathrm{pres}_n(\mathcal{T}) \subset \mathrm{pres}_{n-1}(\mathcal{T})$. Also loc. cit. proves the statement $\mathrm{pres}_{n-1}(\mathcal{T}) = \mathrm{pres}_n(\mathcal{T})$. The implication "(T1') implies (T1")" is trivial since $\mathcal{T}^{\perp} = \mathrm{add}(\mathcal{T}^{\perp})$.
Assume (T1"), i.e. $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} \leq n$ and $\mathcal{T} \subset \mathcal{T}^{\perp} \subset \mathrm{add}(\mathrm{pres}_n(\mathcal{T}))$. To see (T1), it is enough to prove $\mathcal{T}^{\perp} \subset \mathrm{gen}(\mathcal{T})$. Observe that $\mathrm{pres}_{n-1}(\mathcal{T}) \subset \mathcal{T}^{\perp}$ is fulfilled by Lemma 3.4, (b). Let $X \in \mathcal{T}^{\perp}$, then there exists an $L \in \mathcal{E}$ such that $X \oplus L \in \mathrm{pres}_n(\mathcal{T})$ - in particular $L \in \mathcal{T}^{\perp}$ and there is an exact sequence

$$0 \to X_{n+1} \to T_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} T_0 \to X \oplus L \to 0$$

with $T_i \in \mathcal{T}$. Let $X_i := \mathrm{Im}(f_i), X \oplus L := X_0$. For every $j \geq 1, T \in \mathcal{T}$ we have $\mathrm{Ext}^j(T, X_i) \cong \mathrm{Ext}^{j+1}(T, X_{i+1})$. This implies $X_1 \in \mathcal{T}^{\perp}$ since $\mathrm{Ext}^j(T, X_1) \cong \mathrm{Ext}^{n+j}(T, X_{n+1}) = 0$ by assumption. Therefore $T_0 \to X \oplus L$ is a deflation in $\mathcal{T}^{\perp}$. This implies that the composition with the projection onto the summand $T_0 \to X \oplus L \to X$ is a deflation in $\mathcal{T}^{\perp}$ - (since projections onto summands are deflation in every exact category and the composition of two deflations is a deflation). Therefore, (T1) is fulfilled. □

Example 5.6.
(1) If $\mathcal{E}$ is an exact category with enough projectives $\mathcal{P}$, then $\mathcal{P}$ is a tilting subcategory. In fact, this is the only 0-tilting subcategory.
(2) If $\mathcal{T}$ is a subcategory of an exact category $\mathcal{E}$ which fulfills (T1), then $\mathcal{E}' = \mathrm{Cores}(\mathcal{T}^{\perp}) = \mathrm{Thick}(\mathcal{T}^{\perp})$ is a thick subcategory and $\mathcal{T}$ is a tilting subcategory of $\mathcal{E}'$.

(3) A counterexample: Let $\mathcal{T}$ be a skelletally small additive category. Let $\mathcal{E} = \operatorname{mod}_n -\mathcal{T}$ be the category of n-finitely presented $\mathcal{T}$-modules (i.e. for $n=0$ finitely generated, $n=1$ finitely presented, etc.). Assume that $\mathcal{E}$ does not have enough projectives. $\mathcal{E}$ is a fully exact category of the abelian category $\operatorname{Mod}\mathcal{T}$ which contains the finitely generated projectives $\mathcal{P}_\mathcal{T} := \{\operatorname{Hom}_\mathcal{T}(-,T) \mid T \in \mathcal{T}\}$. We have $\mathcal{P}_\mathcal{T}^\perp = \mathcal{E}$ (so (T2) is fulfilled) but since $\mathcal{E}$ does not contain enough projectives, (T1) is not fulfilled and it is not a tilting subcategory.

So, in our definition if you have a ring R which is not left coherent, i.e. $\mathcal{E} = R\operatorname{mod}_1$ is not abelian, then R is not a tiliting module in $\mathcal{E}$. But R is always a tilting module in $R\operatorname{mod}_\infty$ by example (1).

Lemma 5.7. *Let $\mathcal{E}$ be an exact category and $\mathcal{T}$ an n-tilting subcategory. Then, all objects in $\mathcal{T} \subset \mathcal{P}^{<\infty}$ (i.e. all objects in $\mathcal{T}$ have finite projective dimension) and the following are equivalent:*

(1) $\mathcal{E} = \mathcal{P}^{<\infty}$
(2) $\mathcal{T}^\perp \subset \mathcal{P}^{<\infty}$
(3) $\mathcal{T}^\perp = \operatorname{Res}(\mathcal{T})$
(4) $\mathrm{K}^b(\mathcal{T}) = \mathrm{D}^b(\mathcal{T}^\perp)$
(5) $\mathcal{E} = \operatorname{Thick}(\mathcal{T})$
(6) $\mathrm{K}^b(\mathcal{T}) = \mathrm{D}^b(\mathcal{E})$

Proof. Since $\operatorname{pd}_\mathcal{E} T \leq n$ for all $T \in \mathcal{T}$, $\mathcal{T}$ consists of ojects with finite projective dimension. This implies $\operatorname{Thick}(\mathcal{T}) \subset \mathcal{P}^{<\infty}$. Therefore (5) implies (1) trivially. Now assume (1). Then it follows $\mathcal{T}^\perp = \operatorname{Res}(\mathcal{T}) = \operatorname{Thick}_{\mathcal{T}^\perp}(\mathcal{T})$. This implies by (T2') that

$$\mathcal{E} = \operatorname{Thick}(\mathcal{T}^\perp) = \operatorname{Thick}(\operatorname{Thick}_{\mathcal{T}^\perp}(\mathcal{T})) \subset \operatorname{Thick}(\mathcal{T}).$$

This means (1) and (5) are equivalent. The equivalences of (5) and (6) and the one of (3) and (4) follow from [**126**], Lemma 7.1.2 . The equivalence of (4) and (6) follows from the definition of a tilting subcategory. The implications (1) implies (2), (2) implies (3) are clear. □

Out of curiosity, we also prove the following more general statement:

Lemma 5.8. *If $\mathcal{E}$ is an exact category and $\mathcal{T}$ is an n-tilting subcategory, then*

$$\operatorname{Thick}(\mathcal{T}) = \mathcal{P}^{<\infty}$$

In particular, $\mathcal{T}$ is also an n-tilting subcategory of $\mathcal{P}^{<\infty}$.

Proof. Since $\mathcal{T}$ is n-tilting, we have $\mathcal{T} \subset \mathcal{P}^{<\infty}$. Since $\mathcal{P}^{<\infty}$ is a thick subcategory of $\mathcal{E}$, it follows that $\operatorname{Thick}(\mathcal{T}) \subset \mathcal{P}^{<\infty}$.
Now, let $X \in \mathcal{P}^{<\infty}$. There exists an exact sequence

$$0 \to X \to X_0 \to \cdot \to X_n \to 0$$

with $X_i \in \mathcal{T}^\perp$ by the axiom (T2). By Lemma 5.4 we have $\mathcal{T}^\perp = \operatorname{pres}_\infty(\mathcal{T})$, so we can apply Lemma 6.5 to find a short exact sequence $0 \to Z \to L \to X \to 0$ with $Z \in \mathcal{T}^\perp$ and $L \in \operatorname{Cores}(\mathcal{T})$. This implies $L \in \operatorname{Thick}(\mathcal{T}) \subset \mathcal{P}^{<\infty}$ and $Z \in \mathcal{P}^{<\infty}$. Now clearly, $\mathcal{T}^\perp \cap \mathcal{P}^{<\infty} = \mathcal{P}^{<\infty}(\mathcal{T}^\perp) \subset \operatorname{Thick}(\mathcal{T})$ since $\mathcal{T}^\perp$ has enough projectives given by $\mathcal{T}$. This implies $Z \in \operatorname{Thick}(\mathcal{T})$ and therefore also $X \in \operatorname{Thick}(\mathcal{T})$. Since $\mathcal{T}^\perp \cap \mathcal{P}^{<\infty}$ has still enough projectives given by $\mathcal{T}$ we have (T1) trivially true. Secondly, since $\operatorname{Thick}_\mathcal{E}(\mathcal{T}) = \mathcal{P}^{<\infty}$, we have $\operatorname{Thick}_{\mathcal{P}^{<\infty}}(\mathcal{T}) = \mathcal{P}^{<\infty}$ and therefore (T2)' holds. □

Open question 5.9. Using the later theorem 6.3 we can conclude that for exact categories with enough projectives n-tilting subcategories in $\mathcal{E}$ are precisely n-tilting subcategories in $\mathcal{P}^{<\infty}$.
Is this true for all exact categories?

5.1.1. *The Auslander correspondence.* Our definition has this trivial version of the Auslander correspondence as a consequence:

THEOREM 5.10. *(trivial Auslander correspondence) Let $\mathcal{E}$ be an exact category. The assignments $\mathcal{T} \mapsto \mathcal{T}^{\perp}$ and $\mathcal{X} \mapsto \mathcal{P}(\mathcal{X})$ are inverse bijections between*

(1) *the class of $\mathcal{T}$ (n-)tilting subcategory of $\mathcal{E}$*
(2) *the class of $\mathcal{X}$ finitely coresolving subcategory (with $\mathrm{Cores}_n(\mathcal{X}) = \mathcal{E}$) which have enough projectives $\mathcal{P} = \mathcal{P}(\mathcal{X})$ and which are of the form $\mathcal{X} = \mathcal{P}^{\perp}$*

Nevertheless, condition (2) can be reformulated in a more elegant way once we assume extra conditions on the exact category. This are the occurrences in the literature:

(*) [**25**], Theorem 5.5, original version for tilting modules in $\mathrm{mod}\,\Lambda$ for an Artin algebra Λ.
(*) [**126**], Theorem 7.2.18 in the case of a strongly homologically finite exact category $\mathcal{E}$ with enough projectives which are of the form $\mathcal{P} = \mathrm{add}(P)$ for one object P.
(*) [**185**], Theorem 4.15, for an extriangulated category with enough projectives and enough injectives.

Here, we use that the different definitions of tilting are special cases of our definition (cf. section 5: Definitions of tilting). For example, the latest version of the Auslander correspondence is the following:

THEOREM 5.11. *(Auslander correspondence) ([**185**], Theorem 4.15) Let $\mathcal{E}$ be an exact category with enough projectives and enough injectives. The assignments $\mathcal{T} \mapsto \mathcal{T}^{\perp}$ and $\mathcal{X} \mapsto {}^{\perp}\mathcal{X} \cap \mathcal{X}$ are inverse bijections between*

(1) *the class of $\mathcal{T}$ (n-)tilting subcategory of $\mathcal{E}$*
(2) *the class of $\mathcal{X}$ finitely coresolving subcategory (with $\mathrm{Cores}_n(\mathcal{X}) = \mathcal{E}$) which are covariantly finite and closed under summands.*

Open question 5.12. Can the previous result be proven only with the assumption that the exact category has enough projectives?

6. Definitions of tilting

In this section we look at our definition in exact categories with extra assumptions and compare this to existing definitions of tilting. Also, we discuss the relationship to tilting in triangulated subcategories in subsection 6.1.

Proposition 6.1. *If $\mathcal{E}$ is an exact category with enough injectives and $\mathcal{T}$ a full subcategory then $\mathcal{T}$ is n-tilting if and only if it fulfills the following two conditions:*

(i) $\mathrm{pd}_{\mathcal{E}}\, \mathcal{T} \leq n$
(ii) $\mathcal{T} \subset \mathcal{T}^{\perp} \subset \mathrm{pres}_n(\mathcal{T})$

Furthermore, in this case we have $\mathcal{P}(\mathcal{E}) \subset \mathrm{Cores}_n(\mathcal{T})$.

PROOF. Follows from Lem. 5.5, Lem. 5.2 and from Prop. 3.7. ☐

Remark 6.2. This recovers the definition of n-tilting subcategories given in [**185**] (for exact categories with enough projectives and injectives).

THEOREM 6.3. *(and definition) Let $\mathcal{E}$ be an exact category with enough projectives and let $\mathcal{T}$ be a full additive subcategory. We define:*

(t1) *$\mathcal{T}$ is self-orthogonal and closed under summands.*

(t2) $\mathrm{pd}_{\mathcal{E}}\, \mathcal{T} \leq n$
(t3) $\mathcal{P}(\mathcal{E}) \subset \mathrm{Cores}_n(\mathcal{T})$

Then: $\mathcal{T}$ is n-tilting if and only if $\mathcal{T}$ fulfills (t1), (t2), (t3).

Proof. Assume $\mathcal{T}$ is n-tilting. (T1) implies (t1), (t2) follows from Lemma 5.2, (2), (a) and (t3) from Prop. 3.7, (2) with $\mathcal{X} = \mathcal{T}^{\perp}$.
Conversely, assume $\mathcal{T}$ fulfills (t1), (t2), (t3). It is straight forward to see that $\mathrm{pres}_n(\mathcal{T}) \subset \mathrm{pres}_{n-1}(\mathcal{T}) \subset \mathcal{T}^{\perp}$ by dimension shift using (t1) and (t2).
We claim: For every $X \in \mathcal{E}$ there is an exact sequence $X \to L \to Z$ in $\mathcal{E}$ such that $L \in \mathrm{pres}_n(\mathcal{T})$ and $Z \in \mathrm{Cores}(\mathcal{T})$. This short exact sequence implies $X \in \mathrm{Cores}(\mathcal{T}^{\perp})$. Using dimension shift and $\mathrm{pd}_{\mathcal{E}}\, \mathcal{T} \leq n$ one can see $X \in \mathrm{Cores}_n(\mathcal{T})$, so (T2) follows.
The claim follows using the beginning of a projective resolution of X

$$P_n \to \cdots \to P_0 \to X \to 0$$

with $P_i \in \mathcal{P}$. Recall by (t3) we have $P_i \in \mathrm{Cores}(\mathcal{T})$, this implies that we can use Lemma 6.4 below to obtain the short exact sequence.
Now, assume $X \in \mathcal{T}^{\perp}$ and consider again the short exact sequence $X \to L \to Z$ with $L \in \mathrm{pres}_n(\mathcal{T})$ and $Z \in \mathrm{Cores}(\mathcal{T})$. Since $\mathcal{T}^{\perp}$ is inflation-closed we have $Z \in \mathcal{T}^{\perp}$. Now, it is easy to see that $\mathcal{T}^{\perp} \cap \mathrm{Cores}(\mathcal{T}) = \mathcal{T}$, so $Z \in \mathcal{T}$. Therefore we have $\mathrm{Ext}^1_{\mathcal{E}}(Z, X) = 0$ which implies that the sequence splits and we conclude that $X \in \mathrm{add}(\mathrm{pres}_n(\mathcal{T}))$. Thus we proved $\mathcal{T}^{\perp} \subset \mathrm{add}(\mathrm{pres}_n(\mathcal{T}))$. By Corollary 5.5 we conclude that (T1) is fulfilled. □

Lemma 6.4. *Let $\mathcal{E}$ be an exact category and $\mathcal{T}$ a full additive subcategory which is self-orthogonal and $n \geq 1$. Assume that we have an exact sequence*

$$X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \to \cdots \to X_0 \xrightarrow{f_0} X \to 0$$

with $X_i \in \mathrm{Cores}(\mathcal{T})$. Then, there exists an exact sequence

$$T_{n-1} \xrightarrow{g_{n-1}} T_{n-2} \to \cdots \to T_0 \xrightarrow{g_0} L \to 0$$

with T_i in $\mathcal{T}$ and an exact sequence $X \to L \to Z$ with $Z \in \mathrm{Cores}(\mathcal{T})$.

The main ingredient to prove this is the following: The push out of an admissible morphism along an inflation is again admissible with the same kernel and cokernel as the admissible map that we pushed out ([**49**], Prop. 2.15)

Proof. We choose a short exact sequence $X_{n-1} \to T_{n-1} \to Z_{n-1}$ with $T_{n-1} \in \mathcal{T}, Z_{n-1} \in \mathrm{Cores}(\mathcal{T})$. Then take the push out

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{f_{n-1}} & X_{n-2} \\ \downarrow & & \downarrow \\ T_{n-1} & \xrightarrow{h_{n-1}} & Y_{n-2} \end{array}$$

since $\mathrm{Cores}(\mathcal{T})$ is closed under extensions (by Lem 3.4 (a)) we have $Y_{n-2} \in \mathrm{Cores}(\mathcal{T})$ since $\mathrm{coker}(X_{n-2} \to Y_{n-2}) = Z_{n-1}$. By construction h_{n-1} admissible in $\mathcal{E}$ with kernel and cokernel of h_{n-1} coincide with those of f_{n-1}. This means we constructed an exact sequence

$$T_{n-1} \to Y_{n-2} \to X_{n-3} \to \cdots \to X_0 \to X \to 0$$

Now, we pick a short exact sequence $Y_{n-2} \to T_{n-2} \to Z_{n-2}$ with $T_{n-2} \in \mathcal{T}, Z_{n-2} \in \mathrm{Cores}(\mathcal{T})$. Then we push out the admissible morphism $Y_{n-2} \to X_{n-3}$ along the inflation $Y_{n-2} \to T_{n-2}$ and proceed with the same method as before. Once Y_0 is constructed, choose the exact sequence $Y_0 \to T_0 \to Z_0$ with $T_0 \in \mathcal{T}, Z_0 \in \mathrm{Cores}(\mathcal{T})$ and take the push out of the deflation $Y_0 \to X$ along the inflation $Y_0 \to T_0$. This gives another deflation $T_0 \to L$ such that $\mathrm{Im}(T_1 \to Y_0) = \ker(T_0 \to L)$ and an inflation $X \to L$ with $\mathrm{coker}(X \to L) = Z_0 \in \mathrm{Cores}(\mathcal{T})$. □

For an earlier lemma, we need the following modification of the previous lemma: Firstly, observe that we can replace $\mathrm{Cores}(\mathcal{T})$ by $\mathrm{copres}_\infty(\mathcal{T})$ as long as we know that $\mathrm{copres}_\infty(\mathcal{T})$ is extension-closed. Secondly, if in the proof f_{n-1} is an inflation, then we construct in the proof an exact sequence with g_{n-1} an inflation. Thirdly, by passing to the opposite exact category, we have the following dual statement:

Lemma 6.5. *Let $\mathcal{E}$ be an exact category and $\mathcal{T}$ a full additive subcategory which is self-orthogonal, $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} < \infty$ and $n \geq 1$. Assume that we have an exact sequence*

$$0 \to X \to X_0 \to \cdots \to X_{n-1} \to 0$$

with $X_i \in \mathrm{pres}_\infty(\mathcal{T})$. Then, there exists an exact sequence

$$0 \to L \to T_0 \to \cdots \to T_{n-1} \to 0$$

with T_i in $\mathcal{T}$ and an exact sequence $Z \to L \to X$ with $Z \in \mathrm{pres}_\infty(\mathcal{T})$.

Observe that in the previous Lemma: The condition $\mathrm{pd}_{\mathcal{E}}\,\mathcal{T} < \infty$ is needed to assure $\mathrm{pres}_\infty(\mathcal{T})$ is extension-closed - cf. Lem. 3.4.

Remark 6.6. In particular, this generalizes the usual definition of tilting modules of finite projective dimensions over an artin algebra (cf. Happel [**87**]). Furthermore, it includes the generalization of Auslander and Solberg (cf. [**13**]) to so-called relative tilting modules (i.e. tilting objects in a different exact structure on the category of finitely presented modules over an artin algebra.

Inclusion of perpendicular categories defines a partial order on all tilting subcategories on an exact category. The previous theorem has the following application to this partial order. Here, given a fully exact subcategory $\mathcal{X} \subset \mathcal{E}$ we write $\mathrm{Res}_{\mathcal{X},m}(-)$ (resp. $\mathrm{Cores}_{\mathcal{X},m}(-)$) for the category $\mathrm{Res}_m(-)$ (resp. $\mathrm{Cores}_m(-)$) inside the fully exact category $\mathcal{X}$.

Proposition 6.7. *(and definition) Let $\mathcal{E}$ be an exact category and $\mathcal{T}$ an n-tilting subcategory. Let $\tilde{\mathcal{T}}$ be another full subcategory which is self-orthogonal and closed under summands. Then the following are equivalent:*

(a) *$\tilde{\mathcal{T}}$ is a m-tilting subcategory for some $m \geq 0$ of $\mathcal{E}$ and $\tilde{\mathcal{T}}^\perp \subset \mathcal{T}^\perp$.*
(b) *$\tilde{\mathcal{T}}$ is a m-tilting subcategory for some $m \geq 0$ of $\mathcal{E}$ and $\tilde{\mathcal{T}} \subset \mathcal{T}^\perp$.*
(c) *$\tilde{\mathcal{T}}$ is a m-tilting subcategory for some $m \geq 0$ of $\mathcal{T}^\perp$.*
(d) *There is an $m \geq 0$ such that $\tilde{\mathcal{T}} \subset \mathrm{Res}_{\mathcal{T}^\perp,m}(\mathcal{T})(\subset \mathcal{T}^\perp)$ and $\mathcal{T} \subset \mathrm{Cores}_{\mathcal{T}^\perp,m}(\tilde{\mathcal{T}})$.*

In this case, we write $\tilde{\mathcal{T}} \leq \mathcal{T}$.

Before, we give the proof. Let us note the following easy corollary:

Corollary 6.8. *Let $\mathcal{E}$ be an exact category and let $\mathcal{T}$ be an n-tilting subcategory and $\widetilde{\mathcal{T}}$ be an m-tilting subcategory for some $m \geq 0, n \geq 0$. If $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$, then $\widetilde{\mathcal{T}} = \mathcal{T}$.*

Proof. (of Cor. 6.8) The inclusion implies $\mathrm{Ext}^{>0}(\mathcal{T}, \widetilde{\mathcal{T}}) = 0 = \mathrm{Ext}^{>0}(\widetilde{\mathcal{T}}, \mathcal{T})$, therefore $\widetilde{\mathcal{T}} \leq \mathcal{T} \leq \widetilde{\mathcal{T}}$ and since $\leq$ is a partial order, they are equal. □

Proof. We are going to show the equivalences of 1. (c) and (d), 2. (a) and (b) and then 3. (b) and (c).
1. The equivalence is a direct consequence of Theorem 6.3 since $\mathcal{T}^\perp$ is an exact category with enough projectives.
2. Clearly (a) implies (b). So let us assume (b) and let $X \in \tilde{\mathcal{T}}^\perp$ and $T \in \mathcal{T}$. Since $\tilde{\mathcal{T}}^\perp$ has enough projectives given by $\tilde{\mathcal{T}}$, we find short exact sequences (in $\mathcal{E}$) setting $X := X_0$

$$0 \to X_i \to \tilde{T}_i \to X_{i-1} \to 0$$

with $\tilde{T}_i \in \tilde{\mathcal{T}}$ and $X_i \in \tilde{\mathcal{T}}^{\perp}$ for all $i \geq 1$. Then we have $\mathrm{Ext}^j_{\mathcal{E}}(T, X_{i-1}) \cong \mathrm{Ext}^{j+1}(T, X_i)$ for all $j \geq 1, i \geq 1$ implying $\mathrm{Ext}^j_{\mathcal{E}}(T, X_0) \cong \mathrm{Ext}^{j+n}_{\mathcal{E}}(T, X_n) = 0$ since $\mathrm{pd}_{\mathcal{E}}\, \mathcal{T} \leq n$.
3. Assume (b). Since (b) is equivalent to (a), we have $\tilde{\mathcal{T}}^{\perp} \subset \mathcal{T}^{\perp}$. This implies that the perpendicular category of $\tilde{\mathcal{T}}$ inside $\mathcal{T}^{\perp}$ coincides with the perpendicular category $\tilde{\mathcal{T}}^{\perp}$ inside $\mathcal{E}$. This implies (T1). It is straight forward to see that $\mathrm{Cores}_{\mathcal{T}^{\perp},m}(\tilde{\mathcal{T}}^{\perp}) = \mathrm{Cores}_m(\tilde{\mathcal{T}}^{\perp}) \cap \mathcal{T}^{\perp}$ holds and therefore (T2).
Assume (c). Since the equivalence to (d) implies $\mathcal{T} \subset \mathrm{Cores}_m(\tilde{\mathcal{T}})$ one can deduce $\tilde{\mathcal{T}}^{\perp} \subset \mathcal{T}^{\perp}$ (take an exact sequence $0 \to T \to \tilde{T}_0 \to \cdots \to \tilde{T}_s \to 0$, $T \in \mathcal{T}, \tilde{T}_i \in \tilde{\mathcal{T}}$, take $X \in \tilde{\mathcal{T}}^{\perp}$ and apply $\mathrm{Hom}_{\mathcal{E}}(-, X)$ to the sequence to conclude $\mathrm{Ext}^{>0}_{\mathcal{E}}(T, X) = 0$). This implies that the perpendicular category of $\tilde{\mathcal{T}}$ in $\mathcal{T}^{\perp}$ and in $\mathcal{E}$ coincide, therefore (T1) is fulfilled. To show (T2), observe that $\tilde{\mathcal{T}} \subset \mathrm{Res}_m(\mathcal{T})$ implies (since $\mathrm{pd}_{\mathcal{E}}\, \mathcal{T} \leq n$) that there is a $t \geq 0$ such that $\mathrm{pd}_{\mathcal{E}}(\tilde{\mathcal{T}}) \leq t$. Furthermore since $\mathcal{T}^{\perp}$ is coresolving and $\mathcal{T}^{\perp} \subset \mathrm{Cores}_m(\tilde{\mathcal{T}})$ we conclude that $\tilde{\mathcal{T}}^{\perp}$ is coresolving. This already implies that $\mathcal{E} = \mathrm{Cores}_t(\tilde{\mathcal{T}}^{\perp})$, to see this: Let $X_0 \in \mathcal{E}$, then there are short exact sequences $0 \to X_i \to Z_i \to X_{i+1} \to 0$ in $\mathcal{E}$ with $Z_i \in \tilde{\mathcal{T}}^{\perp}$, $i \geq 0$. Let $\tilde{T} \in \tilde{\mathcal{T}}$, then one has by dimension shift $\mathrm{Ext}^j_{\mathcal{E}}(\tilde{T}, X_t) \cong \mathrm{Ext}^{j+t}(\tilde{T}, X_0) = 0$ for all $j \geq 1$ since $\mathrm{pd}\, \tilde{T} \leq t$. □

Of course one has the dual result for cotilting subcategories (passing to opposite categories gives a poset isomorphism between m-tilting subcategories for some m in $\mathcal{E}$ and m-cotilting subcategories for some m in $\mathcal{E}^{op}$).

Remark 6.9. The previous proposition can be used to obtain the following (one-sided) *mutation*: Given an n-tilting subcategory $\mathcal{T} = \mathcal{M} \vee \mathcal{X}$ with $\mathcal{X} \subset \mathrm{gen}(\mathcal{M})$, define $\mathcal{Y} = \Omega^-_{\mathcal{M}}\mathcal{X}$ and $\tilde{\mathcal{T}} = \mathcal{M} \vee \mathcal{Y}$. If $\tilde{\mathcal{T}}$ is self-orthogonal (or equivalently: $\mathcal{Y} \subset \mathrm{cogen}(\mathcal{M}), \mathcal{X} = \Omega_{\mathcal{M}}\mathcal{Y}$), then $\tilde{\mathcal{T}}$ is m-tilting for some m and $\tilde{\mathcal{T}} \leq \mathcal{T}$.
Just observe: $\tilde{\mathcal{T}} \subset \mathrm{Res}_{\mathcal{T}^{\perp},1}(\mathcal{T}) \subset \mathcal{T}^{\perp}$ and $\mathcal{T} \subset \mathrm{Cores}_{\mathcal{T}^{\perp},1}(\tilde{\mathcal{T}})$. Therefore (d) in the previous Proposition applies.

Let $\mathcal{E}$ be an exact category and $n \geq 0$. Let $n - \mathrm{tilt}(\mathcal{E})$ be the class of n-tilting subcategories.

Corollary 6.10. *(of Thm. 6.3) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and $\mathcal{X}$ a resolving subcategory of $\mathcal{E}$. Then one has*

$$n - \mathrm{tilt}(\mathcal{X}) = \{\mathcal{T} \in n - \mathrm{tilt}(\mathcal{E}) \mid \mathcal{T} \subset \mathcal{X}\}.$$

In particular, we also have $n - \mathrm{tilt}(\mathcal{E}) \cong \{\mathcal{T} \subset n - \mathrm{tilt}(\mathrm{mod}_{\infty}\, \mathcal{P}) \mid \mathcal{T} \subset \mathrm{Im}\, \mathbb{P}\}$, $\mathcal{T}' \mapsto \mathbb{P}(\mathcal{T}')$ where $\mathbb{P}\colon \mathcal{E} \to \mathrm{mod}_{\infty} -\mathcal{P}, X \mapsto \mathrm{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{P}}$.

Proof. This is clear from the Theorem 6.3. The second statement follows since $\mathbb{P}$ is fully faithful, exact and $\mathrm{Im}\, \mathbb{P}$ is a resolving subcategory of $\mathrm{mod}_{\infty}\, \mathcal{P}$. □

One of the obvious questions is when can one restrict to tilting objects and when is it necessary to study more general tilting subcategories? In short, at least if you have enough projectives and a Krull-Schmidt category then the category of projectives should tell you the answer. More detailed, we have:

Theorem 6.11. *If $\mathcal{E}$ is a Krull-Schmidt, Hom-finite, exact category with enough projectives.*

(1) *If there is a tilting object T, then we have $|\mathcal{P}(\mathcal{E})| = |T| < \infty$.*
(2) *If $|\mathcal{P}(\mathcal{E})| < \infty$ and there is a tilting category $\mathcal{T}$, then we have $|\mathcal{T}| = |\mathcal{P}(\mathcal{E})| < \infty$.*

Proof. (of Thm 6.11) Let $\Gamma := \mathrm{End}_{\mathcal{E}}(T)$. Since $\mathcal{P} = \mathcal{P}(\mathcal{E}) \subset \mathrm{cogen}^1_{\mathcal{E}}(T)$, we have by [**171**], Lemma 2.1 that $\mathrm{Hom}_{\mathcal{E}}(-, T)\colon \mathcal{P} \to \Gamma - \mathrm{Mod}$ is full and faithful. For every $P, P' \in \mathcal{P}$ we have $0 = \mathrm{Ext}^i_{\mathcal{E}}(P', P) \cong \mathrm{Ext}^i_{\Gamma}(\mathrm{Hom}_{\mathcal{E}}(P, T), \mathrm{Hom}_{\mathcal{E}}(P', T))$ by [**171**], Lemma 3.3. Let Q be the direct sum of all indecomposable projectives appearing in a minimal projective resolution of T. When we apply $\mathrm{Hom}_{\mathcal{E}}(-, T)$ to the projective resolution of T and the exact sequence in (3), we conclude that

$\mathrm{Hom}_{\mathcal{E}}(Q,T)$ is a tilting Γ-module. Since tilting modules are maximal rigid, we have $\mathrm{add}(\mathrm{Hom}_{\mathcal{E}}(Q,T)) = \mathrm{add}(\mathrm{Hom}_{\mathcal{E}}(\mathcal{P},T))$. This implies $|\mathcal{P}| = |Q| = |\Gamma| = |T|$. □

6.1. Tilting in triangulated categories. In this subsection we compare our definition of a tilting subcategory with the definition of a tilting subcategory in the bounded homotopy category of projectives and in the bounded derived category.

Definition 6.12. In a triangulated category $\mathcal{C}$ one defines $T \in \mathcal{C}$ to be tilting if $\mathrm{Hom}(T, \Sigma^i T) = 0$ for $i \neq 0$ and the smallest thick subcategory that contains T is $\mathcal{C}$. Similarly we call a full additive subcategory $\mathcal{T}$ of $\mathcal{C}$ **tilting** if

(Tr1) $\mathrm{Hom}(T, \Sigma^i T') = 0$ for $i \neq 0$ for all T, T' in $\mathcal{T}$ and
(Tr2) $\mathrm{Thick}_\Delta(\mathcal{T}) = \mathcal{C}$

We recall the following Lemma which we want to use:

Lemma 6.13. *([**126**], Lemma 7.1.2) Let $\mathcal{X}$ be a self-orthogonal subcategory in an exact category $\mathcal{E}$. We consider $\mathcal{E} \subset \mathrm{D}^b(\mathcal{E})$ as stalk complexes in degree zero. Then the following are equivalent for an object $X \in \mathcal{E}$:*

(1) $X \in \mathrm{Thick}(\mathcal{X})$
(2) $X \in \mathrm{Thick}_\Delta(\mathcal{X})$

In particular, we have $\mathrm{Thick}(\mathcal{X}) = \mathcal{E}$ *if and only if* $\mathrm{Thick}_\Delta(\mathcal{X}) = \mathrm{D}^b(\mathcal{E})$.

Lemma 6.14. *If $\mathcal{E}$ is an exact category and $\mathcal{T}$ a full additive subcategory. Then the following are equivalent:*

(1) *$\mathcal{T} \subset \mathcal{E} \subset \mathrm{D}^b(\mathcal{E})$ already lies in $\mathrm{K}^b(\mathcal{P}(\mathcal{E}))$ and gives rise to a tilting subcategory in $\mathrm{K}^b(\mathcal{P}(\mathcal{E}))$.*
(2) *$\mathcal{T}$ is self-orthogonal and* $\mathrm{Thick}(\mathcal{P}(\mathcal{E})) = \mathrm{Thick}(\mathcal{T}) \subset \mathcal{E}$
(2') *$\mathcal{T}$ is self-orthogonal and $\mathcal{P}(\mathcal{E}) \subset \mathrm{Cores}(\mathcal{T})$, $\mathcal{T} \subset \mathrm{Res}(\mathcal{P}(\mathcal{E}))$*

Of course, in the situation that $\mathcal{E}$ has enough projectives and $\mathcal{T} = \mathrm{add}(T)$ contravariantly finite then (2') is equivalent to $\mathcal{T}$ being n-tilting (for some n).

PROOF. We have for a self-orthogonal subcategory $\mathcal{T} \subset \mathcal{E}$: $\mathcal{T} \subset \mathrm{K}^b(\mathcal{P}(\mathcal{E}))$ if and only if $\mathcal{T} \subset \mathrm{Thick}(\mathcal{P}(\mathcal{E})) = \mathrm{Res}(\mathcal{P}(\mathcal{E}))$ by Lem. 6.13 ([**126**], Lem 7.1.2). In this case, we have $\mathcal{T}$ self-orthogonal in $\mathcal{E}$ if and only if $\mathcal{T}$ fulfills (Tr1) in $\mathrm{K}^b(\mathcal{P}(\mathcal{E}))$. Furthermore, we also have by loc.cit.: $\mathrm{Thick}(\mathcal{T}) = \mathrm{Thick}(\mathcal{P}(\mathcal{E})) \subset \mathcal{E}$ if and only if $\mathrm{K}^b(\mathcal{P}(\mathcal{E})) = \mathrm{Thick}_\Delta(\mathcal{P}(\mathcal{E})) = \mathrm{Thick}_\Delta(\mathcal{T}) \subset \mathrm{D}^b(\mathcal{E})$. We now observe that for a self-orthogonal category $\mathcal{T}$ we have $\mathcal{P}(\mathcal{E}) \subset \mathrm{Thick}(\mathcal{T})$ implies $\mathcal{P}(\mathcal{E}) \subset \mathrm{Cores}(\mathcal{T})$ by [**126**], Lem. 7.1.6. , this finishes the proof. □

Lemma 6.15. *If $\mathcal{E}$ is an exact category and $\mathcal{T}$ a full subcategory. Then, the following are equivalent*

(1) *$\mathcal{T}$ is a tilting subcategory in $\mathcal{E}$ and* $\mathrm{Thick}(\mathcal{T}) = \mathcal{E}$.
(2) *$\mathcal{T}$ is self-orthogonal and* $\mathrm{Thick}(\mathcal{T}) = \mathcal{E}$
(2') *$\mathcal{T}$ is a tilting subcategory in $\mathrm{D}^b(\mathcal{E})$*

PROOF. Clearly, (1) implies (2). Assume (2), then $\mathcal{T}^\perp = \mathrm{Res}(\mathcal{T})$ by [**126**], Prop. 7.10. This implies that $\mathcal{T} = \mathcal{P}(\mathcal{T}^\perp)$ and that $\mathcal{T}^\perp$ has enough projectives, so (T1). Also $\mathrm{Thick}(\mathcal{T}) = \mathcal{E}$ implies $\mathrm{Cores}(\mathcal{T}^\perp) = \mathrm{Thick}(\mathcal{T}^\perp) = \mathcal{E}$, so (T2) and $\mathcal{T}$ is tilting. The equivalence of (2) and (2') follows from Lem. 6.13. □

Remark 6.16. Lemma 6.15, (2) recovers the definition of tilting from [**126**], Chapter 7 as a special case of our definition.

7. Induced triangle equivalences

Definition 7.1. Let $\mathcal{T}$ be a tilting subcategory in an exact category $\mathcal{E}$. From now on, we call the functor

$$\boxed{f_{\mathcal{T}}\colon \mathcal{T}^{\perp} \to \operatorname{mod}_{\infty} \mathcal{T}, \quad X \mapsto \operatorname{Hom}_{\mathcal{E}}(-,X)|_{\mathcal{T}}}$$

the **tilting functor** of $\mathcal{T}$. Furthermore, since $\mathcal{T}^{\perp}$ is finitely coresolving in $\mathcal{E}$ and $\operatorname{Im} f_{\mathcal{T}}$ is resolving in $\operatorname{mod}_{\infty} \mathcal{T}$, we have an exact functor

$$\boxed{F_{\mathcal{T}}\colon \mathrm{D}^b(\mathcal{E}) \xrightarrow{\cong} \mathrm{D}^b(\mathcal{T}^{\perp}) \xrightarrow{\cong} \mathrm{D}^b(\operatorname{Im} f_{\mathcal{T}}) \to \mathrm{D}^b(\operatorname{mod}_{\infty} \mathcal{T})}$$

which we call the **derived tilting functor** of $\mathcal{T}$.

Derived tilting functors are exact and fully faithful by remark 4.4

Definition 7.2. Let $\mathcal{T}$ be a (n-)tilting subcategory in an exact category $\mathcal{E}$. We say that $\mathcal{T}$ is **ideq** (n-)-tilting if $F_{\mathcal{T}}$ is a triangle equivalence (i.e. essentially surjective). We call $\mathcal{T}$ m-**ideq** (n-)tilting if $\operatorname{Im} f_{\mathcal{T}}$ is m-resolving.

Remark 7.3. If $\operatorname{Im} f_{\mathcal{T}}$ is finitely resolving in $\operatorname{mod}_{\infty} \mathcal{T}$ then $F_{\mathcal{T}}$ is a triangle equivalence. By Lemma 3.17, $\operatorname{Im} f_{\mathcal{T}}$ is finitely resolving iff for every morphism $f\colon T_1 \to T_0$ in $\mathcal{T}$ which admits a sequence of successive weak kernels in $\mathcal{T}$ we find a complex in $\mathcal{T}^{\perp}$

$$0 \to M_m \to M_{m-1} \to \cdots \to M_2 \to T_1 \to T_0$$

such that the induced sequence of restrictions of representable functors
$0 \to \operatorname{Hom}(-,M_m)|_{\mathcal{T}} \to \cdots \to \operatorname{Hom}(-,M_0)|_{\mathcal{T}} \to \operatorname{Hom}_{\mathcal{T}}(-,T_1) \to \operatorname{Hom}_{\mathcal{T}}(-,T_0)$ in $\operatorname{mod}_{\infty} \mathcal{T}$ is exact.

Remark 7.4. If $\mathcal{T}$ is m-ideq n-tilting for some $n, m \geq 0$, then we have induced triangle equivalences also on the unbounded derived category

$$\mathbb{F}_{\mathcal{T}}\colon \mathrm{D}(\mathcal{E}) \cong \mathrm{D}(\mathcal{T}^{\perp}) \xrightarrow{f_{\mathcal{T}}} \mathrm{D}(\operatorname{Im} f_{\mathcal{T}}) \cong \mathrm{D}(\operatorname{mod}_{\infty} \mathcal{T})$$

Example 7.5. Assume that $\mathcal{T}$ is a tilting subcategory such that every map $f\colon T_1 \to T_0$ in $\mathcal{T}$ admits a kernel in $\mathcal{T}^{\perp}$ (the monomorphism on $\ker f$ does not have to be an inflation), then $\mathcal{T}$ is 2-ideq tilting (just use the kernel of the map).
An instance of this is the following: Assume that $\mathcal{T}$ is 1-tilting with $\mathcal{T}$ contravariantly finite in $\mathcal{E}$ (then: $\mathcal{T}^{\perp} = \operatorname{gen}(\mathcal{T}) = \operatorname{pres}(\mathcal{T})$) and assume every morphism in $\mathcal{T}$ factors over a deflation in $\mathcal{E}$ (for example if $\mathcal{E}$ is abelian). Then every morphism in $\mathcal{T}$ has a kernel in $\mathcal{T}^{\perp}$.

7.1. Ideq tilting from equality with $\mathcal{P}^{<\infty}$.

Let us first observe the obvious:

Lemma 7.6. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and $m \geq 0$. Then, the following are equivalent:*

1. $\mathcal{E} = \mathcal{P}^{<\infty}$ *(resp.* $\operatorname{gldim} \mathcal{E} \leq m < \infty$*)*
2. *Every resolving subcategory is finitely resolving (resp. is m-resolving).*

Proof. (2) implies (1): $\mathcal{P}$ is a resolving subcategory, it is finitely resolving, i.e. $\operatorname{Res}(\mathcal{P}) = \mathcal{E}$, (resp. m-resolving, i.e. $\operatorname{Res}_m(\mathcal{P}) = \mathcal{E}$) if and only if $\mathcal{P}^{<\infty} = \mathcal{E}$ (resp. $\operatorname{gldim} \mathcal{E} \leq m < \infty$).
(1) implies (2): If $\mathcal{X}$ is resolving then $\mathcal{P} \subset \mathcal{X}$. Therefore $\operatorname{Res}(\mathcal{P}) \subset \operatorname{Res}(\mathcal{X})$, $\operatorname{Res}_m(\mathcal{P}) \subset \operatorname{Res}_m(\mathcal{X})$. □

Which brings us to this naive question:

Open question 7.7. Is the previous lemma still true if we drop the assumption that $\mathcal{E}$ has enough projectives?

Proposition 7.8. *Let $\mathcal{E}$ be an exact category and $\mathcal{T}$ a tilting subcategory. If $\operatorname{mod}_{\infty} \mathcal{T} = \mathcal{P}^{<\infty}$ (resp. $\operatorname{gldim}(\operatorname{mod}_{\infty} \mathcal{T}) \leq m < \infty$), then $\mathcal{T}$ is ideq tilting (resp. m-ideq tilting) .*

Another common triangle equivalence considered uses *perfect complexes*:

Definition 7.9. For an exact category $\mathcal{E}$ with enough projectives $\mathcal{P}$, we call $\mathrm{D}^b_{perf}(\mathcal{E}) := \mathrm{K}^b(\mathcal{P}) = \mathrm{Thick}_\Delta(\mathcal{P}) \subset \mathrm{D}^b(\mathcal{E})$ the triangulated subcategory of **perfect complexes**.

Observe, that $\mathrm{D}^b_{perf}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$ is equivalent to $\mathcal{E} = \mathcal{P}^{<\infty}$ by Lemma 5.7 since $\mathcal{P}$ is a 0-tilting subcategory.

Lemma 7.10. *If $\mathcal{E}$ is an exact category with $\mathcal{E} = \mathcal{P}^{<\infty}$, and $\mathcal{T}$ tilting subcategory then we have an induced triangle equivalence*

$$\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b_{perf}(\mathrm{mod}_\infty \mathcal{T})$$

Proof. Since $\mathcal{E} = \mathcal{P}^{<\infty}$ implies $\mathcal{T}^\perp = \mathcal{P}^{<\infty}$ and since we have the additive equivalence $\mathcal{P}(\mathcal{T}^\perp) \to \mathcal{P}(\mathrm{mod}_\infty \mathcal{T})$, $T \mapsto \mathrm{Hom}_\mathcal{T}(-, T)$ we obtain induced triangle equivalences

$$\mathrm{D}^b(\mathcal{E}) \cong \mathrm{D}^b(\mathcal{T}^\perp) \cong \mathrm{K}^b(\mathcal{P}(\mathcal{T}^\perp)) \to \mathrm{K}^b(\mathcal{P}(\mathrm{mod}_\infty \mathcal{T})) \cong \mathrm{D}^b_{perf}(\mathrm{mod}_\infty \mathcal{T})$$

□

We also have the following:

Proposition 7.11. *Let $\mathcal{E}$ be an exact category and $\mathcal{T}$ an m-ideq n-tilting subcategory. Then we have:*

$$\mathrm{gldim}\,\mathcal{E} \leq \mathrm{gldim}(\mathrm{mod}_\infty \mathcal{T}) + n, \quad \mathrm{gldim}(\mathrm{mod}_\infty \mathcal{T}) \leq \mathrm{gldim}\,\mathcal{E} + m$$

The proof follows directly from the following Lemma (and its dual statement).

Lemma 7.12. *Let $\mathcal{E}$ be an exact category and $\mathcal{X}$ be a fully exact category. If we have $\mathrm{Res}_n(\mathcal{X}) = \mathcal{E}$ for some $n \in \mathbb{N}$, then we have*

$$\mathrm{gldim}\,\mathcal{X} \leq \mathrm{gldim}\,\mathcal{E} \leq \mathrm{gldim}\,\mathcal{X} + n$$

Proof. The first inequality is clear since $\mathrm{Ext}^i_\mathcal{X} = (\mathrm{Ext}^i_\mathcal{E})|_\mathcal{X}$. For the second, wlog. $\mathrm{gldim}\,\mathcal{X} = s \leq \infty$, let $E, L \in \mathcal{E}$, we claim $\mathrm{Ext}^{>(s+n)}_\mathcal{E}(L, E) = 0$. By assumption, exists an exact sequence $0 \to X_n \to \cdots \to X_0 \to E \to 0$ with $X_i \in \mathcal{X}, 0 \leq i \leq n$. We apply $\mathrm{Hom}(X, -)$ with $X \in \mathcal{X}$ and obtain $\mathrm{Ext}^{>s}_\mathcal{E}(X, E) = 0$. Now, we take the exact sequence $0 \to Y_n \to \cdots \to Y_0 \to L \to 0$ with $Y_i \in \mathcal{X}, 0 \leq i \leq n$, and apply $\mathrm{Hom}(-, E)$ and obtain $\mathrm{Ext}^{>(s+n)}_\mathcal{E}(L, E) = 0$. □

7.2. Ideq tilting in exact categories with enough projectives.

The following is the most important result for this question (it is a generaliztion of Miyashita's theorem [**137**], Thm 1.16):

Theorem 7.13. *(Generalized Miyashita-Thm) Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and let $\mathcal{T}$ be an n-tilting subcategory which is essentially small. We consider the contravariant functor*

$$\Psi_\mathcal{T} \colon \mathcal{E} \to \mathcal{T}\,\mathrm{Mod}, \quad X \mapsto \mathrm{Hom}_\mathcal{E}(X, -)|_\mathcal{T}$$

and the covariant functor

$$\Phi_\mathcal{T} \colon \mathcal{E} \to \mathrm{Mod}-\mathcal{T}, \quad X \mapsto \mathrm{Hom}_\mathcal{E}(-, X)|_\mathcal{T}.$$

Let $\widetilde{\mathcal{T}} := \Psi_\mathcal{T}(\mathcal{P})$ and $\overline{\mathcal{T}} := \Phi_\mathcal{P}(\mathcal{T})$. Then we have:

(1) *$\widetilde{\mathcal{T}}$ is an n-tilting subcategory of $\mathcal{T}\,\mathrm{mod}_\infty$ and $\overline{\mathcal{T}}$ is an n-tilting subcategory of $\mathrm{mod}_\infty \mathcal{P}$.*
(2) *$\Psi_\mathcal{T}$ restricts to an equivalence $\mathcal{P} \cong \widetilde{\mathcal{T}}^{op}$ and $\Phi_\mathcal{P}$ to one $\mathcal{T} \cong \overline{\mathcal{T}}$.*
(3) *The category ${}_\perp\widetilde{\mathcal{T}} := \{M \in \mathrm{mod}_\infty \mathcal{T} \mid \mathrm{Tor}^\mathcal{T}_{>0}(M, \widetilde{\mathcal{T}}) = 0\}$ is a resolving subcategory of $\mathrm{mod}_\infty \mathcal{T}$ with $\mathrm{Res}_n({}_\perp\widetilde{\mathcal{T}}) = \mathrm{mod}_\infty \mathcal{T}$.*
(4) *The functor $\Phi = \Phi_{\overline{\mathcal{T}}} \colon \mathrm{Mod}\,\mathcal{P} \to \mathrm{Mod}\,\mathcal{T}, X \mapsto \mathrm{Hom}_{\mathrm{Mod}\,\mathcal{P}}(\Phi_\mathcal{P}(-), X)|_\mathcal{T}$ has a left adjoint $\Phi' \colon \mathrm{Mod}\,\mathcal{T} \to \mathrm{Mod}\,\mathcal{P}, X \mapsto (P \mapsto X \otimes_\mathcal{T} \Psi_\mathcal{T}(P))$. They restrict to inverse equivalences between*

(i) $\{M \in \operatorname{Mod} \mathcal{P} \mid \operatorname{Ext}^{>0}_{\operatorname{Mod} \mathcal{P}}(\mathcal{T}, M) = 0\}$ *and* $\{N \in \operatorname{Mod} \mathcal{T} \mid \operatorname{Tor}^{\mathcal{T}}_{>0}(N, \widetilde{\mathcal{T}}) = 0\}$
(ii) $\overline{\mathcal{T}}^{\perp}(\subset \operatorname{mod}_\infty \mathcal{P})$ *and* ${}_{\perp}\widetilde{\mathcal{T}}(\subset \operatorname{mod}_\infty \mathcal{T})$.

(5) *We have a commutative triangle of exact functors (restricted to these subcategories)*

$$\begin{array}{ccc} & \mathcal{T}^{\perp} & \\ {}^{\Phi_{\mathcal{P}}}\swarrow & & \searrow^{\Phi_{\mathcal{T}}} \\ \overline{\mathcal{T}}^{\perp} & \xrightarrow{\Phi} & {}_{\perp}\widetilde{\mathcal{T}} \end{array}$$

In particular, $\overline{\mathcal{T}}$ *is* n*-ideq* n*-tilting and we have an induced triangle equivalence* $\mathrm{D}^b(\operatorname{mod}_\infty \mathcal{P}) \to \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{T})$.

Before we prove the previous theorem, let us state this Theorem as a corollary.

THEOREM 7.14. *Let* $\mathcal{E}$ *be an exact category with enough projectives* $\mathcal{P}$*. Then the following are equivalent:*

(1) $\mathcal{E}$ *is equivalent as an exact category to a finitely resolving subcategory of* $\operatorname{mod}_\infty \mathcal{P}$
(2) *There is an* $n \in \mathbb{N}_0$ *and an* n*-tilting subcategory of* $\mathcal{E}$ *which is ideq* n*-tilting.*
(3) *For every* $n \geq 0$*, every* n*-tilting subcategory of* $\mathcal{E}$ *is ideq* n*-tilting.*

PROOF. It is straight-forward to see that (1) is equivalent to $\mathcal{P}$ is ideq 0-tilting. To prove the equivalences we show for a given n-tilting subcategory $\mathcal{T}$, we have: $\mathcal{T}$ is ideq n-tilting if and only if $\mathcal{P}$ is ideq 0-tilting. For this it is enough to show (using Theorem 7.14) that we have a commutative diagram of triangle functors

$$\begin{array}{ccc} & \mathrm{D}^b(\mathcal{E}) & \\ \swarrow & & \searrow \\ \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{P}) & \longrightarrow & \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{T}) \end{array}$$

where $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{P})$ is the derived functor of $\Phi_{\mathcal{P}}$ and $\mathrm{D}^b(\mathcal{E}) \cong \mathrm{D}^b(\mathcal{T}^{\perp}) \to \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{T})$ is the derived functor of $\Phi_{\mathcal{T}}|_{\mathcal{T}^{\perp}}$ and $\mathrm{D}^b(\operatorname{mod}_\infty \mathcal{P}) \to \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{T})$ is the triangle equivalence induced by the equivalence $\overline{\mathcal{T}}^{\perp} \to {}_{\perp}\widetilde{\mathcal{T}}$. But this follows immediately from loc. cit. (5). $\square$

We can prove the even stronger corollary of Theorem 7.14.

Definition 7.15. We define $\operatorname{Tilt}(\mathcal{E}) := \bigoplus_{n \geq 0} n - \operatorname{tilt}(\mathcal{E})$. The relation $\leq$ from Lemma .. defines a poset structure on this set.
Let $\mathcal{E}$ be an exact category. We say $\mathcal{E}$ is **tilting connected** if in the poset $\operatorname{Tilt}(\mathcal{E})$ is non-empty and for every two element $\mathcal{T}$ and $\mathcal{T}'$ there is a finite sequence $\mathcal{T}_0 = \mathcal{T}, \mathcal{T}_1, \ldots, \mathcal{T}_r = \mathcal{T}'$ with $\mathcal{T}_i \leq \mathcal{T}_{i+1}$ or $\mathcal{T}_i \geq \mathcal{T}_{i+1}$, $0 \leq i \leq r-1$.

Example 7.16. If $\mathcal{E}$ has enough projectives $\mathcal{P}$ then $\mathcal{E}$ is tilting connected since $\mathcal{P}$ is a globales maximum.
If the injectives $\mathcal{I}$ in $\mathcal{E}$ happen to be an n-tilting subcategory then $\mathcal{E}$ is tilting connected since $\mathcal{I}$ has to be a global minimum.

Open question 7.17. Are exact categories are always tilting connected?

Corollary 7.18. *Let* $\mathcal{E}$ *be an exact category and* $\mathcal{T}$ *an* n*-tilting subcategory. Then the following are equivalent:*

(1) $\mathcal{T}$ *is ideq* n*-tilting*
(2) *Every* m*-tilting subcategory* $\mathcal{L}$ *with* $\mathcal{L} \leq \mathcal{T}$ *or* $\mathcal{T} \leq \mathcal{L}$ *(i.e.* $\mathcal{L}$ *comparable to* $\mathcal{T}$*) is ideq* m*-tilting.*

In particular, every connected component in $\mathrm{Tilt}(\mathcal{E})$ *is either ideq tilting (i.e. every* n*-tilting subcategory in it is ideq* n*-tilting) or it is not ideq tilting.*

The proof of the main result uses an auxiliary preprint of the author [**171**] in which some technical results are explained.

Proof. (of Theorem 7.13)

(1) We want to see that $\widetilde{\mathcal{T}}$ satisfies (t1),(t2) and (t3): Since $\mathcal{T}$ is an n-tilting subcategory of $\mathcal{E}$ we have by (t2) for every $T \in \mathcal{T}$ an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to T \to 0$$

with $P_i \in \mathcal{P}$. We apply $\Psi_{\mathcal{T}}$ to it and obtain a complex

$$0 \to \Psi_{\mathcal{T}}(T) \to \Psi_{\mathcal{T}}(P_0) \to \cdots \to \Psi_{\mathcal{T}}(P_n) \to 0$$

Since $\mathcal{T}$ fulfills (t1), it follows that this is exact, so $\widetilde{\mathcal{T}}$ fulfills (t3).

Now, by (t3) for $\mathcal{T}$ we have for every $P \in \mathcal{P}$ an exact sequence

$$0 \to P \to T_0 \to \cdots \to T_n \to 0,$$

we apply again $\Psi_{\mathcal{T}}$ and again by (t1) for $\mathcal{T}$ we get an exact sequence

$$0 \to \Psi_{\mathcal{T}}(T_n) \to \cdots \to \Psi_{\mathcal{T}}(T_0) \to \Psi_{\mathcal{T}}(P) \to 0$$

which shows that $\widetilde{\mathcal{T}}$ fulfills (t2). Furthermore, $\mathcal{P} \subset \mathrm{cogen}^{\infty}_{\mathcal{E}}(\mathcal{T})$ implies by [**171**], Lemma 3.3. that $\mathrm{Ext}^i(\Psi_{\mathcal{T}}(P), \Psi_{\mathcal{T}}(P')) = 0$ for $P, P' \in \mathcal{P}$ and $0 < i < \infty$, so (t1) holds for $\widetilde{\mathcal{T}}$. The second claim follows since $\Phi_{\mathcal{P}}$ is exact, fully faithful and preserves extension groups and maps projectives to projectives.

(2) Since $\mathcal{P} \subset \mathrm{cogen}^1_{\mathcal{E}}(\mathcal{T})$ it follows that $\Psi_{\mathcal{T}}$ restricted to $\mathcal{P}$ is fully faithful by [**171**], Lemma 2.1. Since $\Phi_{\mathcal{P}}$ is fully faithful, the second claim is clear.

(3) By the properties of Tor the category ${}_{\perp}\widetilde{\mathcal{T}}$ contains the projectives, is extension closed and deflation-closed, so it is resolving.

The last statement follows when we consider the first n terms of a projective resolution of $X \in \mathrm{mod}_{\infty}\,\mathcal{T}$

$$0 \to \Omega^n \to T_{n-1} \to \cdots \to T_0 \to X \to 0$$

with $T_i \in \mathcal{P}(\mathrm{mod}_{\infty}\,\mathcal{T})$. We claim that $\mathrm{Tor}^{\mathcal{T}}_{>0}(\Omega^n, \widetilde{\mathcal{T}}) = 0$. By dimension shift $\mathrm{Tor}^{\mathcal{T}}_i(\Omega^n, \widetilde{\mathcal{T}}) = \mathrm{Tor}^{\mathcal{T}}_{i+n}(X, \widetilde{\mathcal{T}}) = 0$ since $\mathrm{pd}\,\widetilde{\mathcal{T}} \leq n$.

(4) It is standard to see that these functors form an adjoint pair (it should be seen as a Hom-Tensor adjunction), cf. [**171**], Lemma 3.7.

(i) This is a straight forward generalization of the orginal result [**137**], Thm 1.16. We just mention it for completeness.

(ii) We want to see that both functors restrict to functors as claimed and that they are both fully faithful.

By Lemma 5.5 and [**171**], Lem. 3.13, Rem. 3.14 we have

$$\begin{aligned}\overline{\mathcal{T}}^{\perp} &= \mathrm{gen}^{\mathrm{mod}_{\infty}\mathcal{P}}_{\infty}(\overline{\mathcal{T}}) = \mathrm{pres}^{\mathrm{mod}_{\infty}\mathcal{P}}_{\infty}(\overline{\mathcal{T}}) \\ &= \{X \in \mathrm{mod}_{\infty}\,\mathcal{P} \mid \varphi_X \text{ isom}, \Phi(X) \in \mathrm{mod}_{\infty} -\mathcal{T}, \mathrm{Tor}^{\mathcal{T}}_{>0}(\Phi(X), \widetilde{\mathcal{T}}) = 0\}\end{aligned}$$

therefore, the functor Φ restricts as claimed and is fully faithful since $\mathrm{gen}_{\infty}(\overline{\mathcal{T}}) \subset \mathrm{gen}_1(\overline{\mathcal{T}})$ (again using [**171**], Lemma 1.1.)

We are going to proof the following claim:

(*) $\mathrm{pres}^{\mathrm{mod}_{\infty}\mathcal{P}}_{\infty}(\overline{\mathcal{T}}) = \mathrm{pres}^{\mathrm{Mod}\,\mathcal{P}}_{\infty}(\overline{\mathcal{T}})$

proof of (*): Given an exact sequence $\cdots \to T_m \to \cdots \to T_0 \to X \to 0$ in $\operatorname{Mod}\mathcal{P}$ with $T_i \in \overline{\mathcal{T}}$, we claim $X \in \operatorname{mod}_\infty \mathcal{P}$. Now, we see this as a quasi-isomorphism of complexes (with terms in $\operatorname{Mod}\mathcal{P}$)

$$\begin{array}{cccccccccc} T_*\colon & \cdots \longrightarrow & T_m & \longrightarrow \cdots \longrightarrow & T_1 & \longrightarrow & T_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ X\colon & \cdots \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Since T_* is a complex in $\operatorname{mod}_\infty \mathcal{P}$, we find by [49], Thm 12.7, a quasi-isomorphism $P_* \to T_*$ with P_* is a complex $\cdots \to P_m \to P_{m-1} \to \cdots \to P_0 \to 0$, $P_{-n} = 0$ for all $n > 0$ with terms in $\mathcal{P}$, here a quasi-isomorphism means that the mapping cone of $P_* \to T_*$ is acyclic. Since composition of quasi-isomorphisms are quasi-isomorphsims, we have a quasi-isomorphism $P_* \to X$ which means that the mapping cone which is a complex $\cdots \to P_1 \to P_0 \to X \to 0$ is exact and therefore $X \in \operatorname{mod}_\infty \mathcal{P}$.

Let $Y \in {}_\perp\widetilde{\mathcal{T}}$. Since $Y \in \operatorname{mod}_\infty \mathcal{T}$, e.g. there exists an exact sequence

$$\cdots \to \Phi(T_m) \to \cdots \to \Phi(T_0) \to Y \to 0$$

with $\Phi(T_i) \in \mathcal{P}(\operatorname{mod}_\infty \mathcal{T}) = \Phi(\overline{\mathcal{T}})$. Applying Φ' and using that $\Phi'\Phi(T) \cong T$ for all $T \in \overline{\mathcal{T}}$ yields a complex

$$\cdots \to T_m \to \cdots \to T_0 \to \Phi'(Y) \to 0$$

with $T_i \in \overline{\mathcal{T}}$. Since $\operatorname{Tor}^{\mathcal{T}}_{>0}(Y, \widetilde{\mathcal{T}}) = 0$ and Φ' is right exact, this complex is exact. This implies by (a) that $\Phi'(Y) \in \overline{\mathcal{T}}^{\perp} = \operatorname{gen}^{\operatorname{mod}_\infty \mathcal{P}}_\infty(\overline{\mathcal{T}})$. Therefore, Φ' restricts as claimed. To see that it is fully faithful, we also have the consequence that applying Φ is again exact on this complex

$$\cdots \to \Phi(T_m) \to \cdots \to \Phi(T_0) \to \Phi(\Phi'(Y)) \to 0$$

By the triangle identity of the adjunction, this implies that the unit $Y \to \Phi\Phi'(Y)$ is an isomorphism.

Now, an adjunction with unit and counit isomorphisms (i.e. fully faithful left and right adjoint) is an equivalence of categories.

(5) It follows from the definition that $\Phi \circ \Phi_{\mathcal{P}} = \Phi_{\mathcal{T}}$. Since $\Phi_{\mathcal{P}}$ preserves all extension groups, it is clear that $\Phi_{\mathcal{P}}$ restricts to a functor as claimed, for Φ this has been proven in (4).

$\square$

Example 7.19. If $\mathcal{E}$ is an abelian category with enough projectives, then every n-tilting subcategory is ideq n-tilting. For example, let $\mathcal{P}$ be essentially small, then $\mathcal{E} = \operatorname{Mod}\mathcal{P}$ is an abelian category with enough projectives given by summands of arbitrary direct sums of $\mathcal{P}$, i.e. $\mathcal{P}(\operatorname{Mod}\mathcal{P}) = \operatorname{Add}(\mathcal{P})$. It implies that the functor $\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_\infty(\operatorname{Add}(\mathcal{P}))$ is an exact equivalence.

Example 7.20. Let $\mathcal{E} := \mathcal{P}(\operatorname{mod}_\infty \mathcal{P}) \subset \operatorname{mod}_\infty \mathcal{P}$. We consider $\mathcal{E}$ as a fully exact subcategory of $\operatorname{mod}_\infty \mathcal{P}$, then it is a resolving subcategory. Since the category $\mathcal{E}$ is semi-simple, it has a unique tilting subcategory $\mathcal{T} = \mathcal{P}(\mathcal{E}) = \mathcal{E}$ which is a 0-tilting subcategory. Now, it follows from Thm. 7.14 that $\operatorname{gldim}(\operatorname{mod}_\infty \mathcal{P}) = m < \infty$ (resp. $\operatorname{mod}_\infty \mathcal{P} = \mathcal{P}^{<\infty}$) if and only if $\operatorname{Res}_m(\mathcal{E}) = \operatorname{mod}_\infty \mathcal{P}$ (resp. $\operatorname{Res}(\mathcal{E}) = \operatorname{mod}_\infty \mathcal{P}$) if and only if $\mathcal{E}$ is m-ideq 0-tilting subcategory (resp. ideq 0-tilting) of itself.

7.3. Conjectural generalizations to ideq tilting in arbitrary exact categories. Here, we come accross the following unresolved question:

Open question: Is there an analogue of Rickard's Morita theory for functor categories as follows:

Conjecture 7.21. *(Strong Rickard conjecture) Let $\mathcal{X}, \mathcal{Y}$ be two essentially small idempotent complete additive categories. Then the following equivalent:*

(0) $\mathrm{D}(\operatorname{Mod}-\mathcal{X})$ *and* $\mathrm{D}(\operatorname{Mod}-\mathcal{Y})$ *are triangle equivalent.*
(1) $\mathrm{D}^-(\operatorname{Mod}\mathcal{X})$ *and* $\mathrm{D}^-(\operatorname{Mod}\mathcal{Y})$ *are triangle equivalent.*
(2) $\mathrm{D}^-(\operatorname{mod}_\infty \mathcal{X})$ *and* $\mathrm{D}^-(\operatorname{mod}_\infty \mathcal{Y})$ *are triangle equivalent.*

(3) $\mathrm{D}^b(\mathrm{mod}_\infty \mathcal{X})$ *and* $\mathrm{D}^b(\mathrm{mod}_\infty \mathcal{Y})$ *are triangle equivalent.*
(4) $\mathrm{K}^b(\mathcal{X})$ *and* $\mathrm{K}^b(\mathcal{Y})$ *are triangle equivalent.*
(5) *There is a tilting subcategory* $\mathcal{T}$ *of* $\mathrm{K}^b(\mathcal{X})$ *such that* $\mathcal{T} \cong \mathcal{Y}$ *as additive categories.*

If we do not assume $\mathcal{X}$ *and* $\mathcal{Y}$ *to be small, then (2),(3),(4), (5) are still equivalent. Furthermore, every triangle equivalence in (0)-(3) restricts to a triangle equivalence as in (4).*

For $\mathcal{X}$ and $\mathcal{Y}$ of finite type and idempotent complete, part of the conjecture is Rickard's Morita theorem for derived categories of rings (cf. [**158**], Thm 6.4, Prop. 8.1 - for example (3) implies (4) is proven in loc. cit only for right coherent rings - yet (4) implies (3) is proven for arbitrary rings). An alternative proof can be found in [**126**]. For more general small additive categories partial answers are given by Keller [**118**], Corollary in 9.2 and Asadollahi-Hafezi-Vaheed [**9**], Thm 3.21.
For our purpose, we need the following weaker statement:

Conjecture 7.22. *(Rickard-Lemma) Let* $\mathcal{X}$*,* $\mathcal{Y}$ *be small idempotent complete additive categories. If* $\mathrm{K}^b(\mathcal{X})$ *and* $\mathrm{K}^b(\mathcal{Y})$ *are triangle equivalent, then there exists a triangle equivalence* $\mathrm{D}^b(\mathrm{mod}_\infty \mathcal{X}) \to \mathrm{D}^b(\mathrm{mod}_\infty \mathcal{Y})$ *which restricts to a triangle equivalence* $\mathrm{K}^b(\mathcal{X}) \to \mathrm{K}^b(\mathcal{Y})$*.*

For $\mathrm{add}\,\Lambda$ with Λ an arbitrary ring, the existence is proven in [**158**] after Prop 8.1. One proof of the Rickard-Lemma should be as follows:
Let $\mathcal{T} \subset \mathrm{K}^b(\mathcal{Y})$ be the image of $\mathcal{X}$ under the assumed triangle equivalence. We define $\mathrm{Add}(\mathcal{Y}) := \mathcal{P}(\mathrm{Mod}\,\mathcal{Y})$. Then one shows that $\mathcal{T} \subset \mathrm{K}(\mathrm{Add}(\mathcal{Y}))$ fulfills (P1),(P2),(P3) in [**9**] and the acyclic complexes in $\mathrm{K}(\mathrm{Add}(\mathcal{Y}))$ coincide with the $\mathcal{T}$-acyclic complexes in loc. cit (since $\mathrm{Thick}_\Delta(\mathcal{T})$ in $\mathrm{K}^b(\mathcal{Y})$ equals $\mathrm{K}^b(\mathcal{Y})$)). Therefore, [**9**], Thm 3.21 can be applied to obtain a triangle equivalence $\mathrm{K}^-(\mathrm{Add}(\mathcal{X})) \cong \mathrm{K}^-(\mathrm{Add}(\mathcal{T})) \to \mathrm{K}^-(\mathrm{Add}(\mathcal{Y}))$ which restricts to a triangle equivalence $\mathrm{K}^b(\mathcal{X}) \to \mathrm{K}^b(\mathcal{Y})$. Then, arguments of [**158**] should generalize to intrinsic characterizations of the subcategory inclusions $\mathrm{K}^{b,-}(\mathcal{X}) \subset \mathrm{K}^-(\mathcal{X}) \subset \mathrm{K}^-(\mathrm{Add}(\mathcal{X}))$ - these imply the claimed restricted triangle equivalence $\mathrm{D}^b(\mathrm{mod}_\infty \mathcal{X}) \to \mathrm{D}^b(\mathrm{mod}_\infty \mathcal{Y})$. Unfortunately, these results are not easy to puzzle together, so we leave this as a conjecture.

Corollary 7.23. *(of Rickard-Lemma) Assume that the Conjecture 7.22 holds. Let* $\mathcal{E}$ *be an exact category. Then for every* n*-tilting and* m*-tilting subcategories* $\mathcal{T}$ *and* $\mathcal{T}'$ *which are small there exists a triangle equivalence* $\mathrm{D}^b(\mathrm{mod}_\infty \mathcal{T}) \to \mathrm{D}^b(\mathrm{mod}_\infty \mathcal{T}')$

Proof. We have the two derived tilting functors

$$\mathrm{D}^b(\mathrm{mod}_\infty \mathcal{T}) \longleftarrow \mathrm{D}^b(\mathcal{E}) \longrightarrow \mathrm{D}^b(\mathrm{mod}_\infty \mathcal{T}')$$

Now, the thick subcategory of $\mathrm{D}^b(\mathcal{E})$ that $\mathcal{T}$ and $\mathcal{T}'$ generate is equal to

$$\mathrm{Thick}_\Delta(\mathcal{T}) = \mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) = \mathrm{Thick}_\Delta(\mathcal{T}')$$

by Lemma 5.8. This implies that derived tilting functors restrict to triangle equivalences

$$\mathrm{K}^b(\mathcal{T}) \longleftarrow \mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) \longrightarrow \mathrm{K}^b(\mathcal{T}')$$

Then the claim follows from the conjecture 7.22. □

Remark 7.24. If we could prove the stronger statement that every triangle equivalence $\mathrm{K}^b(\mathcal{T}) \to \mathrm{K}^b(\mathcal{T}')$ can be extended to a triangle equivalence as in the previous corollary, then we would obtain that the existence of one ideq n-tilting subcategory is equivalent to that all m-tilting are ideq m-tilting for all $m \geq 0$. This extension property for arbitrary triangle equivalences $\mathrm{K}^b(\mathcal{X}) \to \mathrm{K}^b(\mathcal{Y})$ would imply the following conjecture.

Conjecture 7.25. *(Generalization of Thm. 7.14) Let* $\mathcal{E}$ *be an exact category and assume that there exists an* $n \geq 0$ *such that there is at least one* n*-tilting subcategory. Then the following are equivalent:*

(a) *There is an* $n \geq 0$ *and an* n*-tilting subcategory which is ideq* n*-tilting*
(b) *For every* $m \geq 0$ *every* m*-tilting subcategory is ideq* m*-tilting.*

(c) *There is a triangle equivalence* $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{S})$ *for some idempotent complete additive category* $\mathcal{S}$ *which restricts to a triangle equivalence* $\mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) \to \mathrm{K}^b(\mathcal{S})$.

Remark 7.26. If every triangle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\operatorname{mod}_\infty \mathcal{S})$ for some idempotent complete additive category $\mathcal{S}$ restricts to a triangle equivalence $\mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) \to \mathrm{K}^b(\mathcal{S})$, then the previous conjecture says for an exact category with at least one n-tilting subcategory: n-tilting = ideq n-tilting for every $n \geq 0$ is equivalent to $\mathcal{E}$ is bounded derived equivalent to a category $\operatorname{mod}_\infty \mathcal{S}$.

7.4. When is the image of the tilting functor the perpendicular of a cotilting subcategory? Enomoto characterized in [73], Thm 2.4.11 when an exact category $\mathcal{E}$ is equivalent to the perpendicular category ${}^\perp\mathcal{C}$ of an m-cotilting subcategory $\mathcal{C}$ inside a functor category $\operatorname{mod}_\infty \mathcal{P}$ (Clearly a perpendicular category of a cotilting subcategory is a finitely resolving subcategory). Observe that a necessary condition is that such an $\mathcal{E}$ has enough projectives and enough injectives. Here Enomoto's notion of *higher kernels* in an additive category plays a crucial role.

Definition 7.27. (cf. [73], Def. 2.4.5) Let $\mathcal{C}$ be an additive category and $n \geq 1$, then we say that $\mathcal{C}$ has n-**kernels** if for every $f\colon C_1 \to C_0$ in $\mathcal{C}$ there is a complex $0 \to C_{n+1} \to \cdots \to C_2 \to C_1 \xrightarrow{f} C_0$ in $\mathcal{C}$ such that

$$0 \to \mathrm{Hom}_{\mathcal{C}}(-, C_{n+1}) \to \cdots \to \mathrm{Hom}_{\mathcal{C}}(-, C_0)$$

is exact in $\operatorname{mod}_\infty \mathcal{C}$.
If $\mathcal{C}$ is additionally an exact category, we say that $\mathcal{C}$ has 0-**kernels** if every morphism f in $\mathcal{C}$ can be factored as $f = id$ for a deflation d and a monomorphism i. We say $\mathcal{C}$ has (-1)-**kernels** if it is abelian.

Example 7.28. If $\mathcal{T}$ is (n-)tilting and has m-kernels for some $m \geq 1$, then $\mathcal{T}$ is $(m-1)$-ideq (n-)tilting (check the definitions).

Proposition 7.29. *([73], Pro. 2.4.6) Let* $\mathcal{C}$ *be essentially small additive category with weak kernels and* $n \geq 1$. *Then:* $\operatorname{mod}_\infty \mathcal{C} = \mathcal{P}^{\leq n+1}$ *if and only if* $\mathcal{C}$ *has n-kernels.*

Theorem 7.30. *(Enomoto's Theorem, [73], Thm 2.4.11) Let* $\mathcal{E}$ *be an idempotent complete, exact category and* $\mathcal{T}$ *a tilting subcategory and* $m \geq 0$. *We write* $f_{\mathcal{T}}\colon \mathcal{T}^\perp \to \operatorname{mod}_\infty \mathcal{T}$ *for the tilting functor* $X \mapsto \mathrm{Hom}_{\mathcal{E}}(-, X)|_{\mathcal{T}}$. *The following are equivalent:*

1. $\mathcal{T}$ *has weak kernels and is* $(m-1)$*-ideq tilting and* $\mathcal{T}^\perp$ *has enough injectives*
2. *There is an m-cotilting subcategory* $\mathcal{C}$ *in* $\operatorname{mod}_\infty \mathcal{T}$ *with* ${}^\perp\mathcal{C} = \operatorname{Im} f_{\mathcal{T}}$
3. $\mathcal{T}^\perp$ *has enough injectives and* $(m-1)$*-kernels*
4. $\mathcal{T}^\perp$ *has enough injectives and there is a category* $\mathcal{T} \subset \mathcal{M} \subset \mathcal{T}^\perp$ *which has* $(m-1)$*-kernels*

In this case, $\operatorname{Im} f_{\mathcal{T}} = {}^\perp\mathcal{C}$ *with* $\mathcal{C} = f_{\mathcal{T}}(\mathcal{I}(\mathcal{T}^\perp))$ *is m-cotilting.*

All examples that I know of this situation follow from Auslander's notion of a dualizing R-variety.

Definition 7.31. (and Lemma) Let R be a commutative ring such that there is a duality D on finite length R-modules (e.g. if R is a field). Let $\mathcal{A}$ be an additive R-category such that $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a finite length R-module for all X, Y in $\mathcal{A}$. Then, $\mathrm{D}\colon \text{Mod-}\mathcal{A} \to \mathcal{A}\,\mathrm{Mod}, F \mapsto F \circ \mathrm{D}$ is a duality (if $\mathcal{A}$ is essentially small). $\mathcal{A}$ is called a **dualizing R-variety** if the $F \mapsto F \circ D$ defines a duality between finitely presented left and right $\mathcal{A}$-modules, i.e. a contravariant equivalence
$\mathrm{D}\colon \operatorname{mod}_1 \mathcal{A} \to \mathcal{A} \operatorname{mod}_1 \colon \mathrm{D}$.
In this case, $\operatorname{mod}_1 \mathcal{A} = \operatorname{mod}_\infty \mathcal{A}$ is an abelian category with enough injectives and projectives and $\mathcal{A}^{op}$ is also a dualizing R-variety.

Corollary 7.32. *(of Miyashita's and Enomoto's Theorem) Let* $\mathcal{E}$ *be an exact category with enough projectives* $\mathcal{P}$. *Let* $\mathcal{T}$ *be an n-tilting subcategory of* $\mathcal{E}$ *and assume that there is a duality (i.e. contravariant equivalence)* $\mathrm{D}\colon \operatorname{mod}_\infty \mathcal{T} \to \mathcal{T} \operatorname{mod}_\infty \colon \mathrm{D}$. *Let* $\widetilde{\mathcal{T}} = \Psi_{\mathcal{T}}(\mathcal{P}) \subset \mathcal{T} \operatorname{mod}_\infty$ *be the n-tilting subcategory of Thm. 7.13, then* $\mathcal{C} := \mathrm{D}\,\widetilde{\mathcal{T}} \subset \operatorname{mod}_\infty \mathcal{T}$ *is an n-cotilting subcategory and we have*

$${}_\perp\widetilde{\mathcal{T}} = {}^\perp\mathcal{C}$$

7.5. Ideq tilting in relative homological algebra. We look at exact substructures with enough projectives on exact catgeories of the form $\operatorname{mod}_\infty \mathcal{P}$ with $\mathcal{P}$ essentially small. Recall, if $\mathcal{E} = (\mathcal{A}, \mathcal{S})$ is an exact category with underlying additive category $\mathcal{A}$ and class of short exact sequences $\mathcal{S}$, then an **exact substructure** is an exact category $\mathcal{E}' = (\mathcal{A}, \mathcal{S}')$ with $\mathcal{S}' \subset \mathcal{S}$.

We prove the following:

THEOREM 7.33. *Let $\mathcal{P}$ be an idempotent complete, additive category. Let $\mathcal{E}$ be an exact substructure of $\operatorname{mod}_\infty \mathcal{P}$, with enough projectives $\mathcal{Q} := \mathcal{P}(\mathcal{E})$. Then, $\mathcal{Q}$ is 2-ideq 0-tilting subcategory of $\mathcal{E}$.*

As a trivial corollary of Theorem 7.14 and the previous Theorem, we obtain:

Corollary 7.34. *Let $\mathcal{P}$ be an idempotent complete, additive category. Let $\mathcal{E}$ be an exact substructure of $\operatorname{mod}_\infty \mathcal{P}$, with enough projectives $\mathcal{Q} := \mathcal{P}(\mathcal{E})$. Then for every $n \geq 0$, every n-tilting subcategory of $\mathcal{E}$ is ideq n-tilting.*

We prove two lemmata for the proof.

Lemma 7.35. *In the previous situation. The functor $\mathbb{P}\colon \operatorname{mod}_\infty \mathcal{P} \to \operatorname{mod}_\infty \mathcal{Q}, X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{Q}}$ has a left adjoint functor $\Phi'\colon \operatorname{mod}_\infty \mathcal{Q} \to \operatorname{mod}_\infty \mathcal{P}$ given by the restriction functor $\Phi'(X) = X|_{\mathcal{P}}$. Furthermore, Φ' is exact.*

PROOF. We consider $\mathcal{Q} \subset \operatorname{mod}_\infty \mathcal{P} \subset \operatorname{Mod} \mathcal{P}$. Then there is an adjoint pair of functors $\Phi\colon \operatorname{Mod} \mathcal{P} \to \operatorname{Mod} \mathcal{Q} \colon \Phi'$ with $\Phi(X) = \operatorname{Hom}_{\operatorname{Mod} \mathcal{P}}(-,X)|_{\mathcal{Q}}$ (cf. [**171**]). By loc. cit. Cor.3.15, we have for $X \in \operatorname{gen}_\infty^{\operatorname{mod}_\infty \mathcal{P}}(\mathcal{Q}) = \operatorname{mod}_\infty \mathcal{P}$ (by assumption) we have $\mathbb{P}(X) = \Phi(X) \in \operatorname{mod}_\infty \mathcal{Q}$ and $\Phi'(X)(P) = X(P)$. Therefore, the restriction functor is the left adjoint if it is well-defined. We claim: If $X \in \operatorname{mod}_\infty \mathcal{Q}$, then $X|_{\mathcal{P}} \in \operatorname{mod}_\infty \mathcal{P}$.
We apply the restriction functor to a projective resolution of X. This gives a right bounded complex Q_* in $\operatorname{mod}_\infty \mathcal{P}$ with terms in $\mathcal{Q}$ which is quasi-isomorphic to the restricted stalk complex of X. Now, by [**49**], Thm 12.7, there exists a quasi-isomorphism $P_* \to Q_*$ with P_* a right bounded complex of projectives in $\operatorname{mod}_\infty \mathcal{P}$. Since compositions of quasi-isomorphisms are quasi-isomorphisms, the quasi-isomorphism P_* to the stalk complex $X|_{\mathcal{P}}$ gives a projective resolution. □

Lemma 7.36. *Let $\mathcal{E}$ be an idempotent complete exact category with enough projectives given by $\mathcal{Q}$. If the functor $\mathbb{P}\colon \mathcal{E} \to \operatorname{mod}_\infty \mathcal{Q}, X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{Q}}$ has an exact left adjoint, then we have* $\operatorname{Res}_2(\operatorname{Im} \mathbb{P}) = \operatorname{mod}_\infty \mathcal{Q}$.

PROOF. Let $X \in \operatorname{mod}_\infty \mathcal{Q}$. We choose a projective presentation $\operatorname{Hom}_{\mathcal{Q}}(-,Q_1) \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}_{\mathcal{Q}}(-,Q_0) \to X \to 0$ and denote $\Omega^2 = \ker(\operatorname{Hom}(-,f))$. We claim that the unit $\Omega^2 \to \mathbb{P}\Phi'(\Omega^2)$ is an isomorphism (then $\Omega^2 \in \operatorname{Im}(\mathbb{P})$ and since the projectives are in $\operatorname{Im} \mathbb{P}$, the claim follows).
First of all, we observe that any object $Z \in \operatorname{Im}(\mathbb{P})$ fulfills that the unit is an isomorphism on Z - this follows from the triangle identity of the adjunction.
In particular, this holds for the projectives which lie in $\operatorname{Im}(\mathbb{P})$. Since $\mathbb{P} \circ \Phi'$ preserves kernels (since $\mathbb{P}$ preserves kernels and Φ' is exact) we can deduce from the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^2 & \longrightarrow & \operatorname{Hom}_{\mathcal{Q}}(-,Q_1) & \longrightarrow & \operatorname{Hom}_{\mathcal{Q}}(-,Q_0) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{P} \circ \Phi'(\Omega^2) & \longrightarrow & \mathbb{P} \circ \Phi'(\operatorname{Hom}_{\mathcal{Q}}(-,Q_1)) & \longrightarrow & \mathbb{P} \circ \Phi'(\operatorname{Hom}_{\mathcal{Q}}(-,Q_0)) \end{array}$$

that the unit $\Omega^2 \to \mathbb{P} \circ \Phi'(\Omega^2)$ is an isomorphism. □

Example 7.37. In the special case, $\mathcal{A} = \Lambda$-mod for an artin algebra Λ, then the definition of a *relative tilting object* has been given in [**13**], the derived equivalence had been proven in [**45**]. The proof in loc. cit. claims, we have an equivalence of additive categories and therefore, we have an equivalence on $\mathrm{K}^{b,-}(-)$ of those - but this triangulated category depends not just on the additive category but on the ambient exact category and therefore a further explanation for the triangle equivalence (as given here, is helpful for the understanding).

8. Examples

We plan to write an extended separate article on this topic. Therefore, we restrict to very view examples. This is my favourite construction of tilting subcategories:

Example 8.1. (special tilting)

Lemma 8.2. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Let $n \geq 1$ and $\mathcal{M}$ be full self-orthogonal subcategory, closed under summands with $\operatorname{pd}_{\mathcal{E}} \mathcal{M} \leq 1$ and assume that*

$$\mathcal{P} \subset \operatorname{cogen}^{n-1}(\mathcal{M}).$$

Let $\Omega^{-n}_{\mathcal{M}}\mathcal{P}$ denote the full subcategory of $\mathcal{E}$ consisting of objects X such that there exists an exact sequence

$$0 \to P \to M_0 \to \cdots \to M_{n-1} \to X \to 0$$

with $M_i \in \mathcal{M}, P \in \mathcal{P}$ and $\operatorname{Hom}_{\mathcal{E}}(-, \mathcal{M})$ exact on it. We define $\mathcal{T}_n := \mathcal{M} \vee \operatorname{add}(\Omega^{-n}_{\mathcal{M}}\mathcal{P})$. Then $\mathcal{T}_n$ is an n-tilting subcategory and

$$\mathcal{T}_n^{\perp} = \operatorname{gen}_{n-1}(\mathcal{M})$$

Proof. The proof of the dual of Lem. 8.3 in [**136**] also carries through to show (t1), (t2) and (t3) (of Thm 6.3). □

Definition 8.3. Let $\mathcal{E}, \mathcal{M}$ be as in the previous lemma. If $\mathcal{M} \subset \mathcal{P}$, then we call $\mathcal{T}_n = \mathcal{M} \vee \operatorname{add}(\Omega^{-n}_{\mathcal{M}}\mathcal{P})$ the $\mathcal{M}$-**special tilting subcategory**.

Example 8.4. (Tilting modules for infinitely presented modules over rings) Let R be an associative unital ring. We set $R\operatorname{Mod} := \operatorname{Mod}-(\operatorname{proj}-\mathrm{R}), \mathrm{R}\operatorname{mod}_\infty := \operatorname{mod}_\infty -(\operatorname{proj}-\mathrm{R})$ where $\operatorname{proj}-\mathrm{R}$ denotes the category of finitely generated projective left R-modules. This notation is justified by the observation that the category of all left R-modules is equivalent to $\operatorname{Mod}-(\operatorname{proj}-\mathrm{R})$, just consider the following functor

$$R\operatorname{Mod} \longrightarrow \operatorname{Mod}-(\operatorname{proj}-\mathrm{R})$$
$$M \mapsto \big(P \mapsto \operatorname{Hom}_R(P, M)\big)$$

It is an equivalence with quasi-inverse given by $F \mapsto F(R)$. Let now $\mathcal{E} := R\operatorname{mod}_\infty$. This is an exact category with enough projectives. Let T be an object in $\mathcal{E}$ and $\Gamma = \operatorname{End}_{\mathcal{E}}(T)^{op}$. We have $\operatorname{add}(T)$ has weak kernels if and only if Γ is left coherent but we do not need to assume this here.
Then T is n-tilting in $\mathcal{E}$ if and only if it satisfies (t1),(t2) and (t3) (cf. Theorem 6.3). By Thm 7.14 we have an induced equivalence on bounded derived categories $\mathrm{D}^b(R\operatorname{mod}_\infty) \to \mathrm{D}^b(\Gamma\operatorname{mod}_\infty)$. This is also implied by Rickard's Morita theory for derived categories ([**162**], Thm 6.4 and Prop. 8.1).

On endomorphism rings of generators one can always find a special 1-tilt:

Let R be a ring and M be a left R-module and Q be a projective left R-module such that there is an epimorphism $Q^n \to M$ for some $n \geq 1$. Let $E = M \oplus Q$ and $\Gamma = \operatorname{End}_R(E)^{\mathrm{op}}$. Then, $P = \operatorname{Hom}_R(Q, E)$ is a projective right Γ-module.
Take the short exact sequence

$$0 \to K = \ker(f) \to Q^{n+1} \to E \to 0$$

and apply the functor $\operatorname{Hom}_R(-, E)$

$$0 \to \Gamma \xrightarrow{F} P^{n+1} \to T_1 := \operatorname{coker}(F) \to 0.$$

We set $T = P \oplus T_1$. Then, it is straight forward to see:

Corollary 8.5. *$T \in \operatorname{mod}_\infty \Gamma$ is a the special 1-tilting module for $\mathcal{M} = \operatorname{add}(P)$. In particular, $\operatorname{gen}(T) = \operatorname{gen}(P)$.*

CHAPTER 8

An application of tilting theory to infinite type A quiver representations

1. Synopsis

Let Q be an infinite quiver, we look at those whose category of finitely presented quiver representations $\operatorname{rep}^+(Q)$ over a field K is an hereditary abelian category with enough projectives which is Hom- and Ext-finite (i.e. these take values in finite-dimensional K-vector spaces). The easiest class of examples of such infinite quivers are quivers Q which are tree-shaped quivers with finitely many branching points. The abelian categories $\operatorname{rep}^+(Q)$ are always one-sided Auslander-Reiten categories, cf. [**34**]. The purpose of this chapter is to start studying tilting subcategories in $\operatorname{rep}^+(Q)$ for Q an infinite quiver with sufficient finiteness conditions. The obvious first example are quivers of type $\mathbb{A}_\infty$.

We prove the following theorem using tilting theory:

THEOREM 1.1. *(cf. Theorem 4.9) Let Q and Q' be two quivers of type $\mathbb{A}_\infty$. Then, there exists a triangle equivalence*

$$\mathrm{D}^b(\operatorname{rep}^+(Q)) \to \mathrm{D}^b(\operatorname{rep}^+(Q'))$$

if and only if one of the following three cases holds

(a) *Q and Q' have a left infinite path*
(b) *Q and Q' have a right infinite path*
(c) *Q and Q' have no infinite path*

In each of the cases the derived equivalence is obtained by composing at most two derived equivalences induced from a tilting subcategory.

The same question can be asked for the other infinite Dynkin types. We give an idea how to answer this in section 5.

2. Preliminaries

We remark that in our situtation tilting subcategories are maximal self-orthogonal:

Remark 2.1. Let $\mathcal{A}$ be an exact category and assume $\operatorname{gldim} \mathcal{A} < \infty$. Let $\mathcal{T}$ be a tilting subcategory and $\mathcal{T} \subseteq \mathcal{S}$ with $\mathcal{S}$ selforthogonal in $\mathcal{A}$. Then we have $\mathcal{T} = \mathcal{S}$, the proof goes as follows:
Since $\operatorname{Ext}^{>0}(\mathcal{T}, \mathcal{S}) = 0$ implies $\mathcal{S} \subset \mathcal{T}^\perp$ and because $\operatorname{gldim} \mathcal{T}^\perp \leq \operatorname{gldim} \mathcal{A} < \infty$ and $\mathcal{T}^\perp$ wep. given by $\mathcal{T}$ it follows that $\operatorname{Ext}^{>0}(\mathcal{S}, \mathcal{T}^\perp) = 0$ and therefore $\mathcal{S} \subset \mathcal{P}(\mathcal{T}^\perp) = \mathcal{T}$.

Lemma 2.2. *Assume that $\mathcal{T}$ is a 1-tilting subcategory in an abelian category $\mathcal{A}$. Then the category of finitely presented functors $\operatorname{mod}_1 \mathcal{T}$ has enough projectives, i.e. equals $\operatorname{mod}_\infty \mathcal{T}$ and is abelian.*

PROOF. We show first the inclusion $\operatorname{mod}_1 \mathcal{T} \subseteq \operatorname{mod}_\infty \mathcal{T}$: Let $f\colon T_1 \to T_0$ be a morphism in $\mathcal{T}$, we denote by $f'\colon T_1 \to \operatorname{Im} f$ the induced morphism to the image. We have $\operatorname{Im} f \in \operatorname{gen}_\infty \mathcal{T}$ (using $\operatorname{pres} \mathcal{T} = \operatorname{gen} \mathcal{T} = \operatorname{gen}_\infty \mathcal{T}$, using [**172**], Lem 4.4). So we find an exact sequence $T_2' \xrightarrow{g} T_1' \xrightarrow{h} \operatorname{Im} f \to 0$

with $T_i' \in \mathcal{T}$ such that $\operatorname{Hom}_{\mathcal{A}}(T, T_2') \to \operatorname{Hom}_{\mathcal{A}}(T, T_1') \to \operatorname{Hom}_{\mathcal{A}}(T, \operatorname{Im} f) \to 0$ for all T in $\mathcal{T}$. Let us form the pullback of h along f' in the abelian category $\mathcal{A}$

$$\begin{array}{ccccc}
 & & \ker f & = & \ker f \\
 & & \downarrow & & \downarrow i \\
\ker h & \xrightarrow{j} & H & \xrightarrow{\tilde{h}} & T_1 \\
\| & & \downarrow & & \downarrow f' \\
\ker h & \longrightarrow & T_1' & \xrightarrow{h} & \operatorname{Im} f
\end{array}$$

One can observe that the second row is even split exact (but we are not going to use this). Now, we look at the induced morphism $g' \colon T_2' \to \operatorname{Im} g = \ker h$. Using the bicartesian commutative diagram we conclude $f'\tilde{h}jg' = 0$ and so by the universal property of the kernel there is a unique morphism $t \colon T_2 \to \ker f$ such that $it = \tilde{h}jg'$. Now we check that $it \colon T_2' \to T_1$ is a weak kernel of f. For that it suffices to see that for every T in $\mathcal{T}$ the map $\operatorname{Hom}(T, T_2') \twoheadrightarrow \operatorname{Hom}(T, \ker f)$ is surjective. Let T be an object in $\mathcal{T}$ and $s \colon T \to \ker f$ a morphism. Using the bicartesian commuting square, we see that $T \to \ker f \to H \to T_1' \xrightarrow{h} \operatorname{Im} f$ is zero, so it has to factor uniquely over a morphism $T \to \ker h$. But as g' is a right $\mathcal{T}$-approximation of $\ker h$, it follows that there exists a morphism $s' \colon T \to T_2'$ with $ts' = s$.
As every morphism has a weak kernel, the claim follows, cf. e.g. [**126**, Lemma 2.1.6].

□

There is also the following other case when we can conclude that $\operatorname{mod}_\infty \mathcal{T}$ is abelian.

Lemma 2.3. *If $\mathcal{T}$ is a contravariantly finite subcategory of an abelian category $\mathcal{A}$, then $\mathcal{T}$ has weak kernels. In particular $\operatorname{mod}_\infty \mathcal{T} = \operatorname{mod}_1 \mathcal{T}$ is abelian with enough projectives.*

Proof. Let $f \colon T \to T'$ be a morphism in $\mathcal{T}$. Take $T_f \to \ker f$ a right $\mathcal{T}$-approximation. Then, consider the composition $g \colon T_f \to \ker f \to T$. It is straightforward to see that g is a weak kernel of f. □

That a tilting subcategory is contravariantly finite is an extra property. It is equivalent to that we have a torsion class associated to it:

Definition 2.4. A pair $(\mathcal{R}, \mathcal{F})$ of full subcategories in an abelian category $\mathcal{A}$ is a torsion pair if:

(TP1) $\operatorname{Hom}(R, F) = 0$ for all $R \in \mathcal{R}, F \in \mathcal{F}$,
(TP2) For each $Z \in \mathcal{A}$ exists a short exact sequence $0 \to X \to Z \to Y \to 0$ with $X \in \mathcal{R}, Y \in \mathcal{F}$.

We recall the following

Corollary 2.5. *(of [**36**, Prop.1.2]) If $\mathcal{R}$ is a torsion class in an abelian category $\mathcal{A}$, then the $\mathcal{R}$ is contravariantly finite in $\mathcal{A}$.*

Proof. By [**36**, Prop. 1.2], the inclusion $i \colon \mathcal{R} \to \mathcal{A}$ has a right adjoint $R \colon \mathcal{A} \to \mathcal{R}$. Then, for $Z \in \mathcal{A}$, the counit $\epsilon_Z \colon iR(Z) \to Z$ of the adjunction is a right $\mathcal{R}$-approximation. □

Lemma 2.6. *Let $\mathcal{T}$ be a 1-tilting subcategory in an abelian category. Then the following are equivalent:*

(1) *$\mathcal{T}^\perp$ is a torsion-class.*
(2) *$\mathcal{T}^\perp$ is contravariantly finite in $\mathcal{A}$.*
(3) *$\mathcal{T}$ is contravariantly finite in $\mathcal{A}$.*

Proof. Assume $\mathcal{T}^{\perp}$ is a torsion class, then $\mathcal{T}^{\perp}$ is contravariantly finite in $\mathcal{A}$ by the previous lemma. Since $\mathcal{T}$ is contravariantly finite in $\mathcal{T}^{\perp}$ (since $\mathcal{T}^{\perp}$ has enough projectives given by $\mathcal{T}$) it follows that $\mathcal{T}$ is contravariantly finite in $\mathcal{A}$. So we have (1) implies (2) implies (3)
Conversely, assume (3), i.e. that $\mathcal{T}$ is contravariantly finite in $\mathcal{A}$. Define $\mathcal{R} = \mathcal{T}^{\perp}$ and $\mathcal{F} = \{F \in \mathcal{A} \mid \operatorname{Hom}(T, F) = 0 \; \forall \, T \in \mathcal{T}\}$. Let $R \in \mathcal{R}, F \in \mathcal{F}$ and $f \colon R \to F$ be a morphism. By definition, there exists an epimorphism $p \colon T \to R$ with $T \in \mathcal{T}$ and $f \circ p = 0$. This implies $f = 0$ and (TP1).
Now let $Z \in \mathcal{A}$ be arbitrary. By assumption, there exists a right $\mathcal{T}$-approximation $f_Z \colon T_Z \to Z$, in particular $X = \operatorname{Im}(f_Z) \in \mathcal{R}$. Let $Y := \operatorname{coker}(f_Z)$. We consider the short exact sequence $0 \to X \xrightarrow{j} Z \to Y \to 0$ and apply $\operatorname{Hom}(T, -)$ with $T \in \mathcal{T}$. We look at $\operatorname{Hom}(T, X) \to \operatorname{Hom}(T, Z)$ and want to see that this map is surjective. So, given $g \colon T \to Z$, we use that there is an $h \colon T \to T_Z$ such that $f_Z \circ h = g$. Since $f_Z \colon T_Z \xrightarrow{q} X \xrightarrow{j} Z$ factors over its image as $f_Z = j \circ q$, it follows $g = j(qh)$ and therefore $\operatorname{Hom}(T, X) \to \operatorname{Hom}(T, Z)$ is surjective. This implies that $Y \in \mathcal{F}$ and therefore (TP2). □

Definition 2.7. We say that an object X in $\mathcal{A}$ is **noetherian** if it satisfies the ascending chain (acc) condition, i.e. whenever there is a chain of subobjects of X

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$

then it eventually stabilizes.
Let $\mathcal{C}$ be a small category, then one says $\operatorname{Mod} \mathcal{C}$ (also denoted by $\operatorname{Rep} \mathcal{C}$) is **noetherian** if every finitely generated $\mathcal{C}$-module X fulfills the (acc) for chains of finitely generated submodules.

Lemma 2.8. *Let k be a field. Let $\mathcal{A}$ be an abelian Hom-finite k-category and $\mathcal{T}$ a 1-tilting subcategory with countably many indecomposables in it. If every object in $\mathcal{A}$ is noetherian, then $\mathcal{T}$ is contravariantly finite in $\mathcal{A}$.*

Proof. Let X be in $\mathcal{A}$. We choose a numbering of the indecomposables of $\mathcal{T}$, e.g. $T_n, n \in \mathbb{N}$. We define $X_n \subseteq X$ to be $\sum_{f \in \operatorname{Hom}_{\mathcal{A}}(\bigoplus_{i=1}^n T_i, X)} \operatorname{Im} f$. By assumption there is an $N \in \mathbb{N}$ such that $X_N = X_n$ for all $n \geq N$. This means that every morphism $T \to X$, $T \in \mathcal{T}$ must factor over X_N. We observe $X_N \in \operatorname{pres}(\mathcal{T}) = \operatorname{gen}(\mathcal{T}) = \mathcal{T}^{\perp}$. Since $\mathcal{T}^{\perp}$ is an exact category with enough projectives given by $\mathcal{T}$ there is a *projective cover* $T \to X_N$, with $T \in \mathcal{T}$ and this is clearly a right $\mathcal{T}$-approximation. □

3. Representations of strongly locally finite infinite quivers

Here we follow [**34**]. We fix a **strongly locally finite quiver** Q, this means every vertex has finitely many arrows arriving and starting at it and for every two (possibly equal) vertices there are only finitely many paths from one to the other.
A representation (over an always fixed field K) of Q is an assignment of a K-vector space V_i to every vertex $i \in Q_0$ and a linear map $V_i \to V_j$ to every arrow $a \colon i \to j$ in Q_1. This defines the objects in an abelian category $\operatorname{Rep}(Q)$. For every vertex $x \in Q_0$ one defines a Q-representation P_x with top S_x (the one-dimensional representation supported at x) and let $\mathcal{P} := \operatorname{proj} Q = \operatorname{add}\{P_x \colon x \in Q_0\}$ be the category of finitely generated projectives in $\operatorname{Rep}(Q)$. A quiver is called **noetherian** if every object in $\mathcal{P}$ defined as above satisfies the ascending chain property (definition of [**69**]). In [**127**] this is called left noetherian and it is shown that this implies that $\operatorname{Rep}(Q)$ is a locally noetherian abelian category in loc. cit. Theorem 1.1.

Lemma 3.1. *Let Q be an infinite quiver. If the underlying graph is a (possibly infinite) tree with only finitely many branching points, then the quiver is noetherian.*

Proof. Since Q is a strongly locally finite quiver and its underlying graph is a tree with only finitely many branch points, it follows that the graph of $P(Q)_v$ has finitely many branch points for any $v \in Q$. We get that $P(Q)_v$ is barren (in the sense of [**69**]) and thus Q is notherian by [**69**, Theorem 3.6]. □

We define

$$\mathrm{rep}^+(Q) := \mathrm{mod}_1\, \mathcal{P}$$

as the finitely presented Q-representations, this is an extension-closed subcategory of $\mathrm{Rep}(Q)$.

Lemma 3.2. *([34, Lem 1.14]) Let K be a field. Then $\mathrm{rep}^+(Q)$ is hereditary abelian and is a Hom-finite K-category with finite-dimensional Ext^1-groups.*

Corollary 3.3. *In particular $\mathcal{P}$ has weak kernels and $\mathrm{rep}^+(Q) = \mathrm{mod}_\infty\, \mathcal{P}$ has enough projectives which are given by $\mathcal{P}$.*

Following [34, Cor 2.2] an abelian Krull-Schmidt category $\mathcal{C}$ is a **right Auslander-Reiten category** if every indecomposable non-projective is ending term of an almost split sequence and all indecomposable projectives have a simple top. It is called a **left Auslander-Reiten category** if its opposite is a right Auslander-Reiten category.
We call a quiver of the form $\bullet \to \bullet \to \cdots$ a **right infinite path** and its opposite a **left infinite path**. We call a quiver $\cdots \to \bullet \to \bullet \to \cdots$ a **double infinite path**.

Theorem 3.4. *([34], Thm 3.7, Cor. 3.8) Let Q be a strongly locally finite quiver. Then*

(1) $\mathrm{rep}^+(Q)$ *is left Auslander-Reiten if and only if Q has no right infinite path.*
(2) $\mathrm{rep}^+(Q)$ *is right Auslander-Reiten if and only if Q has no left infinite path or else Q is a left infinite path or double infinite path.*
(3) $\mathrm{rep}^+(Q)$ *is Auslander-Reiten if and only if Q has no infinite path or Q is a left infinite path.*

Observe that: Q having no right infinite path is equivalent to $\mathrm{rep}^+(Q)$ coinciding with the category of finite dimensional Q-representation (i.e. Q-representations V such that $\dim_K \bigoplus_{i\in Q_0} V_i < \infty$).

We will also use the following definition.

Definition 3.5. Let $\mathcal{A}$ be a left (and/or right) abelian Auslander-Reiten category. A **weak slice** in $\mathcal{A}$ is an additively closed subcategory $\mathcal{X}$ such that the indecomposables in $\mathcal{X}$ fulfill the following:

(1) The indecomposables in $\mathcal{X}$ are all in the same component of the Auslander-Reiten quiver and they are a full representing system of the $\tau^{\pm}$-orbits.
(2) The full subquiver of the Auslander-Reiten quiver defined by the indecomposables of $\mathcal{X}$ is path-closed (i.e. if there is a path given by a sequence of arrows $X_1 \to X_2 \to \cdots \to X_n$ in the Auslander-Reiten quiver with X_1, X_n in $\mathcal{X}$, then all X_i are in $\mathcal{X}$).
(3) Given an almost split sequence $M \rightarrowtail L \twoheadrightarrow N$ with one summand of L in $\mathcal{X}$, then either M or N are in $\mathcal{X}$.

We say a weak slice is a **slice** if it defines a 1-tilting subcategory of $\mathcal{A}$.

Ringel showed in [159], section 4.2: If $\mathcal{A}$ is also hereditary exact and $\mathcal{X}$ is a slice then the subcategory $\mathcal{X}$ is selforthogonal (i.e. there are no non-split n-extensions between any two objects in $\mathcal{X}$ for all $n \geq 1$).

3.1. Reflection at a set of sinks. Let Q be a strongly locally finite quiver and $\mathbf{a} \subset Q_0$ a subset consisting of sinks (this can be a single vertex or it may also be an infinite set). We write $\mu_{\mathbf{a}}(Q)$ for the quiver (Q_0', Q_1') with $Q_0' = Q_0$ and
$Q_1' = \{\alpha\colon i \to j \mid j \notin \mathbf{a}\} \cup \{\alpha^*\colon a \to i \mid \alpha\colon i \to a, a \in \mathbf{a}\}$.
To distinguish the finitely generated projective $\mu_{\mathbf{a}}(Q)$-representations from those for Q, we denote them by $\overline{P}_x$, $x \in (\mu_{\mathbf{a}}(Q))_0 = Q_0$.
We define the reflection functor

$$\mathbb{S}_{\mathbf{a}}\colon \mathrm{Rep}(Q) \to \mathrm{Rep}(\mu_{\mathbf{a}}(Q))$$

as follows, for a Q representation M and every $a \in \mathbf{a}$ we have a linear map $M_\alpha \colon M_i \to M_a$ for every $\alpha \colon i \to a$. This induces a linear map from the direct sum

$$0 \to N_a \to \bigoplus_{\alpha \in Q_1 \colon \alpha \colon i \to a} M_i \to M_a$$

where we call N_a the kernel of this map. We define $(\mathbb{S}_{\mathbf{a}}(M))_x = M_x$ for $x \notin \mathbf{a}$ and $(\mathbb{S}_{\mathbf{a}}(M))_a = N_a$ for every $a \in \mathbf{a}$. On all arrows α in Q_1' not ending at an $a \in \mathbf{a}$, we define $(\mathbb{S}_{\mathbf{a}}(M))_\alpha = M_\alpha$. An arrow $\alpha^* \colon a \to i$ in Q_1' with $a \in \mathbf{a}$, corresponds by definition to an arrow $\alpha \colon i \to a$ in Q_1, therefore we we can define $\mathbb{S}_{\mathbf{a}}(M)_{\alpha^*} \colon N_a \to \bigoplus_{\beta \colon j \to a} M_j \xrightarrow{pr_\alpha} M_i$.
It is clear that this functor restricts to finite-dimensional quiver representations. It is not immediately clear that $\mathbb{S}_{\mathbf{a}}$ would restrict to the subcategory of finitely represented quiver representations.
We look at the special tilting subcategory in $\operatorname{rep}^+(Q)$ with respect to $\mathcal{M} = \operatorname{add}\{\mathcal{P}_x \mid x \notin \mathbf{a}\}$. For $a \in \mathbf{a}$, the following is the $\mathcal{M}$-approximation of P_a

$$0 \to P_a \to \bigoplus_{\alpha \in Q_1 \colon \alpha \colon i \to a} P_i.$$

Let R_a be the cokernel, $\mathcal{T}_{\mathbf{a}} = \mathcal{M} \vee \operatorname{add}\{R_a \mid a \in \mathbf{a}\}$. Observe that:

Lemma 3.6. *The reflection functor restricts to an equivalence of categories*

$$\mathcal{T}_{\boldsymbol{a}} \to \mathcal{P}(\operatorname{rep}^+(\mu_{\boldsymbol{a}}(Q)))$$

mapping $P_x \mapsto \overline{P}_x$ *for* $x \notin \boldsymbol{a}$ *and* $R_a \mapsto \overline{P}_a$ *for* $a \in \boldsymbol{a}$.
We have the following commutative diagram

$$\begin{array}{ccc} \operatorname{Rep}(Q) & \xrightarrow{\mathbb{S}_{\boldsymbol{a}}} & \operatorname{Rep}(\mu_{\boldsymbol{a}}(Q)) \\ & \searrow^{\Phi} & \uparrow \cong \\ & & \operatorname{Mod} \mathcal{T}_{\boldsymbol{a}} \end{array}$$

with $\Phi(M) = \operatorname{Hom}_{\operatorname{Rep}(Q)}(-, M)|_{\mathcal{T}_a}$.

In particular, we can restrict $\mathbb{S}_{\mathbf{a}}$ to $\mathcal{T}_{\mathbf{a}}^\perp = \operatorname{gen} \mathcal{T}_{\mathbf{a}} \subseteq \operatorname{rep}^+(Q)$, i.e. to a functor

$$\mathbb{S}_{\mathbf{a}} \colon \mathcal{T}_{\mathbf{a}}^\perp \to \operatorname{mod}_\infty - \mathcal{T}_a \cong \operatorname{rep}^+(\mu_a(Q))$$

But every indecomposable not in $\operatorname{gen} \mathcal{T}_{\mathbf{a}}$ is a simple S_a, $a \in \mathbf{a}$ and $\mathbb{S}_{\mathbf{a}}(S_a) = 0$ which is finitely presented. Therefore, we have a well-defined reflection functor

$$\mathbb{S}_{\mathbf{a}} \colon \operatorname{rep}^+(Q) \to \operatorname{rep}^+(\mu_{\mathbf{a}}(Q))$$

which can be identified with the *tilting functor* of the tilting category $\mathcal{T}_{\mathbf{a}}$.

Proof. By definition we have that $\mathbb{S}_{\mathbf{a}}(P_x) = \overline{P}_x$ for $x \notin \mathbf{a}$, and $\mathbb{S}_a(R_a) = \overline{P}_a$ for $a \in \mathbf{a}$.
As both categories are Krull-Schmidt categories, it is enough to show that $\mathbb{S}_{\mathbf{a}}$ induces isomorphisms on Hom-spaces of indecomposables. For M in $\mathcal{T}_a$, $x \notin \mathbf{a}$ we have natural isomorphisms

$$\operatorname{Hom}_{\operatorname{rep}^+(Q)}(P_x, M) \cong M_x \cong \operatorname{Hom}_{\operatorname{rep}^+(\mu_{\mathbf{a}}(Q))}(\mathbb{S}_{\mathbf{a}}(P_x), \mathbb{S}_{\mathbf{a}}(M))$$

For $x = a \in \mathbf{a}$, we apply $\operatorname{Hom}_{\operatorname{rep}^+(Q)}(-, M)$ to the short exact sequence
$0 \to P_a \to \bigoplus_{\alpha \colon i \to a} P_i \to R_a \to 0$

$$0 \to \operatorname{Hom}(R_a, M) \to \bigoplus_{\alpha \colon i \to a} \operatorname{Hom}(P_i, M) \to \operatorname{Hom}(P_a, M)$$

Now, by the first natural isomorphism this left exact sequence identifies naturally with

$$0 \to N_a \to \bigoplus_{\alpha \colon i \to a} M_i \to M_a$$

In particular $\operatorname{Hom}_Q(R_a, M) \cong N_a \cong \operatorname{Hom}_{\mu_a(Q)}(\overline{P}_a, \mathbb{S}_{\mathbf{a}}(M))$. □

Corollary 3.7. *In particular, the reflection functor induces a triangle equivalence on the bounded derived categories*

$$\boldsymbol{S}_{\boldsymbol{a}}^{+}\colon \mathrm{D}^b(\mathrm{rep}^+(Q)) \to \mathrm{D}^b(\mathrm{rep}^+(\mu_{\boldsymbol{a}}(Q)).$$

We write $\boldsymbol{S}_{\boldsymbol{a}}^{-}$ *for the quasi-inverse of* $\boldsymbol{S}_{\boldsymbol{a}}^{+}$*.*

Here $\mathbf{S}_{\mathbf{a}}^{-}$ can be constructed dually using the special cotilting subcategory associated to the set of sources.

4. Representations of infinite quivers of type A-infinity

We call orientations of the following graph

$$1 \text{———} 2 \text{———} 3 \text{———} \cdots$$

quivers of type $\mathbb{A}_\infty$. For every such quiver, $a \leq b$ in $\mathbb{N}$, we define the interval module $E_{a,b}$ as the indecomposable module with dimension vector $(\underline{\dim}\, E_{a,b})_i = 1$ if $a \leq i \leq b$ and zero else.

4.1. A-infinity.

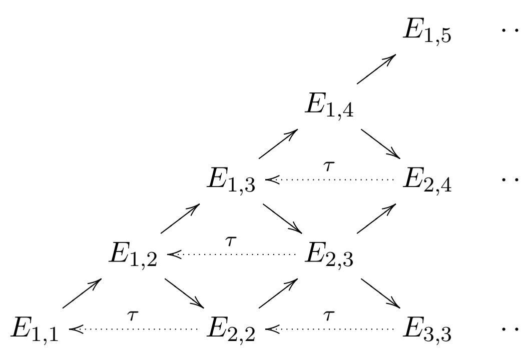
4.1.1. *Left infinite path.* We first look at the following infinite quiver Q

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots$$

Let $\mathcal{A} = \mathrm{rep}^+(Q)$ be the category of finite-dimensional Q-representations. All indecomposables are finite-dimensional interval modules, with projectives

$$P_n = E_{1,n}, \quad n \in \mathbb{N}, \quad \text{set } \mathcal{P} := \mathcal{P}(\mathcal{A}) = \mathrm{add}\{P_n \mid n \in \mathbb{N}\}.$$

In this case, we have an Auslander-Reiten category with Auslander-Reiten quiver



Let $I = [a, b]$ with $a \leq b$ be an intervall in $\mathbb{N}$. We define $\mathcal{C}_I$ to be the full additive subcategory given by objects whose composition factors are simples $S_i, a \leq i \leq b$. Alternatively, $\mathcal{C}_I = \mathrm{add}\{E_{ij} \mid a \leq i \leq j \leq b\}$. This is a fully exact subcategory of $\mathcal{A}$ which is deflation-closed and inflation-closed and even a Serre subcategory. Furthermore, it is an abelian subcategory with enough injectives (indeomposable injectives: $E_{i,b}, a \leq i \leq b$), with enough projectives (indecomposable projectives: $E_{a,j}, a \leq j \leq b$) and unique indecomposble projective injective $E_{a,b}$. It is obvious that $\mathcal{C}_I$ is equivalent to the quiver representations of the full subquiver (of Q) with vertices I. This is a linear oriented quiver of type A.
Furthermore, every tilting subcategory fulfills $\mathcal{T}^\perp = \mathcal{P}^{<\infty}(\mathcal{T}^\perp)$ since $\mathcal{A} = \mathcal{P}^{<\infty}(\mathcal{A})$ by [**172**, Lemma 5.7].

Proposition 4.1. *Let* $\mathcal{A}$ *be the exact category described before. Let* $\mathcal{T}$ *be full additive subcategory in* $\mathcal{A}$ *closed under summands. The following are equivalent:*

(1) $\mathcal{T}$ *is a* 1*-tilting subcategory.*
(2) $|\mathcal{T} \cap \mathcal{P}| = \infty$ *and for every indecomposable* $E_{1,n} \in \mathcal{T}$ *we have that* $\mathcal{T} \cap \mathcal{C}_{[1,n]}$ *is a tilting subcategory of* $\mathcal{C}_{[1,n]}$*.*

(2') $|\mathcal{T} \cap \mathcal{P}| = \infty$ *and for every indecomposable* $E_{a,b} \in \mathcal{T}$ *we have that* $\mathcal{T} \cap \mathcal{C}_{[a,b]}$ *is a tilting subcategory of* $\mathcal{C}_{[a,b]}$.

Proof. (1) implies (2): Assume that $\mathcal{T}$ is 1-tilting (i.e. selforthogonal and $\mathcal{P} \subset \mathrm{Cores}_1(\mathcal{T})$). Since $\mathcal{P} \subset \mathrm{copres}(\mathcal{T})$, it follows that $\mathcal{T}$ contains infinitely many indecomposable projectives. First we look at projectives $E_{1,n} \in \mathcal{T}$. Clearly $\mathcal{T} \cap \mathcal{C}_{[1,n]}$ is selforthogonal in $\mathcal{C}_{[1,n]}$. The inclusion $\mathcal{P} \subset \mathrm{Cores}_1(\mathcal{T})$ implies for $P = E_{1,m}$ with $m \leq n$ that there is an exact sequence $P \rightarrowtail T_0 \twoheadrightarrow T_1$ in $\mathcal{A}$ with $T_i \in \mathcal{T}$. It is easy to see that $\mathrm{Hom}_{\mathcal{A}}(-, T)$ is exact on it for every $T \in \mathcal{T}$. This means, we can choose a minimal left $\mathcal{T}$-approximation $f \colon P \to T_0'$. Let $m < m' \leq n$ be minimal with $E_{1,m'} \in \mathcal{T} \cap \mathcal{C}_{[1,n]}$, then clearly $E_{1,m'} \in \mathrm{add}(T_0')$ and every other morphism $P \to T$ with $T \in \mathcal{T}$ indecomposable, $T \notin \mathcal{C}_{[1,n]}$, must factor over $P \to E_{1,m'}$. This implies that $T_0' \in \mathcal{C}_{[1,n]}$. Let $L := \operatorname{coker} f$, then we have an induced monomorphism $L \to T_1$, this means $L \in \mathrm{pres}(\mathcal{T}) \cap \mathrm{copres}(\mathcal{T}) = \mathcal{T}^{\perp} \cap {}^{\perp}\mathcal{T} = \mathcal{T}$ since $\mathcal{T}^{\perp} = \mathcal{P}^{<\infty}$ (so: being left perpendicular on the projectives in $\mathcal{T}^{\perp}$ implies being projective). Since $\mathcal{C}_{[1,n]}$ is closed under quotients, $L \in \mathcal{T} \cap \mathcal{C}_{[1,n]}$.
(2) implies (2'): For general $E_{a,b} \in \mathcal{T}$ the claim follows since one can find a projective $E_{1,n} \in \mathcal{T}$ with $\mathcal{C}_{a,b} \subset \mathcal{C}_{1,n}$ and since this is well-known for tilting modules over linear oriented A_n quivers (since restrictions of binary trees to a full subtree starting at branching point are binary trees).
(2') implies (1): For any two indecomposable summands X, Y in $\mathcal{T}$ there exists a projective $E_{a,n} \in \mathcal{T}$ such that $X, Y \in \mathcal{C}_{[1,n]}$. By assumption and since $\mathcal{C}_{[1,n]}$ is extension-closed, it follows that $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$. Furthermore, for every projective $P = E_{1,m} \notin \mathcal{T}$ there is a projective $E_{1,n} \in \mathcal{T}$ with $E_{1,j} \notin \mathcal{T}$ for $m < j < n$. Since $\mathcal{T} \cap \mathcal{C}_{[1,n]}$ is tilting in $\mathcal{C}_{[1,n]}$ we have an exact sequence $P \rightarrowtail T_0 \twoheadrightarrow T_1$ for a $T_i \in \mathcal{C}_{[1,n]} \cap \mathcal{T}$. □

Remark 4.2. From the previous result it follows that weak slices only give tilting subcategories if they contain infinitely many indecomposable projectives. (As in the representation finite case, they are precisely the tilting subcategories with $\mathrm{gldim}\, \mathrm{mod}_\infty \mathcal{T} = 1$, the remaining ones fulfill $\mathrm{gldim}\, \mathrm{mod}_\infty \mathcal{T} = 2$.)

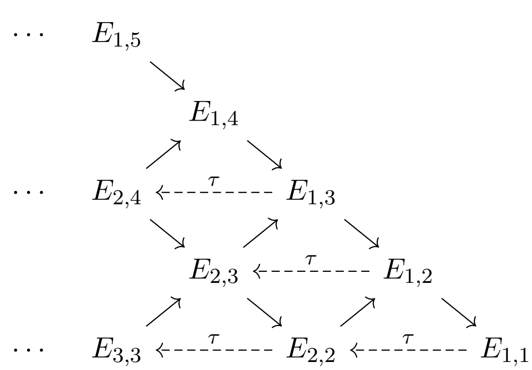
4.1.2. *Right infinite path.* Let Q be the following quiver

$$1 \to 2 \to 3 \to \cdots$$

Its Auslander-Reiten quiver has two components: the preprojective consisting only of the (infinite dimensional) projectives

$$\cdots \to P_3 \to P_2 \to P_1$$

and another component consisiting of all finite-dimensional modules.



Lemma 4.3. *Let* $\mathcal{T}$ *be a tilting subcategory. Then:*

(0) $P_1 \in \mathcal{T}$
(1) *For every* $E_{ij} \in \mathcal{T}$ *we have that* $\mathcal{T} \cap \mathcal{C}_{[i,j]}$ *is tilting in* $\mathcal{C}_{[i,j]}$.
(2) *For* $P_n \in \mathcal{T}$ *we have* $\mathcal{T} \cap \mathcal{C}_{\geq n}$ *is tilting in* $\mathcal{C}_{\geq n}$.
(3) *If* $\mathrm{gen}(P_i) \cap \mathcal{T}$ *contains infinitely many indecomposables, then* $P_i \in \mathcal{T}$ *and* $P_\ell \notin \mathcal{T}$ *for all* $\ell > i$.

(4) *Let $E_{ij} \in \mathcal{T}$. If $P_i \notin \mathcal{T}$ then there is an E_{aj}, $a < i$ or an E_{ib}, $b > j$ in $\mathcal{T}$.*

Proof. (0) As $\mathcal{T}$ is tilting there has to exist an exact sequence $P_1 \rightarrowtail T_0 \twoheadrightarrow T_1$ with $T_i \in \mathcal{T}$. Then we find an inflation $P_1 \rightarrowtail T_1'$ where $T_1' \in \operatorname{add}(T_1)$ is the summand with top supported at $S_1 = E_{1,1}$. As P_1 is infinite-dimensional, at least one summand of T_1' has to be P_1 itself.

(1) Clearly $\operatorname{add}(T) := \mathcal{T} \cap \mathcal{C}_{[i,j]}$ is still selforthogonal. The indecomposable projectives in $\mathcal{C}_{[i,j]}$ are E_{it}, $t \leq j$ and by assumption, the projective-injective $E_{ij} \in \operatorname{add}(T)$. We only show that $E_{it} \in \operatorname{Cores}_1(T)$ for every $t < j$. There exist only finitely many indecomposables modules $E_{a,b}$ with $\operatorname{Hom}(E_{it}, E_{ab}) \neq 0$. Let $\mathcal{Z} := \operatorname{add}\{E_{ab} \mid \operatorname{Hom}(E_{it}, E_{ab}) \neq 0\}$, then $\mathcal{T}_{\mathcal{Z}} := \mathcal{T} \cap \mathcal{Z}$ contains only finitely many indecomposables, so there is a minimal left $\mathcal{T}_{\mathcal{Z}}$-approximation $f\colon E_{it} \to T_0$ with $T_0 \in \mathcal{T}_{\mathcal{Z}}$. Since we have a monomorphism $E_{it} \to E_{ij}$ and $E_{ij} \in \mathcal{T}_{\mathcal{Z}}$, it follows that $T_0 \in \mathcal{C}_{[i,j]}$ and f is a monomorphism. Let $R := \operatorname{coker} f$, since $\mathcal{C}_{[ij]}$ is a wide subcategory it follows that $R \in \mathcal{C}_{[ij]}$. Then we have for every $T \in \mathcal{T}_Z$ an exact sequence

$$\operatorname{Hom}(R,T) \rightarrowtail \operatorname{Hom}(T_0,T) \twoheadrightarrow \operatorname{Hom}(E_{it},T)$$

If we have an indecomposable $T \in \mathcal{T}, T \notin \mathcal{T}_Z$, then it holds $\operatorname{Hom}(E_{it}, T) = 0$. It follows that $R \in {}^{\perp}\mathcal{T}$ and also that $R \in \mathcal{T}^{\perp}$ (the last inclusion can be seen by applying $\operatorname{Hom}(T,-)$ to the short exact sequence $E_{it} \rightarrowtail T_0 \twoheadrightarrow R$). Now, $\mathcal{T}^{\perp}$ is a full exact subcategory with enough projectives given by $\mathcal{T}$ itself. So, for every $Y \in \mathcal{T}^{\perp}$ there is an exact sequence

$$T^1 \rightarrowtail T^0 \twoheadrightarrow Y$$

with $T^i \in \mathcal{T}$, by applying $\operatorname{Hom}(R,-)$ one concludes $\operatorname{Ext}^1(R,Y) = 0$ and therefore $R \in \mathcal{P}(\mathcal{T}^{\perp}) = \mathcal{T}$.

(2) For $m > n$ we have an exact sequence $0 \to P_m \to T_0 \to T_1 \to 0$ with $T_i \in \mathcal{T}$.
Every non-zero homomorphism $P_m \to E_{ij}$ or $P_m \to P_i$ with $i < n$ factors through $P_m \to P_n$. By leaving out the summands we find a left $\mathcal{T}$-approximation $0 \to P_m \to T_0' \to R \to 0$. As in (1) we conclude that $R \in \mathcal{T}$.

(3) Assume that $P_i \notin \mathcal{T}$, then $i > 1$. There exists a short exact sequence $P_i \rightarrowtail T_0 \twoheadrightarrow T_1$ with $T_i \in \mathcal{T}$. As P_i is infinite dimensional there exists an $j < i$ such that $P_j \in \operatorname{add}(T_0)$ and we also assume that there is a summand $E_{jk} \in \operatorname{add}(T_1)$. By assumption there exists $E_{is} \in \mathcal{T}$ with $s > k$. Then $E_{is} \rightarrowtail E_{ik} \oplus E_{js} \twoheadrightarrow E_{jk}$ is a non-split short exact sequence contradicting $\mathcal{T}$ being selforthogonal. This shows $P_i \in \mathcal{T}$. Assume $P_\ell \in \mathcal{T}$ with $\ell > i$, there exists $E_{is} \in \mathcal{T}$ with $s \geq \ell$ but then $\operatorname{Ext}^1(E_{is}, P_\ell) \neq 0$ contradicting $\mathcal{T}$ being selforthogonal.

(4) Observe that for $E_{ab}, E_{cd} \in \mathcal{T}$ we have two binary trees, one in $\mathcal{C}_{[a,b]}$ and one in $\mathcal{C}_{[c,d]}$ then these intervals $[a,b]$ and $[c,d]$ have to be either one contained in the other or they have to be disjoint and if $a \leq b < c \leq d$, then $c - b > 1$. In all other cases we find a non-split extension between E_{ab} and E_{cd}.
Take $E_{ij} \in \mathcal{T}$ and assume that $E_{ab} \notin \mathcal{T}$ for all $[i,j] \subseteq [a,b]$ (this means that $\mathcal{C}_{[i,j]} \cap \mathcal{T}$ is a not a proper subtree of a $\mathcal{C}_{[a,b]} \cap \mathcal{T}$). Then we want to see that $P_i \in \mathcal{T}$. As $\mathcal{T}$ is maximal selforthogonal, it is enough to see that $\operatorname{Ext}^1(\mathcal{T}, P_i) = 0$. For that we look at the indecomposables E_{ts} with $t < i \leq s$ and we need to see that $E_{ts} \notin \mathcal{T}$. But as we remarked before, the next of these binary trees on the right has at least one distance from this one, this implies the claim.

□

Definition 4.4. Let $\Gamma_0 = \Gamma_0^p \cup \Gamma_0^f$ be the set of vertices of the Auslander-Reiten quiver where Γ_0^p denotes the vertices corresponding to the projectives and Γ_0^f the vertices corresponding to the finite-dimensional modules.
A **binary tree** on Γ_0 consists of $\mathbb{T} \cup \mathbb{P}$ with $\mathbb{T} \subset \Gamma_0^f, \mathbb{P} \subset \Gamma_0^p$ such that:

(i) $P_1 \in \mathbb{P}$.
(ii) If there are infinitely many $E_{ij_n} \in \mathbb{T}$, $n \in \mathbb{N}$ then $P_i \in \mathbb{P}$ and $P_\ell \notin \mathbb{P}$ for all $\ell > i$.
(iii) For every $E_{ij} \in \mathbb{T}$ is $\mathcal{C}_{ij} \cap \mathbb{T}$ a binary tree on the Auslander-Reiten quiver of $\mathcal{C}_{ij}$ in the sense of Hille and

either $P_i \in \mathbb{P}$, or
there is an $E_{aj}, a < i$ or an $E_{ib}, b > j$ in $\mathbb{T}$.
If also $E_{ts} \in \mathbb{T}$, then
either $E_{ab} \in \mathbb{T}$ with $a = \min(t, i), b = \max(s, j)$, or
$[t, s] \cap [i, j] = \emptyset$ and $|t - j| \geq 2$.

(iv) Given $\ell \in \mathbb{N}$ assume there is no $t < \ell \leq s$ with $E_{ts} \in \mathbb{T}$ then $P_\ell \in \mathbb{P}$.

Remark 4.5. Given the set $\mathbb{T}$ fulfilling (iii), there is always a unique set $\mathbb{P}$ defined by the properties (i)-(iv), such that the union is a binary tree.

Remark 4.6. As an indexing set we take always $N \subseteq \mathbb{N}$ with $N = \emptyset$ or $N = [1, n]$ or $N = \mathbb{N}$. We have a two types of binary trees:

(a) If $\mathbb{P}$ is infinite, then we have an indexing set N and sequence of pairwise disjoint intervals $i_n \leq j_n$, $n \in N$ with $j_n < i_{n+1} - 1$ such that $\mathcal{C}_{i_n j_n} \cap \mathbb{T}$ is a binary tree and every $\mathbb{T}$ is the union of these.
(b) If $\mathbb{P}$ is finite with $i = \max\{a \mid P_a \in \mathbb{P}\}$, then there is a finite indexing set N and a sequence $i_n \leq j_n < i - 1$, $j_n < i_{n+1} - 1$, $n \in N$ such that $\mathcal{C}_{i_n \cdot j_n} \cap \mathbb{T}$ are binary trees and there is an infinite nested sequence $\mathcal{C}_{i,t_s} \subset \mathcal{C}_{i,t_{s+1}}$ such that $\mathcal{C}_{i,t_s} \cap \mathbb{T}$ is a binary tree. Again $\mathbb{T}$ has to be the union of this finite sequence and the nested sequence of binary trees.

Theorem 4.7. *Let $\mathcal{T}$ be a subcategory, then it is tilting if and only if the vertices give a binary tree on the vertices of the Auslander-Reiten quiver as defined before.*

We first remark the following

Proof. Let $\mathcal{T}$ be a tilting subcategory. By lemma 4.3, we have (i),(ii) and (iii) are fulfilled. Properties (iv) follows since a tilting subcategory is maximal selforthogonal (see remark 2.1). Conversely, consider $\mathcal{T} = \text{add}\{X \mid X \in \mathbb{T} \cup \mathbb{P}\}$ with $\mathbb{T}, \mathbb{P}$ fulfilling the properties (i)-(iv). Then we have $\mathcal{T}$ is selforthogonal- this follows from the easy observation:

$$\text{Ext}^1(E_{ij}, P_\ell) \neq 0 \Leftrightarrow \quad i < \ell \leq j$$

We need to see that $\mathcal{P} \subset \text{copres}_1(\mathcal{T})$. If $P_\ell \notin \mathcal{T}$, then $\ell \geq 2$ and there exists a $E_{ts} \in \mathcal{T}$ with $t < \ell \leq s$ such that E_{ts} is the root of one of the binary trees in $\mathbb{T}$ with also $P_t \in \mathbb{P}$. In case that $E_{\ell s} \in \mathcal{T}$, we have $P_\ell \rightarrowtail E_{\ell s} \oplus P_t \twoheadrightarrow E_{ts}$ shows that claim. Else, as $\mathcal{C}_{[t,s]} \cap \mathcal{T}$ is tilting in $\mathcal{C}_{[t,s]}$, we have an exact sequence $E_{\ell s} \rightarrowtail E_{st} \oplus T_0 \twoheadrightarrow T_1$ with $T_i \in \mathcal{C}_{[t,s]} \cap \mathcal{T}$. Then, we look at the inflation $P_\ell \rightarrowtail P_t \oplus T_0$, the push out along $P_\ell \to E_{\ell s}$ is just the inflation $E_{\ell s} \rightarrowtail E_{st} \oplus T_0$, and both inflation have the same cokernel in $\mathcal{T}$. □

Remark 4.8. In this case, the notion of a slice is empty as all non-projective tilting subcategories have indecomposables from both connected components of the Auslander-Reiten quiver. Assume that we have Q' another orientation differing in an interval $[1, n]$. Now for the interval $[1, n]$ we can realize any orientation of a type $\mathbb{A}_n$ quiver as a binary tree $\mathbb{T}$ in $\mathcal{C}_{[1,n]}$, then define $\mathbb{P} = \{P_1\} \cup \{P_{n+1}, P_{n+2}, \cdots\}$. Then take the corresponding tilting subcategory $\mathcal{T}$ and $\text{mod}_\infty \mathcal{T}$ is $\text{rep}^+(Q')$.

4.1.3. *Derived equivalences between different orientations.*

Theorem 4.9. *Let Q and Q' be two quivers of type $\mathbb{A}_\infty$. Then, there exists a triangle equivalence*

$$\text{D}^b(\text{rep}^+(Q)) \to \text{D}^b(\text{rep}^+(Q'))$$

if and only if one of the following three cases holds:

(a) *Q and Q' have a left infinite path.*
(b) *Q and Q' have a right infinite path.*
(c) *Q and Q' have no infinite path.*

One implication is an immediate corollary of the following result

Theorem 4.10. *([34], Thm 7.11) Let Q be a strongly locally finite infinite quiver, then* $\mathrm{D}^b(\mathrm{rep}^+(Q))$ *has (left/right) almost split triangles if and only if* $\mathrm{rep}^+(Q)$ *has no (right/left) infinite path.*

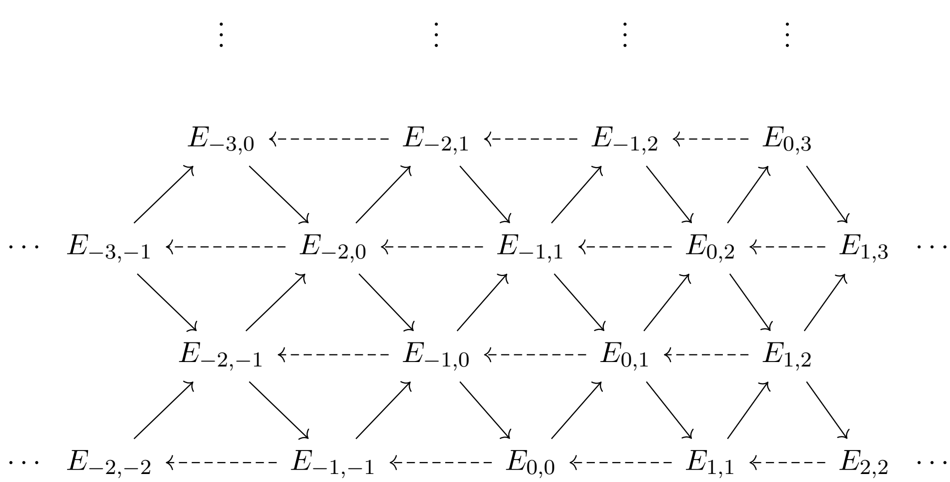
Therefore, it is enough to prove that if Q and Q' in the previous conjecture both fulfill (a) (resp. (b) , resp. (c)), then there exists a triangle equivalence as stated.

Proof. We show that in each of the cases (a), (b) and (c) a derived equivalence between categories of finitely represented quiver representations of two different orientations can be obtained by two tilting derived equivalences.

(a) Let Q'' be the orientation given by one left infinite path and $\mathcal{A} = \mathrm{rep}^+(Q'')$. Then we find two slices for both orientations Q, Q' and the corresponding tilting categories induce then derived equivalence.
$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \leftarrow \mathrm{D}^b(\mathrm{rep}^+(Q'')) \rightarrow \mathrm{D}^b(\mathrm{rep}^+(Q'))$$
(b) Let Q'' be the orientation given by one left infinite path and $\mathcal{A} = \mathrm{rep}^+(Q'')$. We do not find slices in this case but we can find tilting subcategories which fulfill the same task, cf. remark 4.8. Take the two tilting subcategories corresponding to the two different orientations, their tilting functors give derived equivalences
$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \leftarrow \mathrm{D}^b(\mathrm{rep}^+(Q'')) \rightarrow \mathrm{D}^b(\mathrm{rep}^+(Q'))$$
(c) Here we take $\mathcal{A}$ to be the one described below. We will look at another abelian category $\mathcal{A}$ (see below) and find two tilting subcategories inducing derived equivalences
$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \leftarrow \mathrm{D}^b(\mathcal{A}) \rightarrow \mathrm{D}^b(\mathrm{rep}^+(Q'))$$

□

Let from now on $\mathcal{A}$ denote the category $\mathrm{rep}^b(\Delta)$ where Δ is the quiver (of type $\mathbb{A}_\infty^\infty$)
$$\cdots \leftarrow (-3) \leftarrow (-2) \leftarrow (-1) \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \cdots$$
and rep^b denotes the subcategory of all quiver representations of total finite dimension.
Observe that $\mathcal{A}$ is a hereditary abelian Auslander-Reiten category without non-zero projectives or non-zero injectives. Its Auslander-Reiten quiver can be pictured as follows...



A slice has an associated quiver by just taking the full subquiver of the Auslander-Reiten quiver with the vertices given by the slice. We say a slice does not contain a left (resp. right) infinite path if the associated quiver does not contain a right (resp. left) infinite path.

Lemma 4.11. *Weak slices in the Auslander-Reiten quiver of $\mathcal{A}$ give tilting subcategories if and only if the slice does not contain a left or right infinite path.*

PROOF. Let $\mathcal{T}$ be a the full additively closed subcategory of $\mathcal{A}$ with indecomposables given by the vertices of a slice in the Auslander-Reiten category. Let us first see that a slice with an infinite path can not be a tilting subcategory. If the slice does contain a left infinite path, we look at an indecomposable in this path and apply τ^{-1} to it, call this X. Then X is not in $\mathrm{pres}(\mathcal{T})$ and it is not a subobject of an object in $\mathrm{pres}(\mathcal{T})$, therefore (T2) is not fulfilled.
If the slice does not contain a left infinite path, we see $\mathrm{pres}(\mathcal{T})$ as the additive closure of all indecomposables of the slice and of all indecomposables on the right (i.e. after applying τ^{-n}, $n \geq 1$) of it (as going-down arrows in the Auslander-Reiten quiver are all epimorphisms). Assume that the slice contains a right infinite path, and let Y be τ of an indecomposable corresponding to a vertex of the right infinite path. Then Y is not a subobject of an object in $\mathrm{pres}(\mathcal{T})$.
From now on assume that the slice does not contain an infinite path. The description of $\mathrm{pres}(\mathcal{T})$ as above implies $\mathrm{pres}(\mathcal{T}) = \mathcal{T}^{\perp}$. As $\mathcal{T}$ is contravariantly finite, we want to see that the kernel of a right $\mathcal{T}$ approximation of an object in $\mathrm{pres}(\mathcal{T})$ is in $\mathcal{T}^{\perp}$ (this implies (T1)). But this follows directly from applying $\mathrm{Hom}(T, -)$ with $T \in \mathcal{T}$ to the short exact sequence.
Now, take any indecomposable object A in $\mathcal{A}$, we want to see that $A \in \mathrm{Cores}_1(\mathcal{T}^{\perp})$. Wlog. A is in a τ-orbit of an indecomposable in $\mathcal{T}$. We look at the right infinite arrow going-up (i.e. of monomorphisms) starting at A in the Auslander-Reiten quiver. As the slice does not contain a right infinite path, the right infinite path going-up starting at A will eventually meet a vertex in the slice. This gives a monomorphism $A \to T_0$ with $T_0 \in \mathcal{T}$. We look at the short exact sequence $A \rightarrowtail T_0 \twoheadrightarrow B$ and as $B \in \mathrm{pres}(\mathcal{T})$, (T2) follows. □

5. Other Dynkin types?

The same question as in Theorem 4.9 can be probably answered for the other infinite Dynkin types only using sink/source reflections and results from [**34**]. Let us first pose the following question: An infinite quiver is called *strongly locally bounded* (cf. [**34**]) if at every vertex there are only finitely many arrows ending and starting and between any given two vertices there are only finitely many paths.

Open question 5.1. Let Q, Q' be two orientations of a graph, both strongly locally bounded infinite quiver without an infinite path. Is there a finite sequence of sink/source mutations passing from one to the other?

For the rest **assume** that the question has a positive answer for quivers of Dynkin type.

THEOREM 5.2. *(assuming: Yes in 5.1) Let Q and Q' be two quivers of type $\mathbb{D}_{\infty}$. Then we find the same three triangle equivalence classes as in Theorem 4.9.*

PROOF. We sketch the argument as follows: Here, again that we have at least these three equivalence classes follows from [**34**], combine Thm 5.22, Prop. 7.9, Thm 7.10. Inside (a) and (b) we have that (single) sink/source reflection operate transitively, so they are all triangle equivalent. Inside (c), we would need potentially infinite sequences of single sink/source reflections to pass between two orientations (and that is not a valid argument). But by introducing mutation of possibly infinite sets of sources/sinks (cf. subsection 2.1) we can overcome this problem and see that all orientations without an infinite path will induce a triangle equivalence as in the theorem. □

Now, for an infinite quiver Q of type $\mathbb{A}_{\infty}^{\infty}$ we define some numbers:
$\ell :=$ number of maximal left infinite paths in Q,
$r :=$ number of maximal right infinite paths in Q
So $0 \leq r, \ell \leq 2$, $r + \ell \leq 2$ and $r = \ell = 0$ means either Q is a double infinite quiver or has no infinite paths.

In case $r = \ell = 1$, there is a finite number c of arrows in one direction and infinitely in the other.

THEOREM 5.3. *(assuming: Yes in 5.1) Let Q and Q' be two quiver of type $\mathbb{A}_\infty^\infty$. Then, there exists a triangle equivalence*

$$\mathrm{D}^b(\mathrm{rep}^+(Q)) \to \mathrm{D}^b(\mathrm{rep}^+(Q'))$$

if and only if one of the following three cases holds

(a) *Q and Q' are both double infinite paths.*
(b) *Q and Q' have the same numbers ℓ, r and $(\ell, r) \neq (1, 1)$ and are not a double infinite path.*
(c) *Q and Q' have the same numbers $(\ell, r) = (1, 1)$ and c.*

As before, we sketch the proof. To see that these orientations are pairwise non-derived equivalent: Look at the description of the Auslander Reiten quiver components of $\mathrm{D}^b(\mathrm{rep}^+(Q))$ for all quivers of type $\mathbb{A}_\infty^\infty$ in [**34**, Thm. 5.17, Thm 7.9]. Here, in case (c), the number c appears as the number of τ-orbits in the finite wing (cf, Thm 5.17, (4)) and therefore for different c they are pairwise non-derived equivalent.
To see that in each case (a), (b), (c) we have the claimed triangle equivalences: First, observe that the underlying graph automorphism σ which maps the vertices as $\sigma(x) = (-x)$ induces an isomorphism of categories $\mathrm{rep}^+(Q) \cong \mathrm{rep}(\sigma Q)$ and this induces a derived equivalence. Observe that the numbers ℓ, r and also in case (c) the number c are preserved. This shows e.g. that the two double infinite paths in (a) are derived equivalent.
Once we take these isomorphims into account, one can see that in case (c) reflection functors at sinks and sources operate transitively. In case (b) also, but we need reflection functors at infinitely many sinks and sources.

CHAPTER 9

Realization functors in algebraic triangulated categories

This chapter is joint work with Janina Letz, cf. [**133**].

1. Synopsis

Let $\mathcal{T}$ be an algebraic triangulated category and $\mathcal{C}$ an extension-closed subcategory with $\operatorname{Hom}(\mathcal{C}, \Sigma^{<0}\mathcal{C}) = 0$. Then $\mathcal{C}$ has an exact structure induced from exact triangles in $\mathcal{T}$. Keller and Vossieck say that there exists a triangle functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion $\mathcal{C} \subseteq \mathcal{T}$.
What is new? We provide the missing details for a complete proof.

2. Introduction

Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a full additive subcategory with an exact structure. A *realization functor* for $\mathcal{C}$ is a triangle functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion. There are various constructions of a realization functor, all requiring an enhancement and restricting to certain subcategories $\mathcal{C}$. The first realization functor was constructed in [**40**] when $\mathcal{C}$ is the heart of a t-structure in a filtered triangulated category; also see [**156**, Appendix]. A different construction appears in [**142**].

In this chapter we work in algebraic triangulated categories; These include all stable module categories and derived categories. Unlike the works mentioned above we consider exact subcategories of $\mathcal{T}$, not hearts of t-structures. There exist exact categories whose bounded derived category does not admit a bounded t-structure; see [**144**].

The following result appears in [**123**, 3.2 Théorème]:

Theorem 2.1. *Let $\mathcal{T}$ be an algebraic triangulated category and $\mathcal{C}$ an extension-closed full subcategory with $\operatorname{Hom}_{\mathcal{T}}(\mathcal{C}, \Sigma^{-n}\mathcal{C}) = 0$ for $n \geq 1$. Then $\mathcal{C}$ has an exact structure induced from the triangulated structure on $\mathcal{T}$ and there exists a realization functor.*

The article [**123**, 3.2 Théorème] provides a sketch of the proof, referring to a construction later appearing in [**116**]. The main goal of this chapter is to provide the missing details for a complete proof of Theorem 2.1.

The non-negativity condition in Theorem 2.1 for $\mathcal{C}$ is necessary for our construction. It also appears when the realization functor is a triangle equivalence. In fact, whenever the realization functor is fully faithful, then $\mathcal{C}$ has to satisfy the non-negativity condition.

Theorem 2.1 can be considered the standard tool to realize an (algebraic) triangulated as a bounded derived category of an exact category; we provide conditions for when the realization functor is an equivalence in Section 3.4. Therefore, Theorem 2.1 is expected to be used in classifications of exact subcategories of a triangulated category up to (bounded) derived equivalence.

Further, finding a realization functor is an alternative to tilting theory. Tilting subcategories in a triangulated category were defined by Keller; see for example [**122**]. A subcategory $\mathcal{C}$ of $\mathcal{T}$ is *tilting*, if $\mathcal{C}$ is endowed with the split exact structure, hence $\mathrm{D}^b(\mathcal{C}) = \mathrm{K}^b(\mathcal{C})$, and the realization functor

$\mathrm{K}^b(\mathcal{C}) \to \mathcal{T}$ exists and is a triangle equivalence. There exist realization functors that are equivalences that are not induced by tilting theory; for example the inclusion of a small exact category into its weak idempotent completion induces a triangle equivalence on their bounded derived categories; see [**141**, 1.10].

In general it is not known whether a realization functor of a category $\mathcal{C}$ is unique. However, it is unique with respect to the chosen enhancement. Theorem 2.1 is also central to the search for a universal property defining the bounded derived category of an exact category; cf. [**117**] and for derivators [**150**].

3. Realization functor

The bounded derived category of an exact category $\mathcal{C}$ is the Verdier quotient of the homotopy category of the underlying additive category by the full subcategory of bounded $\mathcal{C}$-acyclic complexes $\mathrm{Ac}(\mathcal{C})$; see [**141**] and also [**49**, Section 10]. We fix a triangulated category $\mathcal{T}$ with suspension functor Σ. A *realization functor* for an additive subcategory $\mathcal{C}$ of $\mathcal{T}$ with an exact structure is a triangle functor $\mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion $\mathcal{C} \to \mathcal{T}$.

3.1. Admissible exact subcategories.

In this work we focus on subcategories $\mathcal{C}$ of the triangulated category $\mathcal{T}$ that inherit their exact structure from the triangulated structure of $\mathcal{T}$.

Definition 3.1. A full subcategory $\mathcal{C}$ is called *non-negative* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \Sigma^{<0}\mathcal{C}) = 0$; this means $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y) = 0$ for any $X, Y \in \mathcal{C}$ and $n < 0$. When $\mathcal{C}$ is additionally closed under extensions and direct summands, we say $\mathcal{C}$ is *admissible exact*.

By [**67**], any extension-closed, non-negative subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ inherits an exact structure from the triangulated structure: The short exact sequences $L \xrightarrow{f} M \xrightarrow{g} N$ in $\mathcal{C}$ are precisely those that fit into an exact triangle $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L$.

Remark 3.2. With the notation of 'admissible exact' we follow [**40**, Definition 1.2.5] and [**100**, Section 2]; the former only considers 'admissible abelian', while the latter dropped 'exact'. We use admissible exact to avoid confusion with the notions of left/right admissible in the sense of [**41**, §1].

The crucial condition of admissible exactness is the non-negativity. In fact, when $\mathcal{C}$ is non-negative, then the smallest full subcategory closed under extensions and direct summands containing $\mathcal{C}$ is an admissible exact subcategory.

Example 3.3. We equip an extension-closed subcategory $\mathcal{C}$ of an exact category $\mathcal{E}$ with the induced exact structure; that is $\mathcal{C}$ is a *fully exact* subcategory of $\mathcal{E}$. Then $\mathcal{C}$ is an admissible exact subcategory of $\mathrm{D}^b(\mathcal{E})$.

Example 3.4. The heart of any t-structure on a triangulated category is admissible exact. Any intersection of admissible exact subcategories is admissible exact. Hence the intersection of two hearts is admissible exact; this applies in particular for hearts that are mutations of each other; see [**53**] for HRS tilting and [**43**] for the heart fan of an abelian category.

3.2. Weak realization functor.

Next, we consider triangle functors $\mathrm{K}^b(\mathcal{C}) \to \mathcal{T}$ extending the inclusion for any full subcategory $\mathcal{C}$ of $\mathcal{T}$; such a functor can be considered as a realization functor for $\mathcal{C}$ with the split exact structure. We call such a functor a *weak realization functor*. Under reasonable conditions on the exact structure a weak realization functor induces a realization functor.

Lemma 3.5. *Let $\mathcal{C} \subseteq \mathcal{T}$ be a full subcategory with an exact structure. We assume there exists a weak realization functor $\mathrm{F}\colon \mathrm{K}^b(\mathcal{C}) \to \mathcal{T}$. If any exact sequence $L \to M \to N$ in $\mathcal{C}$ fits into an exact* *triangle $L \to M \to N \to \Sigma L$ in $\mathcal{T}$, then* F *induces a realization functor such that the following diagram commutes*

$$\begin{array}{ccc} \mathrm{K}^b(\mathcal{C}) & \longrightarrow & \mathrm{D}^b(\mathcal{C}) \\ & \searrow & \downarrow \\ & & \mathcal{T} \end{array}$$

In particular, this holds when $\mathcal{C}$ is an admissible exact subcategory.

Proof. It is enough to show that F sends acyclic complexes to zero. For this it is enough to show that complexes of the form

$$(\cdots \to 0 \to L \to M \to N \to 0 \to \cdots) = \operatorname{cone}(\operatorname{cone}(L \to M) \to N)$$

are send to zero when $L \to M \to N$ is an exact sequence in $\mathcal{C}$. But this holds by assumption. □

Remark 3.6. The above condition on F, that any exact sequence in $\mathcal{C}$ fits into an exact triangle in $\mathcal{T}$, means that $\mathcal{C} \to \mathcal{T}$ is a *δ-functor* as defined in [**117**].

In the sequel we construct a weak realization functor. However, we do not know of a general criterion for the existence of a weak realization functor. Our construction requires some form of non-negativity. In particular, a weak realization functor may even exist for $\mathcal{C} = \mathcal{T}$.

Example 3.7. Let k be a field and $\mathcal{T} = \operatorname{vect}(k)$, the category of finite-dimensional k-vector spaces with suspension $\Sigma = \mathrm{id}$. We can view $\mathrm{K}^b(\mathcal{T})$ as the category of finite-dimensional $\mathbb{Z}$-graded k-vector spaces $\operatorname{vect}^{\mathbb{Z}}(k)$ with suspension the shift of the grading. The forgetful functor from graded k-vector spaces to ungraded k-vector spaces is a weak realization functor for $\mathcal{C} = \mathcal{T}$. As $\mathrm{D}^b(\mathcal{T}) = \mathrm{K}^b(\mathcal{T})$ we obtain the realization functor

$$\mathrm{D}^b(\mathcal{T}) = \operatorname{vect}^{\mathbb{Z}}(k) \to \operatorname{vect}(k) = \mathcal{T}\,.$$

3.3. Existence. A *Frobenius category* is an exact category with enough projectives and with enough injectives and the projectives and injectives coincide. Let $\mathcal{E}$ be a Frobenius category with $\mathcal{P}$ the full subcategory of projective-injective objects. The ideal quotient $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ with respect to the morphisms factoring through $\mathcal{P}$ has a natural triangulated structure by [**87**, I.2]. A triangulated category is *algebraic*, if it is triangle equivalent to $\underline{\mathcal{E}}$ for some Frobenius category; see [**120**, 3.6].

The key observation for the proof of Theorem 2.1 is the following result, which is stated in [**123**, 3.2].

Proposition 3.8. *Let $\mathcal{E}$ be a Frobenius category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be a non-negative full subcategory and set $\mathcal{B} := q^{-1}(\mathcal{C})$. Then the functor $\mathcal{B} \to \mathcal{C}$ induces an equivalence of triangulated categories*

$$\mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{C})\,.$$

Note, that in the equivalence connects the Verdier quotient of the homotopy category and the homotopy category of an ideal quotient. We postpone the proof to Section 4.

Remark 3.9. In the Proposition we show that the tilting subcategory $\mathcal{B}$ in $\mathrm{K}^b(\mathcal{B})$ is send to the tilting subcategory $\mathcal{C}$ under the Verdier quotient functor $\mathrm{K}^b(\mathcal{B}) \to \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P})$. In general, Verdier quotients need not preserve tilting subcategories.

Without the assumption that the subcategory $\mathcal{C}$ is non-negative the Proposition 3.8 is false in general:

Example 3.10. Let k be a field and $A = k[x]/(x^2)$. Then $\mathcal{E} = \ \operatorname{mod} A$ is a Frobenius category and $\underline{\mathcal{E}} = \ \operatorname{mod} k$ is the category of finite-dimensional vector spaces which is a triangulated category with $\Sigma = \mathrm{id}$. We show below that $\mathrm{K}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A)$ is not equivalent to $\mathrm{K}^b(\operatorname{mod} k)$, that is that the conclusion of Proposition 3.8 does not hold for $\mathcal{C} = \underline{\mathcal{E}}$, which is not non-negative. Observe first that

$\mathrm{K}^b(\operatorname{mod} k) = \mathrm{D}^b(\operatorname{mod} k)$ has no non-trivial thick subcategory. But on the other hand $\mathrm{K}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A)$ admits a non-trivial Verdier quotient

$$\mathrm{K}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A) \to \mathrm{D}^b(\operatorname{mod} A)/\mathrm{K}^b(\operatorname{proj} A) \cong \underline{\mathcal{E}}\,;$$

see [**48**, Theorem 4.4.1]. In particular, the kernel of this Verdier quotient is a non-trivial thick subcategory. Therefore they can not be triangle equivalent.

Proposition 3.11. *Let $\mathcal{E}$ be a Frobenius category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q\colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be an admissible exact subcategory. Then there exists a weak realization functor $\mathrm{K}^b(\mathcal{C}) \to \underline{\mathcal{E}}$.*

Proof. Set $\mathcal{B} := q^{-1}(\mathcal{C})$. By Proposition 3.8 there exists an equivalence of triangulated categories

$$\mathrm{F}\colon \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{C})\,.$$

There is also an equivalence

$$\mathrm{B}\colon \underline{\mathcal{E}} \to \mathrm{D}^b(\mathcal{E})/\mathrm{K}^b(\mathcal{P})\,;$$

this has been stated in [**123**, Example 2.3] with proofs provided in [**108**, Corollary 2.2] or [**126**, Proposition 4.4.18]. Then the following composition involving the quasi-inverses of the above functors yields the claim

$$\mathrm{K}^b(\mathcal{C}) \xrightarrow{\mathrm{F}^{-1}} \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{E})/\mathrm{K}^b(\mathcal{P}) \xrightarrow{\mathrm{B}^{-1}} \underline{\mathcal{E}}\,. \qquad \square$$

Proof of Theorem 2.1. By Proposition 3.11 there exists a weak realization functor, and it induces a realization functor by Lemma 3.5. $\square$

From Proposition 3.8 we can also deduce the following corollary.

Corollary 3.12. *Let $\mathcal{C}$ be an admissible exact subcategory of $\underline{\mathcal{E}}$. Then $\mathcal{B} = q^{-1}(\mathcal{C})$ is extension-closed in $\mathcal{E}$ and the functor $q\colon \mathcal{B} \to \mathcal{C}$ sends exact sequences to exact triangles. In this case q induces a triangle equivalence*

$$\mathrm{D}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{D}^b(\mathcal{C})\,.$$

Proof. It is straightforward to check that $\mathrm{K}^b(\mathcal{P})$ and $\mathrm{Ac}^b\mathcal{B}$ are Hom-orthogonal in $\mathrm{K}^b(\mathcal{B})$. Then $\mathrm{Ac}^b\mathcal{B}$ is a full subcategory of $\mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P})$ by [**113**, Proposition 1.6.10]. So it is enough to show that the equivalence from Proposition 3.8 restricts to an equivalence of the acyclic complexes $\mathrm{Ac}^b\mathcal{B} \to \mathrm{Ac}^b\mathcal{C}$.

The fully faithfullness of the restriction holds as $\mathrm{Ac}^b\mathcal{B}$ is a full subcategory of $\mathrm{D}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P})$. Essentially surjectivity holds as

$$\mathrm{Ext}^1_{\mathcal{B}}(X,Y) \cong \mathrm{Hom}_{\underline{\mathcal{E}}}(X,\Sigma Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(X,Y)$$

for any $X, Y \in \mathcal{B}$. $\square$

3.4. Fully faithfulness and equivalence. Let $\mathcal{C}$ be an admissible exact subcategory of a triangulated category $\mathcal{T}$. In this section we discuss when a realization functor

$$\mathrm{R}\colon \mathrm{D}^b(\mathcal{C}) \to \mathcal{T}$$

is fully faithful and even an equivalence. The realization functor R induces natural group homomorphisms

$$\Phi_n(X,Y) := (\mathrm{Ext}^n_{\mathcal{C}}(X,Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{D}^b(\mathcal{C})}(X,\Sigma^n Y) \xrightarrow{\mathrm{R}} \mathrm{Hom}_{\mathcal{T}}(X,\Sigma^n Y))$$

for $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$. Here $\mathrm{Ext}^n_{\mathcal{C}}$ are the groups of Yoneda extensions for $n \geq 0$ and we set $\mathrm{Ext}^n_{\mathcal{C}} := 0$ for $n < 0$. For the isomorphism see for example [**126**, Proposition 4.2.11]. These natural morphisms have been considered in [**55**, Lemma 2.11] for hearts of t-structures and in [**151**, A.8] for exact subcategories. The morphism $\Phi_n(X,Y)$ is an isomorphism for $n < 0$ as $\mathcal{C}$ is non-negative, for

$n = 0$ as $\mathcal{C}$ is full, and for $n = 1$ by [**151**, Corollary A.17]. Further, for $n = 2$ it is a monomorphisms by [**151**, Corollary A.17]. The following result appears in [**40**, Remarque 3.1.17] and [**55**, Lemma 2.11] when $\mathcal{C}$ is the heart of a bounded t-structure.

Lemma 3.13. *Let $\mathcal{C}$ be an admissible exact subcategory of $\mathcal{T}$ and let* R *be a realization functor of $\mathcal{C}$. Then the following are equivalent*

(1) R *is fully faithful;*
(2) *$\Phi_n(X, Y)$ is an isomorphism for all $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$;*
(3) *$\Phi_n(X, Y)$ is surjective for all $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$;*
(4) *For every $X, Y \in \mathcal{C}$, $n \geq 1$ and every morphism $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ there exists a $\mathcal{C}$-deflation $d\colon Z \to X$ with $f \circ d = 0$ in $\mathcal{T}$; and*
(4^{op}) *For every $X, Y \in \mathcal{C}$, $n \geq 1$ and every morphism $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ there exists an $\mathcal{C}$-inflation $i\colon Y \to W$ such that $\Sigma^n i \circ f = 0$ in $\mathcal{T}$.*

Proof. The implication (1) $\implies$ (2) is clear and the converse is an application of *dévissage* using $\mathrm{D}^b(\mathcal{C}) = \mathrm{thick}_{\mathrm{D}^b(\mathcal{C})}(\mathcal{C})$; see for example [**126**, Lemma 3.1.8].

The implication (2) $\implies$ (3) is clear and the converse is shown in [**151**, Corollary A.17].

A standard construction shows that (2) is equivalent to

(5) Every $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ with $X, Y \in \mathcal{C}$ and $n \geq 1$ decomposes as
$$X = X_0 \to \Sigma X_1 \to \Sigma^2 X_2 \to \cdots \to \Sigma^n X_n = \Sigma^n Y$$
for $X_i \in \mathcal{C}$;

see for example [**55**, Lemma 2.1] for the abelian case. Moreover, by induction over n this is also equivalent to

(6) Every $f\colon X \to \Sigma^n Y$ in $\mathcal{T}$ with $X, Y \in \mathcal{C}$ and $n \geq 1$ decomposes as $X \to \Sigma U \to \Sigma^n Y$ for some $U \in \mathcal{C}$.

So it is enough to show that (4) and (6) are equivalent. For the backward direction it is enough to observe that any morphism $X \to \Sigma U$ in $\mathcal{T}$ with $X, U \in \mathcal{C}$ induces an exact sequence $U \to Z \xrightarrow{d} X$ in $\mathcal{C}$. For the forward direction let $f\colon X \to \Sigma^n Y$ be a morphism in $\mathcal{T}$ with $X, Y \in \mathcal{C}$ and $n \geq 1$. Then there exists a deflation $d\colon Z \to X$ such that $f \circ d = 0$. We complete d to an exact sequence $U \to Z \xrightarrow{d} X$ in $\mathcal{C}$. Then f factors through the induced morphism $X \to \Sigma U$. This shows (6).

The equivalence of (2) and (4) holds by an analogous argument. □

Remark 3.14. The previous Lemma can be strengthened to yield an explicit description of the image of $\Phi_n(X, Y)$. That is, the subgroup $\mathrm{Im}(\Phi_n(X, Y))$ is the set of all morphisms $f\colon X \to \Sigma^n Y$ with $X, Y \in \mathcal{C}$ such that there exists a $\mathcal{C}$-deflation $d\colon Z \to X$ such that $f \circ d = 0$.

For a subcategory $\mathcal{C}$ of a triangulated category $\mathcal{T}$ we denote by $\mathrm{thick}_{\mathcal{T}}(\mathcal{C})$ the smallest thick subcategory of $\mathcal{T}$ that contains $\mathcal{C}$.

Corollary 3.15. *Let $\mathcal{C}$ be an admissible exact subcategory of $\mathcal{T}$. A realization functor of $\mathcal{C}$ is an equivalence of triangulated categories if and only if it is fully faithful and $\mathrm{thick}_{\mathcal{T}}(\mathcal{C}) = \mathcal{T}$.* □

Example 3.16. Let $\mathcal{C}$ be a fully exact subcategory of $\mathcal{E}$. Then the induced functor $\mathrm{F}\colon \mathrm{D}^b(\mathcal{C}) \to \mathrm{D}^b(\mathcal{E})$ is a realization functor for $\mathcal{C} \subseteq \mathrm{D}^b(\mathcal{E})$. The functor F is fully faithful if and only if the inclusion $\mathcal{C} \subseteq \mathcal{E}$ induces isomorphism on the Ext-groups. For example, the latter condition is satisfied by resolving subcategories; see [**25**, Section 2] and also [**90**, Definition 5.1].

The functor F is an equivalence if additionally $\mathcal{E}$ is the smallest additively-closed subcategory closed under the 2-out-of-three property containing $\mathcal{C}$. For example, this is satisfied by finitely resolving subcategories; cf. [**92**, Theorem 3.11(2)].

4. Proof of the main Proposition

For clarity we use different notations for the suspension in the stable category and the homotopy category. We write Σ for the suspension or shift functor in $\underline{\mathcal{E}}$ where $\mathcal{E}$ is a Frobenius exact category. By construction, we have $q(\Omega^n X) = \Sigma^{-n} X$ for any $X \in \mathcal{E}$ where Ω is the syzygy functor. On the other hand, for an additive category $\mathcal{A}$ we write $\mathrm{Ch}(\mathcal{A})$ for the category of chain complexes. In $\mathrm{Ch}(\mathcal{A})$ and the homotopy category $\mathrm{K}(\mathcal{A})$, we denote the degree n shift of a complex X by $X[n]$; this is the complex given by

$$X[n]^\ell = X^{\ell+n} \quad \text{and} \quad d_{X[n]} = (-1)^n d_X \,.$$

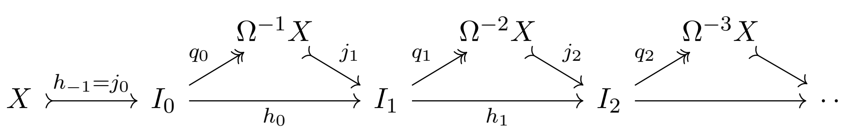
For a *map of complexes* $f \colon X \to Y$ we write

$$\partial(f) = d^Y f - f[-1] d^X \colon X \to Y[-1]\,.$$

The map f is a *chain map* if and only if $\partial(f) = 0$. Note, that a map of complexes need not commute with the differential, while a chain map does.

Lemma 4.1. *Let $\mathcal{E}$ be a Frobenius exact category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q \colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $\mathcal{C} \subseteq \underline{\mathcal{E}}$ be a non-negative full subcategory and set $\mathcal{B} := q^{-1}(\mathcal{C})$. For any chain map $f \colon q(X) \to q(Y)$ in $\mathrm{Ch}(\underline{\mathcal{E}})$ with $X \in \mathrm{Ch}^+(\mathcal{B})$ and $Y \in \mathrm{Ch}^-(\mathcal{B})$ there exist chain maps $g \colon \hat{X} \to Y$ and $s \colon \hat{X} \to X$ with $\hat{X} \in \mathrm{Ch}^+(\mathcal{B})$ and $\mathrm{cone}(s) \in \mathrm{Ch}^b(\mathcal{P})$ such that $q(g) = f \circ q(s)$.*

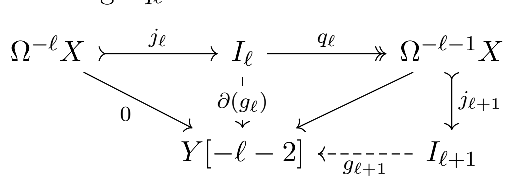
Proof. First we construct an injective resolution I of X in the category of complexes. By [**116**, 4.1, Lemma, b)], there exists a left bounded complex I_0 of projective-injective objects and a chain map $j_0 \colon X \to I_0$ that is an inflation in each degree. We denote the cokernel of j_0 by $q_0 \colon I_0 \to \Omega^{-1} X$. Continuing this process, we obtain a sequence of chain maps



We set $h_{-1} := j_0$ and $h_\ell := j_{\ell+1} q_\ell$. As X is left bounded we may assume that there exists an integer s such that $(I_\ell)^{\leq s} = 0$ for all ℓ; that is s is a universal lower bound. Since the maps j_ℓ are degreewise inflations, every map from $\Omega^{-\ell} X$ to a complex of projective-injective objects factors through j_ℓ.

We take a lift of f to a map of complexes $\hat{f} \colon X \to Y$ in $\mathrm{Ch}(\mathcal{E})$. This map need not commute with the differential. However, as it is the lift of a chain map in $\underline{\mathcal{E}}$ the map $\partial(\hat{f})$ factors through a complex of projective-injective objects. So there exists a map $g_0 \colon I_0 \to Y[-1]$ such that $\partial(\hat{f}) = g_0 j_0$. For convenience we set $q_{-1} := \mathrm{id}_X$ and $g_{-1} := \hat{f}$. We now inductively construct maps $g_\ell \colon I_\ell \to Y[-\ell-1]$ with $\partial(g_{\ell-1}) = g_\ell j_\ell q_{\ell-1}$.

We assume that we have constructed the maps for any integer $\leq \ell$ for some $\ell \geq 0$. Then $0 = \partial(g_\ell) j_\ell q_{\ell-1}$ as j_ℓ and $q_{\ell-1}$ are chain maps. As $q_{\ell-1}$ consists of deflations in each degree, we get $0 = \partial(g_\ell) j_\ell$. Hence $\partial(g_\ell)$ factors through q_ℓ and we obtain the commutative diagram



By the non-negativity of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathrm{Ch}(\underline{\mathcal{E}})}(q(\Omega^{-\ell-1} X), q(Y[-\ell-2])) = \mathrm{Hom}_{\mathrm{Ch}(\underline{\mathcal{E}})}(\Sigma^{\ell+1} q(X), q(Y[-\ell-2])) = 0\,.$$

Hence the map $\Omega^{-\ell-1}X \to Y[-\ell-2]$ factors through $j_{\ell+1}$. Note, that $g_{\ell+1}$ need not be a chain map. We continue this process until the map $g_{\ell+1}$ is a chain map. As Y is right bounded and the I_ℓ's have a universal upper bound, this will happen eventually.

Let t be an integer such that $Y^{\geq t} = 0$. We replace I_ℓ by the truncation $(I_\ell)^{\geqslant t-\ell-1}$. This does not effect the properties of the g_ℓ's, as they are zero in the other degrees. To summarize, we have a sequence of maps

$$\begin{array}{ccccccccccc} X & \xrightarrow{h_{-1}} & I_0 & \xrightarrow{h_0} & I_1 & \longrightarrow & \cdots & \longrightarrow & I_{n-1} & \xrightarrow{h_{n-1}} & I_n \\ \downarrow{\scriptstyle \hat{f}=g_{-1}} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & & & \downarrow{\scriptstyle g_{n-1}} & & \downarrow{\scriptstyle g_n} \\ Y & & Y[-1] & & Y[-2] & & \cdots & & Y[-n] & & Y[-n-1] \end{array}$$

where each I_ℓ is a bounded complex of projective-injective objects, g_n is a chain map and $\partial(g_{\ell-1}) = g_\ell h_{\ell-1}$ and $h_\ell h_{\ell-1} = 0$ for $0 \leq \ell \leq n$.

We take the total complex J of $I_0 \to \cdots \to I_n$. This means as graded module $J = \bigoplus I_i[i]$ with differential

$$d_J|_{I_i[i]} = d_{I_i[i]} + (-1)^i h_i[i]\,.$$

For convenience we use a nonstandard sign convention. We set

$$v := \sum_i g_i[i] \colon J \to Y[-1]\,.$$

This is a chain map, as

$$\begin{aligned} (vd_J)|_{I_i[i]} &= g_i[i-1]d_{I_i[1]} + (-1)^i g_{i+1}[i]h_i[i] \\ &= (-1)^i(g_i[-1]d_{I_i} + g_{i+1}h_i)[i] \\ &= (-1)^i(d_{Y[-i-1]}g_i)[-i] = d_{Y[-1]}g_i[i] = (d_{Y[-1]}v)\big|_{I_i[i]}\,. \end{aligned}$$

One can similarly check that the composition $u := (X \to I_0 \to J)$ is a chain map. By construction we have $\partial(\hat{f}) = vu$. Then $\hat{X} = \Sigma^{-1}\mathrm{cone}(u)$ and $g = (-v, \hat{f})$ and the natural map $s \colon \hat{X} \to X$ satisfy the desired properties. □

Lemma 4.2. *Let $\mathcal{E}$ be a Frobenius exact category with $\mathcal{P}$ the full subcategory of projective-injective objects and $q \colon \mathcal{E} \to \underline{\mathcal{E}}$ the canonical functor. Let $X \in \mathrm{K}^b(\mathcal{E})$. If $q(X) = 0$ in $\mathrm{K}^b(\underline{\mathcal{E}})$, then $X \in \mathrm{K}^b(\mathcal{P})$.*

Proof. It is enough to show the claim for a complex of the form

$$X = (\cdots \to 0 \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \to \cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \to 0 \to \cdots)$$

for any $n \geq 0$. We use induction on n.

For $n = 0$, the assumption $q(X) = 0$ implies $q(X^0) = 0$. Hence $X^0 \in \mathcal{P}$.

Let $n \geq 1$. As $q(X) = 0$, the morphism $q(d^0)$ is a split monomorphism in $\underline{\mathcal{E}}$ and there exists a morphism $s \colon X^1 \to X^0$ such that $sd^0 - \mathrm{id}_{X^0} = ba$ for some morphisms $X^0 \xrightarrow{a} P \xrightarrow{b} X^0$ with $P \in \mathcal{P}$. We view P as a complex concentrated in degree zero and set

$$X' := \mathrm{cone}(\Sigma^{-1}a) = (\cdots \to 0 \to X^0 \xrightarrow{\left(\begin{smallmatrix} d^0 \\ a \end{smallmatrix}\right)} X^1 \oplus P \xrightarrow{(d^1\ 0)} X^2 \xrightarrow{d^2} X^3 \to \cdots)\,.$$

Since $\left(\begin{smallmatrix} s & -b \end{smallmatrix}\right) \circ \left(\begin{smallmatrix} d^0 \\ a \end{smallmatrix}\right) = \mathrm{id}_{X^0}$, the zero differential of X' is a split monomorphism in $\mathcal{E}$. Therefore, in $\mathrm{K}^b(\mathcal{E})$, the complex X' is isomorphic to a complex Y concentrated between degrees 1 and n. In $\mathrm{K}^b(\underline{\mathcal{E}})$ we have $q(Y) \cong q(X') \cong q(X) = 0$. By induction hypothesis we have $X' \cong Y \in \mathrm{K}^b(\mathcal{P})$. By construction there is an exact triangle $X' \to P \to X \to \Sigma X'$, and as $X', P \in \mathrm{K}^b(\mathcal{P})$, so is X. □

Proof of Proposition 3.8. As $q(\mathrm{K}^b(\mathcal{P})) = 0$ in $\mathrm{K}^b(\underline{\mathcal{E}})$, the functor $q \colon \mathrm{K}^b(\mathcal{B}) \to \mathrm{K}^b(\mathcal{C})$ induces a triangle functor

$$\overline{q} \colon \mathrm{K}^b(\mathcal{B})/\mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{C})\,.$$

We claim that $\overline{q}$ is an equivalence of triangulated categories. For this we need to show that $\overline{q}$ is full, faithful and essentially surjective.

The functor $\overline{q}$ is full by Lemma 4.1. By Lemma 4.2, whenever $\overline{q}(X) = 0$ then $X = 0$. As we already know that $\overline{q}$ is full, this implies that $\overline{q}$ is faithful by [**158**, p. 446]; also see [**163**, 4.3, 4.4].

It remains to show that $\overline{q}$ is essentially surjective. The essential image of $\overline{q}$ is a thick subcategory containing the complexes concentrated in degree zero. As these complexes generate $\mathrm{K}^b(\mathcal{C})$, the functor $\overline{q}$ is essentially surjective. □

CHAPTER 10

Fragments of derived Morita theory for exact categories with enough projectives

1. Synopsis

Given $\mathcal{T}$ a tilting subcategory $\mathcal{E}$, then as $\mathcal{T}^{\perp}$ is an exact category with enough projectives, one always finds a bounded derived equivalence to $\mathcal{E}' = \operatorname{mod}_S \mathcal{T}$ where S are $\mathcal{T}^{\perp}$-admissible morphisms. This suggest that for a derived equivalence between two exact categories with enough projectives, we need to find a selforthogonal subcategory $\mathcal{T}$ which is generating ($\operatorname{Hom}(\mathcal{T}, \Sigma^n X) = 0 \; \forall n \in \mathbb{Z}$ implies $X = 0$) and a class of morphisms which we call Δ-suitable morphisms (cf. Def. 1.8). The definition is still suboptimal as it is difficult to verify, but Δ-suitability is used to show that we can find $\mathcal{C} = \operatorname{mod}_S \mathcal{T} \subseteq \Delta$ as h-admissible exact subcategory (i.e. admissible exact and $\operatorname{Ext}^n(X, Y) \cong \operatorname{Hom}_\Delta(X, \Sigma^n Y)$ for all X, Y in $\mathcal{C}$, $n > 1$). Fully Δ-suitable means additionally $\operatorname{Thick}_\Delta(\mathcal{C}) = \Delta$. In this case, if we assume Δ algebraic, then the realization functor for $\mathcal{C}$ is a triangle equivalence, cf. [**133**].
Using this, the following characterization of the triangulated categories triangle equivalent to $D^b(\operatorname{mod}_S \mathcal{T})$ is then immediate.

THEOREM 1.1. *Let Δ be an algebraic triangulated category and assume that there exists $\mathcal{T} \subseteq \Delta$ selforthogonal, generating and $S \subseteq \operatorname{Mor} - \mathcal{T}$ fully Δ-suitable. For every other algebraic triangulated category Δ' the following are equivalent.*

(a) *Δ and Δ' are triangle equivalent.*
(b) *There exists $\mathcal{T}' \subseteq \Delta'$ selforthogonal, generating together with $S' \subseteq \operatorname{Mor} - \mathcal{T}'$ fully Δ-suitable and an additive equivalence $F : \mathcal{T} \to \mathcal{T}'$ with $F(S) = S'$.*

We also conclude that a triangulated category is triangle equivalent to a bounded derived category of an exact category with enough projectives if and only if it contains a selforthogonal generating subcategory which admits fully Δ-suitable morphisms.

1.1. Exact categories with enough projectives revisited. We recall the following notion from chapter 4.

Definition 1.2. Let $\mathcal{C}$ be an idempotent complete additive category. We call a class of morphisms $S \subseteq \operatorname{Mor} - \mathcal{C}$ **homotopy-closed** if $s \in S$ and $\operatorname{coker} \operatorname{Hom}_{\mathcal{C}}(-, s) \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{C}}(-, t)$ in $\operatorname{Mod} \mathcal{C}$ implies $t \in S$.
We say that S is **suitable** if it is homotopy closed and $\operatorname{mod}_S \mathcal{C} = \{F : \operatorname{coker} \operatorname{Hom}(-, s) \mid s \in S\}$ is a resolving subcategory of $\operatorname{mod}_\infty \mathcal{C}$.

Lemma 1.3. *We fix an essentially small idempotent complete additive category $\mathcal{P}$ and look at the following sets:*

(1) *Exact categories $\mathcal{E}$ with enough projectives $\mathcal{P}(\mathcal{E}) = \mathcal{P}$.*
(2) *Suitable classes of morphisms $S \subseteq \operatorname{Mor} - \mathcal{P}$.*

The assignements $\mathbb{S}(\mathcal{E}) = \{\mathcal{E} - \text{admissible morphisms in } \mathcal{P}\}$ and $\mathbb{M}(S) := \operatorname{mod}_S \mathcal{P}$ are mutually inverse bijections.

Proof. For $\mathcal{E}$ with enough projectives $\mathcal{P}$, the functor $\mathcal{E} \to \operatorname{mod}_S \mathcal{P}$, $X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{P}}$ with $S = S_{adm}$ the $\mathcal{E}$-admissible morphisms in $\mathcal{P}$ is an equivalence of exact categories (cf. Chapter 4, Cor. 3.16). This and (b) in Chapter 4, Lemma 3.13 imply the bijection. □

Definition 1.4. We consider classes of morphisms in $\mathcal{P}$ with respect to inclusion, then for suitable morphsims S, S' we have

$$S' \subseteq S \Leftrightarrow \operatorname{mod}_{S'} \mathcal{P} \subseteq \operatorname{mod}_S \mathcal{P}.$$

We consider exact categories with enough projectives given by $\mathcal{P}$ with the partial order $\mathcal{E}' \leq \mathcal{E}$ if and only if $\mathcal{E}'$ is equivalent to a fully exact subcategory of $\mathcal{E}$. When we consider the sets in the previous Lemma with these poset structures, it is straightforward to see that the bijection in the previous lemma becomes an isomorphism of posets.

Remark 1.5. The (unique) maximal suitable morphisms are all morphisms S which admit weak kernels because then $\operatorname{mod}_S \mathcal{P} = \operatorname{mod}_\infty \mathcal{P}$.
The (unique) minimal suitable morphisms are the ones admissible in the split exact structure on $\mathcal{P}$. Then $\operatorname{mod}_S \mathcal{P} = \mathcal{P} \subseteq \operatorname{mod}_\infty \mathcal{P}$.

1.2. Selforthogonal subcategories in triangulated categories with suitable morphisms.

Let Δ be an algebraic triangulated category. We look at the full subcategory of *non-negative* objects in Δ

$$\Delta^{nn} := \{X \in \Delta \mid \operatorname{Hom}_\Delta(X, \Sigma^{<0} X) = 0\}.$$

This is closed under taking direct summands but not under direct sums (it can be just $\{0\}$). We have the following easy observation: Subcategories of Δ^{nn} which are closed under direct sums are precisely the same as additive subcategories of Δ which are non-negative. Recall from [**133**], a full additive subcategory $\mathcal{C}$ of Δ is **admissible exact** if it is non-negative ($\operatorname{Hom}(C, \Sigma^{<0}C') = 0$ for all C, C' in $\mathcal{C}$) and extension-closed.
If $\mathcal{X}$ is a non-negative subcategory of Δ, then its extension-closure is also non-negative and an admissible exact subcategory. This follows from the easy observation: If $X \to Y \to Z \to \Sigma X$ is a triangle in Δ with $X, Z, X \oplus Z \in \Delta^{nn}$ then $Y \in \Delta^{nn}$.

Remark 1.6. If $\mathcal{C}$ is an extension-closed subcategory in Δ and $\mathcal{C} \cap \Delta^{nn}$ is additively closed, then it is the unique largest admissible exact subcategory in $\mathcal{C}$.
(By the previous discussion, we have that $\mathcal{C} \cap \Delta^{nn}$ is extension-closed, the rest is obvious.)

Now, we assume that $\mathcal{T} \subseteq \Delta$ is an essentially small full additively closed subcategory which is **selforthogonal** (i.e $\operatorname{Hom}_\Delta(T, \Sigma^n T') = 0$ for all $n \neq 0$, T, T' in $\mathcal{T}$) and **generating** (this means: $\operatorname{Hom}_\Delta(T, \Sigma^n X) = 0$ for all $n \in \mathbb{Z}$, T in $\mathcal{T}$ implies $X = 0$ in Δ).

Remark 1.7. There are many conditions called *generating* in a triangulated category, our definition is from [**115**, Def. 5.2.1], in the stacks project this is called *weakly generates*. The main example for us is the following: If $\Delta = \mathrm{D}^b(\mathcal{E})$ is the the bounded derived category of an exact category with enough projectives $\mathcal{P}$, then $\mathcal{P}$ generates Δ. It will be a necessary condition for us to assume on $\mathcal{T}$.

Thirdly, we take a class of suitable morphisms $S \subseteq \operatorname{Mor}\mathcal{T}$. We start with

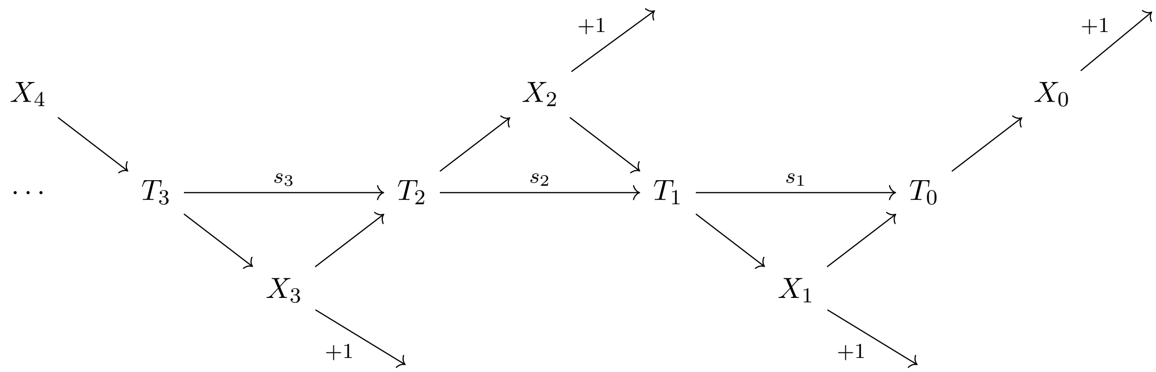
$$\mathcal{T}^\perp := \{X \in \Delta \mid \operatorname{Hom}_\Delta(T, \Sigma^n X) = 0 \text{ for all } n \neq 0, T \in \mathcal{T}\}$$

This is an extension-closed subcategory of Δ so we can see this as an extriangulated category. We consider the functor

$$\Phi \colon \mathcal{T}^\perp \to \operatorname{Mod}\mathcal{T}, \quad X \mapsto \operatorname{Hom}_\Delta(-,X)|_\mathcal{T} =: (-,X)|_\mathcal{T}$$

The functor Φ maps triangles to short exact sequences by definition of $\mathcal{T}^\perp$.

Definition 1.8. Given a suitable class of morphisms S on an additive subcategory $\mathcal{T}$ in a triangulated category Δ. We call S **Δ-suitable** if for every sequence $(s_n)_{n\in\mathbb{N}}$ in S with s_{n+1} is a weak kernel of s_n for every n there exists triangles



with $X_n \in \Delta^{nn} \cap \mathcal{T}^{\perp}$ for all $n \in \mathbb{N}$ and s_n factors over X_n for all $n \in \mathbb{N}$.

Given a suitable class of morphisms $S \subseteq \mathrm{Mor} - \mathcal{T}$. Can we extend $\mathcal{T} \subseteq \Delta$ to an admissible exact category $\mathrm{mod}_S \mathcal{T} \subseteq \Delta$?
In general not. It follows easily from the definition:

Lemma 1.9. *Let $\mathcal{T}$ be selforthogonal in a triangulated category Δ and $S \subseteq \mathrm{Mor} - \mathcal{T}$ a suitable class of morphisms. If $\mathcal{C} := \mathrm{mod}_S \mathcal{T}$ is an admissible exact subcategory of Δ extending the inclusion $\mathcal{T} \subseteq \Delta$, then*

(1) *S is Δ-suitable.*
(2) *$\mathrm{mod}_S \mathcal{T}$ is h-admissible exact (i.e. $\mathrm{Ext}^n_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\Delta}(X, \Sigma^n Y)$ are isomorphism for all X, Y in $\mathcal{C}$).*

Proof. (1) As $\mathcal{C}$ is a resolving category, every X has a projective resolution. Split it into short exact sequences, call the syzygies X_n, $n \in \mathbb{N}$. As $\mathcal{C}$ is admissible exact, these short exact sequence are part of distinguished triangles and all $X_n \in \mathcal{C}$ are non-negative. Now, every consecutive weak kernel sequence arises in this way, so S is Δ-suitable.
(2) As $\mathcal{C}$ is admissible exact, we have an isomorphism $\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\Delta}(X, \Sigma Y)$ for all X, Y in $\mathcal{C}$. Now, choose a projective resolution of X, call the syzygies X_n, $n \in \mathbb{N}$. Now, we can use dimension shift twice, once to see $\mathrm{Ext}^{n+1}_{\mathcal{C}}(X,Y) \cong \mathrm{Ext}^1_{\mathcal{C}}(X_n, Y) \cong \mathrm{Hom}_{\Delta}(X_n, \Sigma Y)$ and a second time apply $\mathrm{Hom}(-, \Sigma Y)$ to the triangles $X_n \to T_{n-1} \to X_{n-1} \xrightarrow{+1}$ to conclude $\mathrm{Hom}(X_n, \Sigma Y) \cong \mathrm{Hom}(\Sigma^{-n} X, \Sigma Y) \cong \mathrm{Hom}(X, \Sigma^{n+1} Y)$.

□

Now, this is the main thing to prove:

Lemma 1.10. *(Extension-Lemma) If S is a Δ-suitable class of morphisms in a selforthogonal subcategory $\mathcal{T}$ which generates a triangulated category Δ.*
We look at the subcategory $\mathcal{C} := \{X \in \Delta \mid \exists (s_n)_n, X_n$ as above such that $X = X_0\}$. Then we claim: $\mathcal{C}$ is an admissible exact subcategory equivalent to $\mathrm{mod}_S \mathcal{T}$ and $\mathcal{C} \subseteq \Delta$ extends $\mathcal{T} \subseteq \Delta$.

Before we give the proof in the next section, let us state the consequence.

Definition 1.11. If $(\mathcal{T}, S)$ with $\mathcal{T}$ selforthogonal in Δ and S Δ-suitable. Then we say that S is **fully** Δ-suitable if $\mathrm{Thick}_{\Delta}(\mathrm{mod}_S \mathcal{T}) = \Delta$.

Clearly triangle equivalences map fully Δ-suitable morphisms to fully Δ-suitable morphisms, so Theorem 1.1 follows trivially.

Example 1.12. Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ - we see $\mathcal{P}$ as stalk complexes in $\Delta = D^b(\mathcal{E})$, then this is a selforthogonal subcategory. We take S to be the $\mathcal{E}$-admissible morphisms (they are suitable) in $\mathcal{P}$ and also Δ-suitable.

Example 1.13. If $\mathcal{E}$ is an exact category and $\mathcal{T} \subseteq \mathcal{E}$ is a tilting subcategory of $\mathcal{E}$ (cf. [**172**]), this means

$$\mathcal{T}^{\perp_\mathcal{E}} := \mathcal{T}^\perp \cap \mathcal{E} = \{X \in \mathcal{E} \mid \operatorname{Ext}^{>0}_\mathcal{E}(T, X) = 0 \quad \forall T \in \mathcal{T}\}$$

has enough projective given by $\mathcal{T}$ and every object in $\mathcal{E}$ has a finite coresolution by objects in $\mathcal{T}^{\perp_\mathcal{E}}$. Let S be the class of $\mathcal{T}^{\perp_\mathcal{E}}$-admissible morphisms then S is fully Δ-suitable in $\mathcal{T} \subseteq D^b(\mathcal{E})$ and $\mathcal{T}^{\perp_\mathcal{E}} = \operatorname{mod}_S \mathcal{T}$

Example 1.14. Let $\mathcal{T}$ be an additive category. We consider it as self-orthogonal subcategory inside $K^b(\mathcal{T})$. Then, all Δ-suitable morphisms in $\mathcal{T}$ are fully Δ-suitable and they are precisely the suitable morphisms S such that $\operatorname{mod}_S \mathcal{T} = \mathcal{P}^{<\infty}(\operatorname{mod}_S \mathcal{T})$. There is a maximal Δ-suitable class of morphisms given by $\operatorname{mod}\,_S\mathcal{T} = \mathcal{P}^{<\infty}(\operatorname{mod}_\infty \mathcal{T}) = \operatorname{Res}(\mathcal{T}) \subseteq \operatorname{mod}_\infty \mathcal{T}$.
More generally, if Δ is triangulated, $\mathcal{T} \subseteq \Delta$ self-orthogonal and $\operatorname{Thick}_\Delta(\mathcal{T}) = \Delta$ (i.e. $\mathcal{T}$ a tilting subcategory in a triangulated category in the sense of Keller), the same statement holds true.

Example 1.15. Let $\mathcal{T}$ be an additive category. We consider it as self-orthogonal subcategory inside $K^+(\mathcal{T})$. Then all suitable morphisms in $\mathcal{T}$ are Δ-suitable but none are fully Δ-suitable:
Given fully Δ-admissible morphisms S in $\mathcal{T} \subseteq \Delta$, since $\Delta \cong D^b(\operatorname{mod}_S \mathcal{T}) \cong K^{+,b}(\mathcal{T}) \subseteq K^+(\mathcal{T})$, we necessarily have a full triangulated subcategory of $K^+(\mathcal{T})$ with $K^{+,b}(\mathcal{T}) \neq K^+(\mathcal{T})$.

1.3. Proof of the Extension-Lemma. Recall, $\mathcal{T}$ selforthogonal and generating in Δ, $S \subseteq \operatorname{Mor}\mathcal{T}$ is Δ-suitable and $\mathcal{C} := \{X \in \Delta \mid \exists (s_n)_n, X_n \text{ as above such that } X = X_0\}$.
Claim: $\mathcal{C}$ is an admissible exact subcategory and with the admissible exact structure equivalent as exact category to $\operatorname{mod}_S \mathcal{T}$.
So, divide and conquer, we show one property after the other, in this order

(i) $\mathcal{C}$ is closed under direct sums
(ii) $\mathcal{C}$ is non-negative
(iii) additive equivalence to $\operatorname{mod}_S \mathcal{T}$
(iv) $\mathcal{C}$ is extension-closed (and so admissible exact in Δ)
(v) exact equivalence to $\operatorname{mod}_S \mathcal{T}$

We look at the composition $\varphi\colon \mathcal{C} \subseteq \mathcal{T}^\perp \xrightarrow{\Phi} \operatorname{Mod}\mathcal{T}$ defined by $X \mapsto \operatorname{Hom}_\Delta(-, X)|_\mathcal{T}$. As $\mathcal{T}$ is generating, the functor φ reflects isomorphism, this can be used to see that

(i) $\mathcal{C}$ is closed under direct sums:
Assume X, Y in $\mathcal{C}$, pick the first morphisms $s^X, s^Y \in S$ in the definition of $\mathcal{C}$. As $\operatorname{mod}_S \mathcal{T}$ is resolving and S homotopy closed, we have that $s := s_X \oplus s_Y \in S$. We extend s to a sequence of consecutive weak kernels in S and as S is Δ-suitable, we can factor s as $T_1 \to Z_1 \to T_0$ such that we have a distinguished triangle Let $Z_1 \to T_0 \to Z \xrightarrow{+1}$ with $Z \in \mathcal{C}$. Now, by definition $\operatorname{Hom}_\Delta(-, Z)|_\mathcal{T} \cong \operatorname{coker}\operatorname{Hom}_\mathcal{T}(-, s) \cong \operatorname{Hom}_\Delta(-, X \oplus Y)|_\mathcal{T}$ and as φ reflects isomorphism we conclude $X \oplus Y \cong Z$ is non-negative.
(ii) Also, it implies that $\mathcal{C}$ is a non-negative subcategory (as $C \oplus C'$ non-negative implies $\operatorname{Hom}(C, \Sigma^{<0} C') = 0$).
(iii) Now as $\mathcal{C}$ is a non-negative subcategory we can easily deduce that φ is a fully faithful functor: For X, Y in $\mathcal{C}$ we choose again $s^X : T_1^X \to T_0^X$ and $s^Y : T_1^Y \to T_0^Y$ from the definition of $\mathcal{C}$.
First observe that φ gives an isomorphism whenever both objects are in $\mathcal{T}$ (by Yoneda) and also if the first object is in $\mathcal{T}$, because $\operatorname{Hom}_\Delta(T, Y) = (\operatorname{coker}\operatorname{Hom}(-, s^Y))(T) = \operatorname{Hom}_{\operatorname{Mod}\mathcal{T}}(\varphi(T), \operatorname{coker}\operatorname{Hom}(-, s^Y)) = \operatorname{Hom}(\varphi(T), \Phi(Y))$ for every $T \in \mathcal{T}$. Now apply $\operatorname{Hom}_\Delta(-, Y)$ to the triangles for X, we find a left exact sequence (as $\mathcal{C}$ is non-neg.)

$$0 \to \operatorname{Hom}_\Delta(X, Y) \to \operatorname{Hom}_\Delta(T_0^X, Y) \to \operatorname{Hom}(T_1^X, Y)$$

Now, to see that φ induces an isomorphism on $\mathrm{Hom}(X,Y)$ it suffices to see that

$$0 \to \mathrm{Hom}_\Delta(\varphi(X),\varphi(Y)) \to \mathrm{Hom}_\Delta(\varphi(T_0^X),\varphi(Y)) \to \mathrm{Hom}(\varphi(T_1^X),\varphi(Y))$$

is also exact, but φ maps triangles to exact sequences, so this claim follows.

We observe that for $X \in \mathcal{C}$ (defined by (s_n) in S)

$$\varphi(X) = \mathrm{Hom}_\Delta(-,X)|_\mathcal{T} \cong \operatorname{coker} \mathrm{Hom}_\mathcal{T}(-,s_1)$$

so, φ induces an equivalence of additive categories $\varphi\colon \mathcal{C} \to \mathrm{mod}_S\,\mathcal{T}$ which maps triangles to exact sequences.

(iv) Next, we claim $\mathcal{C}$ is extension-closed: For this we first observe that for every short exact sequence $\sigma\colon \varphi(X) \rightarrowtail \varphi(Y) \twoheadrightarrow \varphi(Z)$, X,Y,Z in $\mathcal{C}$ in $\mathrm{mod}_S\,\mathcal{T}$ there exists a triangle $\delta\colon X \to Y \to Z \xrightarrow{+1}$ with $\varphi(\delta) \cong \sigma$. Just take $C := \mathrm{cone}(X \to Y)$ and look at the standard triangle $X \to Y \to C \xrightarrow{+1}$ applying $\mathrm{Hom}(T,-)$ with $T \in \mathcal{T}$ implies that $\mathrm{Hom}_\Delta(-,C)|_\mathcal{T} \cong \mathrm{Hom}_\Delta(-,Z)|_\mathcal{T}$ but as $\mathcal{T}$ is generating this implies $C \cong Z$.

Now, this easily implies $\mathcal{C}$ is extension-closed, take a triangle $X \to Y \to Z \xrightarrow{+1}$ with X,Z in $\mathcal{C}$. As $X, Z, X \oplus Z$ are non-negative, this implies Y is non-negative. We apply Φ implies that $\Phi(Y) = \varphi(Y) \in \mathrm{mod}_S\,\mathcal{T}$. Now, we take the short exact sequences from a projective resolution of $\varphi(Y)$, by the first consideration there exist the triangles as required for $Y \in \mathcal{C}$.

(v) To the that φ is an equivalence of exact categories, it is enough to show that it induces a surjection on Ext^1's. But that had just been discussed in (iv).

Part 3

Singular and stable equivalence

CHAPTER 11

Non-commutative resolutions of singularities using exact substructures

1. Synopsis

We introduce (bounded) singularity categories for arbitrary exact categories. An exact category is regular if its singularity category is zero. We recall the known Buchweitz theorem for a Gorenstein exact categories with enough projectives. Then we explore a new concept of a *noncommutative resolution of singularities* (NCR) of a given exact category as an exact substructure which is regular. There exist various alternative versions of non-commutative resolutions in the literature. Our aims here are:

(1) Partially unify and simplify the theory (singularity categories, non-commutative resolutions of singularities and relative singularity categories) for module categories of rings and for coherent sheaves on a quasi-projective variety.
(2) Characterize NCRs corresponding to *cluster tilting* subcategories (as a candidate for a 'minimal' NCR).

What is new? The concept to see NCRs as exact substructures and the generality of our approach.

2. Definitions and notations

We recall some of the previous definitions. Let $\mathcal{E}$ be an exact category in the sense of Quillen. Recall, for an object X in $\mathcal{E}$, the projective dimension $\operatorname{pd} X$ is defined as the infimum of all $n \in \mathbb{N}_0$ such that $\operatorname{Ext}^{\geq n}_{\mathcal{E}}(X,-)=0$. Dually $\operatorname{id} X$ is defined as the infimum of all $n \in \mathbb{N}_0$ such that $\operatorname{Ext}^{\geq n}_{\mathcal{E}}(-,X)=0$. We consider

$$\mathcal{P}^{\leq n} := \{X \in \mathcal{E} \mid \operatorname{pd}_{\mathcal{E}} X \leq n\},$$
$$\mathcal{I}^{\leq n} := \{X \in \mathcal{E} \mid \operatorname{id}_{\mathcal{E}} X \leq n\},$$

and $\mathcal{P}^{<\infty} = \bigcup_{n\geq 0} \mathcal{P}^{\leq n}$, $\mathcal{I}^{<\infty} = \bigcup_{n \geq 0} \mathcal{I}^{\leq n}$. We also use the notation $\mathcal{P}^{<\infty}(\mathcal{E}), \mathcal{P}^{\leq n}(\mathcal{E})$ etc. for these categories.
Using long exact sequences on the Ext-groups it is easy to see that all these full subcategories are extension closed and that $\mathcal{P}^{\leq n}$ is deflation-closed and $\mathcal{I}^{\leq n}$ is inflation-closed, $\mathcal{P}^{<\infty}, \mathcal{I}^{<\infty}$ are thick subcategories. Throughout: All underlying additive categories of exact categories are assumed to be idempotent complete.
The suspension functor in $D^b(\mathcal{E})$ will be denoted by [1] from now on.

3. Singularity categories for exact categories

Summary: We will define singularity categories for exact categories in a naive manner which leaves the question if it is a derived invariant. There are different ways to address this - one way is to try to generalize Rickard's results for rings, more precisely: Given a two derived equivalent exact categories $\mathcal{E}$ and $\mathcal{E}'$. Is there a triangle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}')$ which restricts to $\operatorname{Thick}_\Delta(\mathcal{P}^{\infty}(\mathcal{E})) \to \operatorname{Thick}_\Delta(\mathcal{P}^{<\infty}(\mathcal{E}'))$? We follow an alternative idea of Orlov and define singularity

category for triangulated categories. The question here is to characterize exact categories for which these two coincide.

Definition 3.1. We say the $\mathcal{E}$ is **regular** if $\mathcal{E} = \mathcal{P}^{<\infty}(\mathcal{E})$.
We say X is **homologically finite** if for every $Y \in \mathcal{E}$ there exists an n such that $\operatorname{Ext}_{\mathcal{E}}^{\geq n}(X,Y) = 0$. We write $\mathcal{E}^{hf}$ for the full subcategory of homologically finite objects. We say that $\mathcal{E}$ is **Δ-regular** if $\mathcal{E} = \mathcal{E}^{hf}$.

Observe that we always have thick subcategories $\mathcal{P}^{<\infty}(\mathcal{E}) \subseteq \mathcal{E}^{hf} \subseteq \mathcal{E}$ (in the exact category sense, i.e. they fulfill the 2 out of 3 property for short exact sequences and are closed under summands).

Definition 3.2. If $\mathcal{T}$ is a triangulated category and $X \in \mathcal{T}$, we say that X is **homological finite** if for every $Y \in \mathcal{T}$ there is a finite subset $I \subseteq \mathbb{Z}$ such that $\operatorname{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for all $i \notin I$. We denote by $\mathcal{T}^{hf}$ the full subcategory of homological finite objects.
More generally, given a full additive category $\mathcal{C}$ of $\mathcal{T}$ we say that $X \in \mathcal{C}$ is **$\mathcal{C}$-homological finite** if for every $Y \in \mathcal{C}$ there is a finite subset $I \subseteq \mathbb{Z}$ such that $\operatorname{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for all $i \notin I$. We denote by $\mathcal{C}^{hf}$ the full subcategory of homological finite objects.

Observe that $\mathcal{T}^{hf}$ is thick in $\mathcal{T}$ (in the triangulated sense, i.e. it is a triangulated subcategory closed under summands).

We make the following easy observation:

Lemma 3.3. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ be a full additive category whose extension-closure is $\mathcal{T}$. Then $\mathcal{C} = \mathcal{C}^{hf}$ if and only if $\mathcal{T} = \mathcal{T}^{hf}$.*

PROOF. We assume $\mathcal{C} = \mathcal{C}^{hf}$. Let $X \in \mathcal{C}$. We show that $X \in \mathcal{T}^{hf}$: Let $Y \in \mathcal{T}$. By assumption there is a triangle $Y_1 \to Y \to Y_2 \to Y_1[1]$ with $Y_i \in \mathcal{C}$. Then there exist finite subsets I_1, I_2 of $\mathbb{Z}$ with $\operatorname{Hom}(X, Y_i[k]) = 0$ for $k \notin I_i$, $i \in \{1,2\}$. Then, just take $I = I_1 \cup I_2$ and for $k \notin I$ we conclude that $\operatorname{Hom}(X, Y[k]) = 0$. Therefore $X \in \mathcal{T}^{hf}$. It follows that $\mathcal{T} = \operatorname{Thick}_{\Delta}(\mathcal{C}) \subseteq \mathcal{T}^{hf}$. The other implication is trivially true. □

Corollary 3.4. *Let $\mathcal{E}$ be an exact category. Then $\mathcal{E} = \mathcal{E}^{hf}$ if and only if $\mathrm{D}^b(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})^{hf}$.*

Definition 3.5. Let $\mathcal{E}$ be an exact category. We define the **singularity category** as the Verdier quotient

$$\mathrm{D}_{sg}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E}) / \operatorname{Thick}_{\Delta}(\mathcal{P}^{<\infty}(\mathcal{E}))$$

For a triangulated category $\mathcal{T}$ we define the **Δ-singularity category** as the Verdier quotient

$$\mathcal{T}_{sg} = \mathcal{T} / \mathcal{T}^{hf}$$

Then every triangle equivalence $\mathcal{T} \to \mathcal{S}$ between triangulated categories induces a triangle equivalence $\mathcal{T}_{sg} \to \mathcal{S}_{sg}$. Clearly, since for an exact category $\mathcal{E}$ we have $\operatorname{Thick}_{\Delta}(\mathcal{P}^{<\infty}(\mathcal{E})) \subseteq \operatorname{Thick}_{\Delta}(\mathcal{E}^{hf}) \subseteq \mathrm{D}^b(\mathcal{E})^{hf}$ (for the second inclusion see next lemma) we get an induced Verdier quotient

$$\mathrm{D}_{sg}(\mathcal{E}) \to (\mathrm{D}^b(\mathcal{E}))_{sg}$$

Definition 3.6. We say $\mathcal{E}$ has **Δ-singularities** if this map is an equivalence.

Open question 3.7. When are the two singularity categories locally small (i.e. have Hom-sets)? If $\mathcal{E}$ is essentially small, then $D^b(\mathcal{E})$ is also essentially small and hence it holds. And more generally? When are they idempotent complete?

So if $\mathcal{E}$ and $\mathcal{E}'$ have Δ-singularities and are derived equivalent, then their singularity categories are equivalent.

Example 3.8. ([184, Example 3.3]) This is an example of two derived equivalent exact categories one is regular and the other one not. Furthermore, one has Δ-singularities and the other one not. Let $R = k[x_0, \dots, x_n]/\langle x_i^2, x_ix_j + x_jx_i\rangle$ be the exterior algebra and $S = k[x_0, \dots, x_n]$ a polynomial ring for k a field, both are graded algebras with $\mathrm{degx_i} = 1$, $0 \leq i \leq n$.
We consider the categories of graded modules $\mathcal{E} = \mathrm{grR}$ and $\mathcal{E}' = \mathrm{grS}$ with finite-dimensional graded parts. Then BGG-correspondence provides a triangle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}')$.
But $\mathrm{gldim}\,\mathcal{E}' < \infty$ and $\mathrm{gldim}\,\mathcal{E} = \infty$ as R is self-injective. This implies that $\mathrm{D}_{sg}(\mathcal{E}') = 0 = (\mathrm{D}^b(\mathcal{E}'))_{sg}$ but $\mathrm{D}_{sg}(\mathcal{E}) \neq 0 = (\mathrm{D}^b(\mathcal{E}))_{sg}$.

Observe that $\mathcal{E}$ has Δ-singularities if and only if $\mathcal{P}^{<\infty}(\mathcal{E}) = \mathcal{E}^{hf}$ and $\mathrm{Thick}_\Delta(\mathcal{E}^{hf}) = \mathrm{D}^b(\mathcal{E})^{hf}$. We ask if the last equality is always true?
Here is the answer in a special case:

Lemma 3.9. *Let $\mathcal{E}$ be an exact category. Then we have:*

(a) $\mathcal{E}^{hf} = \mathrm{D}^b(\mathcal{E})^{hf} \cap \mathcal{E}$ *where we consider $\mathcal{E} \subset \mathrm{D}^b(\mathcal{E})$ as stalk complexes in degree zero.*
(b) *If $\mathcal{E}$ is an exact category with enough projectives then* $\mathrm{Thick}_\Delta(\mathcal{E}^{hf}) = \mathrm{D}^b(\mathcal{E})^{hf}$.

Proof. (a) It is enough to show: $\mathcal{E}^{hf} \subseteq \mathrm{D}^b(\mathcal{E})^{hf}$. Let $X \in \mathcal{E}^{hf}$ and $Y \in \mathrm{D}^b(\mathcal{E})$. Assume there exists an infinite set $I \subseteq \mathbb{Z}$ such that $\mathrm{Hom}(X, Y[i]) \neq 0$ for $i \in I$. Since $\mathrm{Thick}_\Delta(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$ we may assume $Y \in \mathrm{Thick}^n_\Delta(\bigvee_m \mathcal{E}[m])$ and that there exists $Y_{n-1}, Y'_{n-1} \in \mathrm{Thick}^{n-1}_\Delta(\bigvee_m \mathcal{E}[m])$ and a triangle $Y_{n-1} \to Y \to Y'_{n-1} \xrightarrow{+1}$. Since $\mathrm{Hom}(X, -)$ is a cohomological functor, either for Y_{n-1} or for Y'_{n-1} there exists an infinite subset of $I_{n-1} \subseteq \mathbb{Z}$ with $\mathrm{Hom}(X, ..[i]) \neq 0$ for all $i \in I_{n-1}$. Then inductively, we can produce a $Y_0 \in \mathcal{E}$ such that there exists an infinite set $I_0 \subseteq \mathbb{Z}$ with $\mathrm{Hom}(X, Y_0[i]) \neq 0$ for all $i \in I_0$. Since $\mathrm{Hom}(X, Y[< 0]) = 0$ it follows that $I_0 \subseteq \mathbb{N}$ and therefore a contradiction to $X \in \mathcal{E}^{hf}$.
(b) We need to see $\mathrm{D}^b(\mathcal{E})^{hf} \subseteq \mathrm{Thick}_\Delta(\mathcal{E}^{hf})$. We identify $\mathrm{D}^b(\mathcal{E})$ with $\mathrm{K}^{b,-}(\mathcal{P})$ where $\mathcal{P}$ are the projectives in $\mathcal{E}$. Take $n \in \mathbb{Z}$, then one shows that $X \in \mathrm{K}^{b,-}(\mathcal{P})^{hf}$ is equivalent to $\sigma_{\leq n}X$ and $\sigma_{>n}X$ are homologically finite. Furthermore, we observe $X \in \mathrm{Thick}_\Delta(\sigma_{\leq n}X, \sigma_{>n}X)$. Since for $|n| >> 0$ we have that $\sigma_{\leq n}X$ is quasi-isomorphic to a shifted stalk complex - this has to lie in $\mathrm{Thick}_\Delta(\mathcal{E}^{hf})$. By definition $\sigma_{>n}X \in \mathrm{K}^b(\mathcal{P}) \subseteq \mathrm{Thick}_\Delta(\mathcal{E}^{hf})$ and therefore $X \in \mathrm{Thick}_\Delta(\mathcal{E}^{hf})$. □

Then let us state the obvious:

Lemma 3.10. *The following are equivalent:*

(1) *$\mathcal{E}$ is regular.*
(2) $\mathrm{D}_{sg}(\mathcal{E}) = 0$.

Furthermore, the following are equivalent:

(a) *$\mathcal{E}$ is Δ-regular.*
(b) $(\mathrm{D}^b(\mathcal{E}))_{sg} = 0$.

Proof. The implication '(1) implies (2)' is obvious since $\mathrm{Thick}_\Delta(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$.
Assume (2), i.e. we assume $\mathcal{E} \subseteq \mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}(\mathcal{E}))$. We look at

$$\mathrm{D}^{<\infty,\mathcal{E}}(\mathcal{E}) := \{X \in \mathrm{D}^b(\mathcal{E}) \mid \exists m \in \mathbb{Z} \text{ such that } \mathrm{Hom}(X, \mathcal{E}[n]) = 0 \; \forall n \geq m\}$$

this is a thick subcategory of $\mathrm{D}^b(\mathcal{E})$. It contains $\mathcal{P}^{<\infty}$ and so by assumption we have $\mathcal{E} \subseteq \mathrm{Thick}(\mathcal{P}^{<\infty}(\mathcal{E})) \subseteq \mathrm{D}^{<\infty,\mathcal{E}}(\mathcal{E})$. This implies $\mathcal{E} \subseteq \mathcal{P}^{<\infty}$.
The implication '(a) implies (b)' is obvious as in the previous proof. Now assume that (b), i.e. $(\mathrm{D}^b(\mathcal{E}))^{hf} = \mathrm{D}^b(\mathcal{E})$. We intersect with the stalks to get $\mathcal{E} = \mathrm{D}^b(\mathcal{E})^{hf} \cap \mathcal{E} \subseteq \mathcal{E}^{hf}$. □

Corollary 3.11. *Let $f\colon \mathcal{E} \to \mathcal{A}$ be an exact functor between exact categories. If the derived functor $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ is a triangle equivalence, then: $\mathcal{E}$ regular if and only if $\mathcal{A}$ is regular.*

PROOF. Assume $\mathcal{E}$ is regular. $f\colon \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ is a triangle equivalence which restricts to $\mathcal{E} \to \mathcal{A}$ on stalk complexes. Therefore (using the definition of the previous proof) it restricts to a triangle functor $\mathrm{D}^{<\infty,\mathcal{E}} \to \mathrm{D}^{<\infty,\mathcal{A}}$. Since $\mathcal{E}$ is regular, it follows as in the previous proof that $\mathrm{D}^b(\mathcal{E}) = \mathrm{D}^{<\infty,\mathcal{E}}$. This implies that the essential image $\mathrm{D}^b(\mathcal{A}) = f(\mathrm{D}^b(\mathcal{E})) \subseteq \mathrm{D}^{<\infty,\mathcal{A}}$. In particular, it follows $\mathcal{A} \subseteq \mathrm{D}^{<\infty,\mathcal{A}}$ and this imples $\mathcal{A} = \mathcal{P}^{<\infty}(\mathcal{A})$.
If $\mathcal{A}$ is regular, then since f is homologically exact (cf. Chapter 1), it follows that $\mathcal{E}$ is also regular. □

Remark 3.12. If $\mathcal{E}$ is an exact category with enough projectives $\mathcal{P}$ then we have

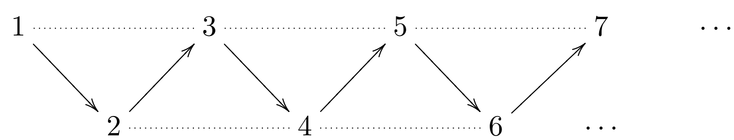
$$\mathrm{D}_{sg}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})/\mathrm{K}^b(\mathcal{P})$$

Lemma 3.13. *If* $\mathcal{E} = \mathrm{Filt}(\mathrm{M}_1, \dots, \mathrm{M}_\mathrm{n})$ *then* $\operatorname{pd} X = \max(m_i \mid 1 \le i \le n)$ *with* $m_i := \inf\{m \in \mathbb{N}_{\ge 0} \mid \mathrm{Ext}^{>m}(X, M_i) = 0\} \in \mathbb{N}_{\ge 0} \cup \{\infty\}$.
This implies $\mathcal{P}^{<\infty}(\mathcal{E}) = \mathcal{E}^{hf}$. *If* $\mathcal{E}$ *has enough projectives then we get that* $\mathcal{E}$ *has* Δ*-singularities.*

Lemma 3.14. *Let* $\mathcal{E}$ *be an exact category with infinite coproducts. Then* $\mathcal{P}^{<\infty}(\mathcal{E}) = \mathcal{E}^{hf}$. *So if* $\mathcal{E}$ *has also enough projectives then it has* Δ*-singularities.*

PROOF. Let $\operatorname{pd} X = \infty$. There exists an infinite subset $I \subset \mathbb{N}$ and objects $Y_n, n \in I$ such that $\mathrm{Ext}^n_{\mathcal{E}}(X, Y_n) \neq 0$. So let $Y = \bigoplus_{n \in I} Y_n \in \mathcal{E}$. Then for every $n \in I$ we have $\mathrm{Ext}^n_{\mathcal{E}}(X, Y) = \mathrm{Ext}^n_{\mathcal{E}}(X, Y_n) \oplus \mathrm{Ext}^n_{\mathcal{E}}(X, \bigoplus_{m \in I, m \neq n} Y_m) \neq 0$. Therefore X is not homological finite. □

Remark 3.15. Here is an example which is Δ-regular but not regular: Let $\mathcal{E}$ be the abelian category of all representations (over some field) of the following infinite quiver with relations that any two consecutive arrows are zero



Indecomposables are either projectives or simples, all simples have infinite projective dimension. Nevertheless all indecomposables are homologically finite in $\mathcal{E}$.

Example 3.16. ([**48**, Theorem 4.4.1]) If R is any left and right noetherian ring, then Buchweitz introduced the singularity category for $\mathcal{E} = R\,\mathrm{Mod}$ and showed the theorem for $R\,\mathrm{mod}$ (f.g. R-modules).
Of course this can be defined for every ring. If R is left coherent and semiperfect, then $R\,\mathrm{mod}$ has Δ-singularities, cf. [**126**, Prop. 9.2.14].
If R is any ring then $R\,\mathrm{Mod}$ has Δ-singularities by loc. cit. Lemma 9.2.3.

Example 3.17. (Orlov, 2004 in [**147**]) Now we consider the following geometric situation: Let X be a scheme over a field K which is separated, noetherian, of finite Krull dimension and coh(X) has enough locally frees - following Orlov [**147**] we will call these properties (ELF). (The last assumption is also called the resolution property cf. [**175**, Tag 0F85]) Orlov introduced in [**147**] the singularity category of X as the Verdier quotient

$$\mathrm{D}_{sg}(\mathrm{coh(X)}) = \mathrm{D^b}(\mathrm{coh(X)})/\mathrm{D^{perf}}(\mathrm{X})$$

If X is ELF, then coh(X) has Δ-singularities (Orlov [**148**], Prop.1.11).

Here are some of my open questions to this subsection:

(3.1) If $\mathcal{E}$ is an exact category, is $\mathcal{P}^{<\infty}(\mathcal{E})$ always regular (as fully exact category of $\mathcal{E}$)? Or stronger: Is $\mathcal{P}^{<\infty}(\mathcal{E}) \subseteq \mathcal{E}$ always homologically exact?
(3.2) If we consider $\mathcal{E}^{hf}$ as homologically finite objects in $\mathcal{E}$. Are all objects in $\mathcal{E}^{hf}$ homologically finite in $\mathcal{E}^{hf}$? Or stronger: Is $\mathcal{E}^{hf} \subseteq \mathcal{E}$ homologically exact?
(3.3) Is $\mathrm{Thick}_\Delta(\mathcal{E}^{hf}) = \mathrm{D}^b(\mathcal{E})^{hf}$? This would imply: $\mathcal{P}^{<\infty} = \mathcal{E}^{hf}$ is equivalent to $\mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) = \mathrm{D}^b(\mathcal{E})^{hf}$?
(3.5) For which exact categories $\mathcal{E}$ do we have $\mathrm{Thick}_\Delta(\mathcal{P}^{<\infty}) = \mathrm{D}^b(\mathcal{E})^{hf}$?

4. Descriptions as stable categories - Buchweitz theorem

We start with the following definition:

Definition 4.1. Let $n \geq 0$. An exact category $\mathcal{E}$ is called **Gorenstein** if $\mathcal{I}^{<\infty} = \mathcal{P}^{<\infty}$. We say it is n-**Gorenstein** if we have $\mathcal{I}^{\leq n} = \mathcal{I}^{<\infty} = \mathcal{P}^{<\infty} = \mathcal{P}^{\leq n}$.

This is by definition a *symmetric* condition (it holds for $\mathcal{E}$ if and only if it holds for $\mathcal{E}^{op}$).

Remark 4.2. Of course, one can define dually, the *injective* singularity category

$$D_{sg-inj}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E}) / \operatorname{Thick}_\Delta(\mathcal{I}^{<\infty})$$

Then $\mathcal{E}$ is Gorenstein if and only if we have $\mathrm{D}^b_{sg}(\mathcal{E}) = \mathrm{D}^b_{sg-inj}(\mathcal{E})$ (= here means they are Verdier quotients of $\mathrm{D}^b(\mathcal{E})$ with the same kernels). In general, we do not know when these two singularity categories are triangle equivalent.

Remark 4.3. One could define *Gorenstein* for triangulated categories as $\mathcal{T}^{hf} = \mathcal{T}^{chf}$ (where cohomologically finite elements are defined dually to homologically finite), so imposing the symmetry condition of the previous remark for Orlov's singularity categories.

We recall from [**172**] the following definition: A full subcategory $\mathcal{P} \subseteq \mathcal{E}$ is called cotilting (resp. n-cotilting) if and only if the following hold

(C1) ${}^\perp\mathcal{P}$ has enough injectives given by $\mathcal{P}$ itself and
(C2) $\operatorname{Res}({}^\perp\mathcal{P}) = \mathcal{E}$.

(resp. (C1) and $(\mathrm{C2})_n$ $\operatorname{Res}_n({}^\perp\mathcal{P}) = \mathcal{E}$).

Lemma 4.4. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. If $\mathcal{P}$ is cotilting then ${}^\perp\mathcal{P}$ is Frobenius exact with enough injectives given by $\mathcal{P}$ and we have $\mathcal{P}^{<\infty} \cap {}^\perp\mathcal{P} = \mathcal{P}$.*

Proof. By definition this category has enough injectives given by $\mathcal{P}$, an easy check shows that ${}^\perp\mathcal{P}$ is resolving in $\mathcal{E}$, so it also has enough projectives given by $\mathcal{P}$.
If $X \in \mathcal{P}^{<\infty} \cap {}^\perp\mathcal{P}$ then there exists an $n \in \mathbb{N}$ such that $\operatorname{Ext}^{>n}_{\mathcal{E}}(X, -) = 0$. This implies, as ${}^\perp\mathcal{P}$ is homologically exact in $\mathcal{E}$, that $\operatorname{Ext}^{>n}_{{}^\perp\mathcal{P}}(X, -) = 0$. This implies that $\operatorname{Ext}^1_{{}^\perp\mathcal{P}}(X, \Omega^{-n}Y) = 0$ for all $Y \in {}^\perp\mathcal{P}$. But every object in ${}^\perp\mathcal{P}$ is an n-th cosyzygy, so X in $\mathcal{P}({}^\perp\mathcal{P}) = \mathcal{P}$. □

Proposition 4.5. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then we have*

(1) *If $\mathcal{P}$ is n-cotilting then $\mathcal{E}$ is n-Gorenstein*
(2) *If $\mathcal{E}$ is n-Gorenstein and ${}^\perp\mathcal{P} \subseteq \operatorname{cogen}_{\mathcal{E}}(\mathcal{P})$ then $\mathcal{P}$ is n-cotilting*

It would be much nicer if we had an equivalence in (1) but we could not see how to prove that n-Gorenstein implies ${}^\perp\mathcal{P} \subseteq \operatorname{cogen}(\mathcal{P})$. But in special situations this is fulfilled.

Proof. (1) If $\mathcal{P}$ is n-cotilting then $\operatorname{Thick}(\mathcal{P}) = \mathcal{I}^{<\infty}$ follows from [**172**, Lem. 5.8]. But $\operatorname{Thick}(\mathcal{P}) = \mathcal{P}^{<\infty}$ then implies that $\mathcal{E}$ is Gorenstein. Now, we show $\mathcal{P}^{<\infty} \subseteq \mathcal{P}^{\leq n}$. Take $X \in \mathcal{P}^\infty$ a projective resolution

$$\Omega^n X \rightarrowtail P_{n-1} \to \cdots \to P_0 \twoheadrightarrow X$$

Then by dimension shift $\operatorname{Ext}^{>0}_{\mathcal{E}}(\Omega^n X, P) = \operatorname{Ext}^{>n}(X, P) = 0$ for all $P \in \mathcal{P}$, therefore $\Omega^n X \in {}^\perp\mathcal{P}$ and so $\Omega^n X \in \mathcal{P}^{<\infty} \cap {}^\perp\mathcal{P} = \mathcal{P}$ by Lemma 4.4.
Now, we want to see that $\mathcal{I}^\infty \subseteq \mathcal{I}^{\leq n}$: We have $\mathcal{I}^{<\infty} = \mathcal{P}^{\leq n}$. Assume $Y \in {}^\perp\mathcal{P}$, then we easily verify $\operatorname{Ext}^{>n}_{\mathcal{E}}(Y, X) = 0$ (using the projective resolution of X). If we look at an arbitrary Y in $\mathcal{E}$, then clearly $\operatorname{Ext}^{>2n}_{\mathcal{E}}(Y, X) = 0$ (using $\operatorname{pd}_{\mathcal{E}} X \leq n, \operatorname{id}_{\mathcal{E}} \mathcal{P} \leq n$). Assume $\operatorname{Ext}^{m+n}_{\mathcal{E}}(Y, X) \neq 0$ for some

$m \in \{1, \dots n\}$. Then $\Omega^n Y \in {}^\perp\mathcal{P}$ and there exists an $Y' \in {}^\perp\mathcal{P}$ such that $\Omega^n_{\mathcal{E}} Y = \Omega^n_{\mathcal{E}} Y'$ (use the first bit of the injective coresolution of $\Omega^n Y$ to find Y'); an easy dimension shift shows

$$\mathrm{Ext}^{m+n}_{\mathcal{E}}(Y, X) \cong \mathrm{Ext}^m_{\mathcal{E}}(\Omega^n_{\mathcal{E}} Y, X) \cong \mathrm{Ext}^{m+n}_{\mathcal{E}}(Y', X) \neq 0$$

This contradicts our previous observation.
(2) assume $\mathrm{id}_{\mathcal{E}}\, \mathcal{P} \leq n$, take X in $\mathcal{E}$ arbitrary and look at the beginning of a projective resolution of X:

$$\Omega^n X \rightarrowtail P_{n-1} \to \cdots \to P_0 \twoheadrightarrow X$$

then $\Omega^n X \in {}^\perp\mathcal{P}$ and $X \in \mathrm{Res}_n({}^\perp\mathcal{P})$, so (C2) holds. Condition (C1) is implied by $\mathcal{P}^\perp \subseteq \mathrm{cogen}_{\mathcal{E}}\, \mathcal{P}$: Clearly $\mathcal{P}$ are injective objects in ${}^\perp\mathcal{P}$ (as ${}^\perp\mathcal{P}$ is a resolving subcategory, it is homologically finite). For $X \in {}^\perp\mathcal{P}$ there is an $\mathcal{E}$-short exact sequence $X \rightarrowtail P \twoheadrightarrow Y$ such that $\mathrm{Hom}(-, P')$ is exact on it for all $P' \in \mathcal{P}$ (this follows by the definition of $\mathrm{cogen}(\mathcal{P})$). But then it follows $Y \in {}^\perp\mathcal{P}$ and this shows that we have enough injectives given by $\mathcal{P}$, so (C1) follows. □

This is the classical result for rings.

Example 4.6. Let R be a left and right noetherian ring and $R\,\mathrm{mod}$ (resp. $\mathrm{mod}\, R$) the category of finetely generated left (resp. right) R-module. In this case, we say R is n-**Iwanaga-Gorenstein** if $\mathrm{id}\, {}_R R \leq n$ and $\mathrm{id}\, R_R \leq n$. Then the following are equivalent:

(1) R is n-Iwanaga-Gorenstein
(2) $R\,\mathrm{Mod}$ and $\mathrm{Mod}\, R$ are n-Gorenstein
(3) $R\,\mathrm{mod}$ and $\mathrm{mod}\, R$ are n-Gorenstein

The implication (1) implies (2) and (1) implies (3) are a famous result of Iwanaga [**101**, Theorem 2]. The implication (2) implies (1) is trivial, and (3) implies (1) follows from $\mathrm{id}_{R\,\mathrm{mod}}\, R = \mathrm{id}_R\, R$ and $\mathrm{id}_{\mathrm{mod}\, R}\, R = \mathrm{id}\, R_R$.
Observe that $R\,\mathrm{mod}$ and $\mathrm{mod}\, R$ are abelian categories with enough projectives but in general not with enough injectives.

Lemma 4.7. *Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough projectives and enough injectives. If $\mathcal{E}$ is n-Gorenstein and $\mathcal{P}$ is covariantly finite in ${}^\perp\mathcal{P}$, then $\mathcal{P}$ is n-cotilting.*

Proof. By Prop. 4.5, (2), it is enough to show ${}^\perp\mathcal{P} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{P})$. As $\mathcal{P}$ is assumed covariantly finite in ${}^\perp\mathcal{P}$, it is enough to show that ${}^\perp\mathcal{P} \subseteq \mathrm{copres}_{\mathcal{E}}(\mathcal{P})$. For $X \in {}^\perp\mathcal{P}$ take an $\mathcal{E}$-inflation $i \colon X \rightarrowtail I$ with I in $\mathcal{I}(\mathcal{E})$. Then take a deflation $p \colon P \twoheadrightarrow I$ with $P \in \mathcal{P}(\mathcal{E})$. As $\mathcal{E}$ is n-Gorenstein, and $I, P \in \mathcal{P}^{<\infty}$, it follows that $L := \ker p \in \mathcal{P}^{<\infty}$. Using a finite projective resolution of L, one sees $\mathrm{Ext}^1_{\mathcal{E}}(X, L) = 0$. This implies that i factors as $i = pf$. By the obscure axiom ([**49**, Prop. 7.6]), we conclude that $f \colon X \rightarrowtail P$ is an inflation. □

Open question 4.8. Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$. We also assume that $\mathcal{P} \subseteq {}^\perp\mathcal{P}$ is covariantly finite and $\mathcal{I}$ contravariantly finite in $\mathcal{I}^\perp$. Then the following are equivalent:

(i) $\mathcal{E}$ is n-Gorenstein
(ii) $\mathrm{id}_{\mathcal{E}}\, \mathcal{P} \leq n$ and $\mathrm{pd}_{\mathcal{E}}\, \mathcal{I} \leq n$.
(iii) $\mathcal{E}^{op}$ is n-Gorenstein.
(iv) There exists a subcategory which is s-tilting and t-cotilting for some $s, t \geq 0$.
(v) A subcategory is s-cotilting for some s if and only if it is t-cotilting for some t.

Observe that we have already seen that (i),(ii),(iii) are equivalent, and (i) implies (iv). In (ii), do we also have $\mathrm{pd}_{\mathcal{E}}\, \mathcal{I} = \mathrm{id}_{\mathcal{E}}\, \mathcal{P}$?

Definition 4.9. Given an exact category $\mathcal{E}$ and we define $\mathcal{P} = \mathcal{P}(\mathcal{E})$ be its projectives. The category of **Gorenstein projectives** (denoted by $\mathbf{Gp}(\mathcal{E})$) are the full subcategory of objects X such that there exists an exact complex of projectives

$$\cdots \to P_{-1} \to P_0 \to P_1 \to \cdots$$

such that

(1) $\cdots \to \mathrm{Hom}(P_n, P) \to \mathrm{Hom}(P_{n-1}, P) \to \cdots$ is exact for all P in $\mathcal{P}$
(2) $\mathrm{Im}(P_{-1} \to P_0) = X$

Proposition 4.10. *Let $\mathcal{E}$ be an exact category and $\mathcal{P} := \mathcal{P}(\mathcal{E})$.*

(1) *Then $\mathbf{Gp}(\mathcal{E})$ is extension-closed, closed under taking summands and deflation-closed (i.e. closed under taking kernels of deflations) and we have $\mathbf{Gp}(\mathcal{E}) \subseteq {}^{\perp}\mathcal{P}$. With this exact structure it is a Frobenius exact category with projectives $\mathcal{P}$.*
(2) *$\mathbf{Gp}(\mathcal{E})$ is resolving if and only if $\mathcal{E}$ has enough projectives.*
(3) *$\mathbf{Gp}(\mathcal{E})$ is finitely resolving (resp. n-resolving) if and only if $\mathcal{E}$ has enough projectives and $\mathcal{P}$ is cotilting (resp. n-cotilting). In these cases we have $\mathbf{Gp}(\mathcal{E}) = {}^{\perp}\mathcal{P}$.*

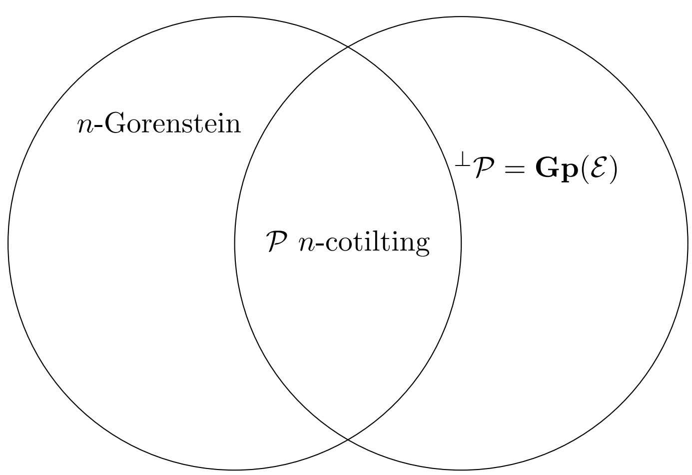
Proof. (1) The proof from [54, Prop. 2.1.7 (1),(2),(3)] can also be used to prove extension-closedness, summand-closedness and deflation-closedness. By definition, $X \in \mathbf{Gp}(\mathcal{E})$ implies all $X_n = \mathrm{Im}(P_{n-1} \to P_n) \in \mathbf{Gp}(\mathcal{E})$ and all short exact sequence $X_n \rightarrowtail P_n \twoheadrightarrow X_{n+1}$ are short exact in $\mathbf{Gp}(\mathcal{E})$, apply $\mathrm{Hom}(-, P)$ with $P \in \mathcal{P}$ to these short exact sequences to conclude $\mathrm{Ext}^{>0}(X, P) = 0$, so $\mathcal{P} \subseteq \mathcal{I}(\mathbf{Gp}(\mathcal{E}))$. Then just use the defining exact sequence in $\mathcal{P}$ to conclude that $\mathbf{Gp}(\mathcal{E})$ is Frobenius.
(2) If $\mathcal{E}$ has enough projectives then these have to be in $\mathbf{Gp}(\mathcal{E})$ and by (1) it is resolving. If $\mathbf{Gp}(\mathcal{E})$ is resolving, we also know it has enough projectives $\mathcal{P} = \mathcal{P}(\mathbf{Gp}(\mathcal{E}))$ (by (1)).
(3) If $\mathcal{P}$ is cotilting then (C1) implies ${}^{\perp}\mathcal{P} \subseteq \mathrm{cogen}_{\infty}(\mathcal{P})$ and the other inclusion follows from the definition of $\mathrm{cogen}_{\infty}(\mathcal{P})$. Then for $X \in {}^{\perp}\mathcal{P} = \mathrm{cogen}_{\infty}(\mathcal{P})$ splice together the projective resolution in $\mathcal{E}$ with the injective coresolution, this shows $X \in \mathbf{Gp}(\mathcal{E})$. By definition $\mathbf{Gp}(\mathcal{E}) \subseteq \mathrm{cogen}_{\infty}(\mathcal{P})$, so we conclude in this case $\mathbf{Gp}(\mathcal{E}) = {}^{\perp}\mathcal{P}$ is finitely resolving (n-resolving if $\mathcal{P}$ was n-cotilting).
Conversely, if $\mathbf{Gp}(\mathcal{E})$ is finitely (resp. n-)resolving, we already know from (1) that $\mathbf{Gp}(\mathcal{E}) \subseteq \mathcal{P}^{\perp}$. We need to see the other inclusion, this follows immediately from the Lemma 4.11. But then all properties for $\mathcal{P}$ being (resp. n-)cotilting are fulfilled.

□

Lemma 4.11. *Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Given an exact sequence $E_1 \rightarrowtail E_0 \twoheadrightarrow X$ with $X \in {}^{\perp}\mathcal{P}, E_0, E_1 \in \mathbf{Gp}(\mathcal{E})$. Then $X \in \mathbf{Gp}(\mathcal{E})$.*

Proof. This can be shown with the same argument as [54, Prop. 2.1.7 (4)]. □

in exact categories wep. $\mathcal{P}$:



Lemma 4.12. *Assume that $\mathcal{E}$ is a weakly idempotent complete exact category with enough projectives $\mathcal{P}$ and assume $\mathcal{E} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{I}^{<\infty})$. If $\mathcal{E}$ is n-Gorenstein, then $\mathcal{P}$ is n-cotilting.*

The proof is very similar to Lemma 4.7.

PROOF. We show that ${}^{\perp}\mathcal{P} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{P})$ (the rest follows from Prop. 4.5, (2)).
We assume $\mathcal{E} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{I}^{<\infty})$: So, for $X \in {}^{\perp}\mathcal{P}$, we take a short exact sequence $X \overset{i}{\rightarrowtail} J \twoheadrightarrow Q$ with $J \in \mathcal{I}^{<\infty}$. We choose a short exact sequence $J_1 \rightarrowtail P \overset{p}{\twoheadrightarrow} J$ with $P \in \mathcal{P}$. As $\mathcal{I}^{<\infty} = \mathcal{P}^{<\infty}$ is deflation-closed, we find that $J_1 \in \mathcal{P}^{<\infty}$ and one easily checks $\mathrm{Ext}^{>0}(X, J_1) = 0$ for all $X \in {}^{\perp}\mathcal{P}$ (using the finite projective resolution of J_1). This implies $\mathrm{Hom}(X, P) \to \mathrm{Hom}(X, J)$ is surjective, pick a morphism $f \colon X \to P$ that maps to i, say $fp = i$. By the obscure axiom f is an $\mathcal{E}$-inflation. Now, we need to see that $\mathrm{Hom}(f, P)$ is surjective for every $P \in \mathcal{P}$. But since $\mathrm{Hom}(i, P) = \mathrm{Hom}(f, P) \circ \mathrm{Hom}(p, P)$, this follows. Then we see ${}^{\perp}\mathcal{P} \subseteq \mathrm{cogen}_{\mathcal{E}}(\mathcal{P})$. □

Example 4.13. From [**70**]: If R is an n-Iwanaga-Gorenstein ring and $\mathcal{F} = R - \mathrm{Mod}$, then $\mathcal{F} \subseteq \mathrm{cogen}_{\mathcal{F}}(\mathcal{I}^{<\infty})$ and this implies $\mathbf{Gp}(R\,\mathrm{Mod}) = {}^{\perp}(\mathrm{Proj}(R))$.
This implies for $\mathcal{E} = R\,\mathrm{mod}$, i.e. the category of finitely generated left R-modules, that $\mathbf{Gp}(\mathcal{E}) = {}^{\perp_{\mathcal{E}}}R =: {}^{\perp}R$, to see this, recall that we only needed to see ${}^{\perp}R \subseteq \mathrm{cogen}_{\mathcal{E}}(R)$. But by the previous result we have ${}^{\perp}R \subseteq \mathrm{cogen}_{R\,\mathrm{Mod}}(\mathrm{ADD}(R))$ and observe that every finitely generated submodule of a free R-module is contained in a finitely generated free summand, this implies the claim.

Here we have the following result

THEOREM 4.14. *([**108**, Cor 2.2], [**123**, Ex. 2.3]) Let $\mathcal{E}$ be a weakly idempotent complete Frobenius exact category and let $\mathcal{P} = \mathcal{P}(\mathcal{E})$ be the projectives in $\mathcal{E}$. Then the functor $\mathcal{E} \to \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}_{sg}(\mathcal{E})$ induces a triangle equivalence*

$$\underline{\mathcal{E}} \to \mathrm{D}_{sg}(\mathcal{E})$$

As a corollary, follows the following result of Kvamme (in the special case of weakly idempotent complete exact categories). Just take $\mathbf{Gp}(\mathcal{E})$ as Frobenius exact category and use Prop. 4.10 (observe this implies: if $\mathcal{E}$ has enough projectives, then $\mathrm{D}^b(\mathbf{Gp}(\mathcal{E})) \to \mathrm{D}^b(\mathcal{E})$ is fully faithful. If $\mathbf{Gp}(\mathcal{E})$ is finitely resolving, it is a triangle equivalence, cf. [**92**]).

THEOREM 4.15. *([**131**]) Let $\mathcal{E}$ be an exact category with enough projectives. Then $\mathbf{Gp}(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b_{sg}(\mathcal{E})$ induces a fully faithful triangulated functor*

$$\underline{\mathbf{Gp}(\mathcal{E})} \to \mathrm{D}_{sg}(\mathcal{E}).$$

This is an equivalence if $\mathbf{Gp}(\mathcal{E})$ is finitely resolving in $\mathcal{E}$.

We prefer to reformulate this last statement to:

THEOREM 4.16. *(Buchweitz Theorem)*
Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and assume that $\mathcal{P}$ is n-cotilting. Then, the functor $\mathbf{Gp}(\mathcal{E}) \to \mathrm{D}^b_{sg}(\mathcal{E})$ induces a triangle equivalence

$$\underline{\mathbf{Gp}(\mathcal{E})} \to \mathrm{D}_{sg}(\mathcal{E})$$

Now, if $\mathcal{E}$ is an exact category with enough projectives, $\mathrm{D}_{sg}(\mathcal{E})$ can be realized as the Heller stabilisation $\mathbb{Z}\underline{\mathcal{E}}$ of the stable category of $\mathcal{E}$ seen as a left triangulated category [**131**, Thm 3.4]. Since the stabilization is functorial, an equivalence of left triangulated stable category $\underline{\mathcal{E}} \cong \underline{\mathcal{E}'}$ implies $\mathcal{E}$ and $\mathcal{E}'$ are singular equivalent (cf. [**131**]), but it also implies that $\mathcal{E}$ and $\mathcal{E}'$ are stable equivalent (investigated in the next chapter). That is the only connection between singular and stable equivalence that we know of.

Here is the my main open questions in this subsection

(4.1) Can we find singular invariants? Can we find classes of singular equivalent exact categories which are not derived equivalent (inspired by Knörrer periodicity)?

5. Non-commutative resolutions from exact substructures

Let $\mathcal{A}$ be an exact category. We fix an exact substructure $\mathcal{E}$ of $\mathcal{A}$. We observe that this gives a Verdier localization sequence

$$\mathrm{Ac}(\mathcal{A})/\mathrm{Ac}(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$$

If $\mathcal{E}$ is regular then we want to interpret $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ as a *categorical desingularization* (following Orlov's definition - only that Orlov required that $\mathcal{E}$ is also abelian).

Definition 5.1. We fix an exact category $\mathcal{A}$ and an exact substructure $\mathcal{E}$. Let $d \geq 0$ be an integer. We will write **NCR** as a shorthand for *non-commutative resolution* throughout the rest of the chapter.

(*) We call $\mathcal{E}$ a **weak (d-)NCR** if $\mathcal{E}$ is regular (resp. $\operatorname{gldim} \mathcal{E} \leq d$).
(*) We call $\mathcal{E}$ a **(d-)NCR** if $\mathcal{E}$ is regular (resp. $\operatorname{gldim} \mathcal{E} \leq d$) and has enough projectives.
(*) We call $\mathcal{E}$ a **strong (d-)NCR** if $\mathcal{E}$ and has enough projectives $\mathcal{Q}$ and $\mathcal{Q}$ has pseudo-kernels and $\operatorname{mod}_\infty \mathcal{Q}$ is regular (resp. $\operatorname{gldim}(\operatorname{mod}_\infty \mathcal{Q}) \leq d$).

Naively, we expect to find weak NCRs in algebraic geometric situations and NCRs when studying certain module or functor categories. For $\mathcal{A}$ Frobenius exact category, what we call a strong $d-$NCR is defined in [**112**] just as an NCR.
The existence of a strong d-NCR for a Frobenius category has the consequence that it is equivalent to the Gorenstein-projectives in $\operatorname{mod}_\infty \mathcal{P}$.

Theorem 5.2. *Let $\mathcal{A}$ be an idempotent complete exact category with enough projectives $\mathcal{P}$ and assume it has a strong d-NCR, then*

$$\mathbb{P} \colon \mathcal{A} \to \operatorname{mod}_\infty \mathcal{P}, \quad X \mapsto \operatorname{Hom}(-, X)|_{\mathcal{P}}$$

has a finitely resolving image.
Furthermore, if $\mathcal{P}$ is n-cotilting in $\mathcal{A}$, then $\mathcal{P}$ is also $(n+d)$-cotilting in $\operatorname{mod}_\infty \mathcal{P}$ and $\mathbb{P}$ restricts to an equivalence of exact categories

$$\mathbb{GP} \colon \mathbf{Gp}(\mathcal{A}) \to \mathbf{Gp}(\operatorname{mod}_\infty \mathcal{P}).$$

Before we give the proof, let us remark the following corollary which shows that for some exact categories a strong d-NCR can not exist (because their bounded derived category does not have a t-structure).

Corollary 5.3. *If $\mathcal{A}$ is an idempotent complete exact category with enough projectives and $\mathcal{A}$ admits a strong d-NCR, then $\mathcal{A}$ is derived equivalent to an abelian category with enough projectives.*

Since the inclusion of a finitely resolving subcategory induces a triangle equivalence on bounded derived categories, the corollary follows.

Proof. Let $\mathcal{E}$ be a strong d-NCR and $\mathcal{Q} = \mathcal{P}(\mathcal{E})$. We have that the restriction functor

$$f = (-)|_{\mathcal{P}} \colon \operatorname{mod}_\infty \mathcal{Q} \to \operatorname{mod}_\infty \mathcal{P}$$

is exact and essentially surjective because $\mathcal{P} \subseteq \mathcal{Q}$ is a full subcategory. Now, we look at the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathbb{P}_{\mathcal{A}}} & \operatorname{mod}_\infty \mathcal{P} \\ \downarrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle f} \\ \mathcal{E} & \xrightarrow{\mathbb{P}_{\mathcal{E}}} & \operatorname{mod}_\infty \mathcal{Q} \end{array}$$

Since $\operatorname{gldim}(\operatorname{mod}_\infty \mathcal{Q}) \le d$, we have $\operatorname{Im} \mathbb{P}_\mathcal{E}$ is d-resolving. Since $\mathbb{P}_\mathcal{A} = f \circ \mathbb{P}_\mathcal{E}$ and f is exact and essentially surjective, it follows that $\operatorname{Im} \mathbb{P}_\mathcal{A}$ is also d-resolving.
Now assume additionally that $\mathcal{P}$ is n-cotilting, clearly the functor $\mathbb{P}_\mathcal{A}$ maps complete resolution by projectives into complete resolutions by projectives, therefore it restricts to a functor on Gorenstein projectives (call this $\mathbb{GP}$). Since $\mathbf{Gp}(\mathcal{A})$ is n-resolving in $\mathcal{A} \cong \operatorname{Im} \mathbb{P}$ which is d-resolving in $\operatorname{mod}_\infty \mathcal{P}$, we conclude $\operatorname{Im} \mathbb{GP}$ is finitely resolving in $\operatorname{mod}_\infty \mathcal{P}$. This implies $\mathbf{Gp}(\operatorname{mod}_\infty \mathcal{P})$ is finitely resolving in $\operatorname{mod}_\infty \mathcal{P}$ and therefore $\operatorname{mod}_\infty \mathcal{P}$ is $\mathcal{P}$ cotilting in $\operatorname{mod}_\infty \mathcal{P}$ by Prop. 4.10. As $\mathcal{A} = \operatorname{Im} \mathbb{P}$ is d-resolving in $\operatorname{mod}_\infty \mathcal{P}$ and $\operatorname{id}_\mathcal{A} \mathcal{P} \le n$, it follows easily by dimension shift that $\operatorname{id}_{\operatorname{mod}_\infty \mathcal{P}} \mathcal{P} \le n + d$ and therefore $\mathcal{P}$ is $(n + d)$-cotilting.
We still need to see that $\mathbb{GP}$ is essentially surjective. But $\mathbb{GP}$ induces an triangle equivalence (since the image is finitely resolving) which induces a triangle equivalence on the singularity categories. But since these are the stable categories we conclude that $\mathbb{GP}$ is essentially surjective. □

Corollary 5.4. *([**112**] and an old result by Auslander) If $\mathcal{A}$ is Frobenius exact and has a strong d-NCR, then the functor $\mathbb{P}$ induces an equivalence of exact categories $\mathcal{A} \to \mathbf{Gp}(\operatorname{mod}_\infty \mathcal{P})$*

Example 5.5. Let A be a left noetherian ring and $\mathcal{A} = A - \operatorname{mod}$ the category of finitely generated left A-modules. Take a generator $E = M \oplus A \in \mathcal{A}$ and assume that $\operatorname{add}(E)$ is contravariantly finite in $\mathcal{A}$ and that $\Gamma = \operatorname{End}_A(E)^{op}$ is again left noetherian of finite global dimension. Take the idempotent $e \in \Gamma$ corresponding to the summand A in E and the exact functor $e \colon \Gamma - \operatorname{mod} \to \mathcal{A}, e(X) := eX$. It has a fully faithful left adjoint and a fully faithful left adjoint ℓ and a fully faithful right adjoint $r = \operatorname{Hom}_A(E, -)$ (the right adjoint is well-defined since $\operatorname{add}(E)$ is contravariantly finite). So by Chapter 1, we get three exact substructures $\mathcal{S}_\ell, \mathcal{S}_r, \mathcal{S}_c$ where $c = \operatorname{Im}(\ell \to r)$ is the intermediate extension functor.
Now we look at $\mathcal{E} = (A - \operatorname{mod}, \mathcal{S}_r)$, as an exact category this is equivalent to $\operatorname{Im} r$ (seen as extension-closed subcategory of $\Gamma - \operatorname{mod}$. It is deflation-closed and contains $\Gamma = r(E)$. Therefore it is a resolving subcategory. Since we assume $\operatorname{gldim} \Gamma < \infty$ it is finitely resolving and we get that the composition is a traingle equivalence $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\operatorname{Im} r) \to \mathrm{D}^b(\Gamma - \operatorname{mod})$ such that the composition $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\Gamma - \operatorname{mod}) \xrightarrow{e} \mathrm{D}^b(\mathcal{A})$ equals the natural map $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{A})$ induced by the identity on $A - \operatorname{mod}$.

Example 5.6. Every exact category has a unique 0-NCR given by the split exact structure. Therefore, we usually look for d-NCRs with $d \ge 1$.

If $\mathcal{E}$ is an NCR for $\mathcal{A}$ with enough projectives $\mathcal{P}(\mathcal{E}) =: \mathcal{P}$, then the previous Verdier localization sequence is triangle equivalent to the Verdier localization sequence

$$\mathrm{K}_{ac}(\mathcal{P}) \to \mathrm{K}^b(\mathcal{P}) \to \mathrm{K}^b(\mathcal{P})/\mathrm{K}_{ac}(\mathcal{P}).$$

where $\mathrm{K}_{ac}(\mathcal{P})$ is the thick subcategory given by complexes which are $\mathcal{A}$-acyclic.

Example 5.7. Let G be a finite group and k a field of characteristic p dividing the order of the group. Let $E := \bigoplus_{H \subset G} k(G/H)$ and $\mathcal{P} = \operatorname{add}(E) \subset \operatorname{mod} kG$. Let $\mathcal{A} = \operatorname{mod} kG$ and $\mathcal{E}$ the exact substructure with $\mathcal{P}(\mathcal{E}) = \mathcal{P}$. This is an NCR of $\mathcal{A}$. This way (up to idempotent completion) the triangle equivalence between $\mathrm{D}^b(\mathcal{A})$ and $\mathrm{K}^b(\mathcal{P})/\mathrm{K}^b_{ac}(\mathcal{P})$ from a previous remark is the main result in [**30**].

We also want to keep track on how $\mathcal{A}$-self-orthogonal the projectives $\mathcal{P}(\mathcal{E})$ are, so we define (inspired by the works of [**52**])

Definition 5.8. Let $n \in \mathbb{N}_{>0}$. Given a full subcategory $\mathcal{M}$ of $\mathcal{A}$, we say that $\mathcal{M}$ is n-**rigid** if $\operatorname{Ext}^{1\sim n}_\mathcal{A}(\mathcal{M}, \mathcal{M}) = 0$ (this is a shorthand notation for $\operatorname{Ext}^i_\mathcal{A}(M, M') = 0$ for all $M, M' \in \mathcal{M}$, $i \in \{1, \dots, n\}$). We say it is $n\mathbb{Z}$-**rigid** if $\operatorname{Ext}^i_\mathcal{A}(M, M') = 0$ for all $M, M' \in \mathcal{M}$, $i \in \mathbb{N}_{>0} \setminus n\mathbb{N}$
Let $\mathcal{E}$ be an exact substructure of $\mathcal{A}$. We define the $\mathcal{A}$-**rigidity** of $\mathcal{E}$ (or more accurately of $\mathcal{P}(\mathcal{E})$) to be

$$\operatorname{rig}_\mathcal{A}(\mathcal{E}) = \sup(\{\mathrm{m} \in \mathbb{N}_{>0} \mid \mathcal{P}(\mathcal{E}) \text{ is m-rigid}\} \cup \{0\}) \quad \in \mathbb{N}_{\ge 0} \cup \{\infty\}$$

If $\operatorname{gldim} \mathcal{A} < \infty$ we have $\operatorname{rig}_\mathcal{A}(\mathcal{E}) \in \{0, 1, \dots, \operatorname{gldim} \mathcal{A} - 1\} \cup \{\infty\}$.

We define the **(projective) rigidity dimension** of $\mathcal{A}$ to be

$$\mathrm{rdim}(\mathcal{A}) = \sup\{\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \mid \mathcal{E} \text{ NCR } \}$$

If $\mathcal{A}$ is regular with enough projectives, it follows $\mathrm{rdim}\mathcal{A} = \infty$ since we may take $\mathcal{E} = \mathcal{A}$.

We have the following (version of the Auslander-Reiten formulation of the Nakayama conjecture)

Proposition 5.9. *(Nakayama Conjecture for NCRs) If $\mathcal{A}$ is exact category and $\mathcal{E}$ is an NCR with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) = \infty$ then $\mathcal{A} = \mathcal{E}$ (in particular $\mathcal{A}$ is also regular with enough projectives).*

Proof. We have $\mathcal{P} := \mathcal{P}(\mathcal{E})$ is homologically exact in $\mathcal{A}$ since it is self-orthogonal. This implies that $\mathrm{D}^b(\mathcal{E}) \cong \mathrm{K}^b(\mathcal{P}) \to \mathrm{D}^b(\mathcal{A})$ is fully faithful. Therefore, the inclusion of the exact substructure $\mathcal{E} \to \mathcal{A}$ is homologically exact implying it is the identity, cf. Chapter 1. □

Corollary 5.10. *If $\mathcal{A}$ is not regular and $\mathcal{E}$ NCR, then $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) < \infty$.*

Corollary 5.11. *If $\mathcal{A}$ is hereditary and Krull Schmidt and $\mathcal{E}$ an NCR which is not equal to $\mathcal{A}$ then $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) = 0$ (i.e. $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{P}(\mathcal{E}), \mathcal{P}(\mathcal{E})) \neq 0$)*

Remark 5.12. For modules over rings: Via generator correspondence and Müller correspondence this is very much related to the so-called dominant dimension, cf. correspondences explained in [**136**]. We added the adjective *projective* since the rigidity dimension of a finite-dimensional algebra is defined using generator-cogenerators and not just generators.

The main reason to introduce $\mathcal{A}$-rigidity for $\mathcal{E}$ is the following easy observation:

Lemma 5.13. *Let $\mathcal{A}$ be an exact category. If $\mathcal{E}$ is an exact substructure with enough projectives $\mathcal{P}(\mathcal{E})$. If we have $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{n}$, then we have $\mathrm{Res}^{\mathcal{E}}_n(\mathcal{P}(\mathcal{E})) = \mathrm{Res}^{\mathcal{A}}_n(\mathcal{P}(\mathcal{E}))$.*

Proof. Let $\mathcal{P} := \mathcal{P}(\mathcal{E})$ be n-rigid (in $\mathcal{A}$). The inclusion $\mathrm{Res}^{\mathcal{E}}_n(\mathcal{P}) \subseteq \mathrm{Res}^{\mathcal{A}}_n(\mathcal{P})$ is trivial. Let $X \in \mathrm{Res}^{\mathcal{A}}_n(\mathcal{P})$. By definition we have an $\mathcal{A}$-exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

with $P_i \in \mathcal{P}$. To see that it is exact in $\mathcal{E}$, we split it in short exact sequences and show that $\mathrm{Hom}(\mathcal{P}, -)$ is exact on it: Set $X = P_{-1}$, let $Q_i = \ker(P_i \to P_{i-1})$, $i = 0, \ldots, n-1$, observe $P_n = Q_{n-1}$. By dimension shift we have $\mathrm{Ext}^1_{\mathcal{A}}(P, Q_i) \cong \mathrm{Ext}^2_{\mathcal{A}}(P, Q_{i+1}) \cong \cdots \cong \mathrm{Ext}^{n-i}_{\mathcal{A}}(P, P_n) = 0$ for all $P \in \mathcal{P}$, $i \in \{0, \ldots, n-1\}$. □

Theorem 5.14. *Let $\mathcal{A}$ be an exact category. The assignment $\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ gives a bijection between:*

(1) *d-NCRs $\mathcal{E}$ with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$.*
(2) *d-rigid subcategories $\mathcal{P}$ of $\mathcal{A}$ with $\mathrm{Res}^{\mathcal{A}}_d(\mathcal{P}) = \mathcal{A}$.*

Remark 5.15. Let $\mathcal{P}$ be a subcategory as in (2).

(1) If $\mathcal{P}$ is also $(d+1)$-rigid, then $\mathcal{P}$ is deflation closed and a d-resolving subcategory of $\mathcal{A}$, this implies $K^b(\mathcal{P}) = D^b(\mathcal{P}) \to D^b(\mathcal{A})$ is a triangle equivalence. If $\mathcal{E}$ is the d-NCR with $\mathcal{P}(\mathcal{E}) = \mathcal{P}$, then Prop.5.9 implies $\mathcal{E} = \mathcal{A}$. In particular, $\mathcal{P} = \mathcal{P}(\mathcal{A})$ is then even self-orthogonal in $\mathcal{A}$.
(2) If $\mathcal{P}$ is not $(d+1)$-rigid, then it can not be deflation-closed (because else, we can apply the same argument as in (1) to deduce that $\mathcal{P}$ has to be even selforthogonal).

Proof. The assignment $\mathcal{E} \mapsto \mathcal{P}(\mathcal{E})$ is one to one between exact substructures with enough projectives and admissibly contravariantly finite subcategories. If we now additionally assume d-NCR with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$ then we clearly get a subcategory as in (2). For the converse we just use the previous lemma to see that the exact structure has gldim $\leq d$. □

Definition 5.16. ([**132**, Def. 3.1]) A d-rigid subcategory $\mathcal{P}$ in an exact category $\mathcal{A}$ is called

(1) **left maximal d-rigid** if $\mathrm{Res}_d^{\mathcal{A}}(\mathcal{P}) = \mathcal{A}$ and
(2) **right maximal d-rigid** if $\mathrm{Cores}_d^{\mathcal{A}}(\mathcal{P}) = \mathcal{A}$

Observe, given an exact category $\mathcal{A}$ with enough projectives $\mathcal{P} = \mathcal{P}(\mathcal{A})$, then $\mathcal{P}$ is left maximal d-rigid iff gldim $\mathcal{A} \leq d$. But $\mathcal{P}$ is right maximal rigid implies $\mathcal{P} = \mathrm{Cores}_d^{\mathcal{A}}(\mathcal{P}) = \mathcal{A}$ because $\mathcal{P}$ is projective.

Proposition 5.17. *If $\mathcal{A}$ has enough projectives $\mathcal{P}$ and let $\mathcal{A}' = \mathbf{Gp}(\mathcal{A})$. Then, restricting exact substructures from $\mathcal{A}$ to $\mathcal{A}'$ gives an injective map from (a) to (b) where*

(a) *d-NCRs $\mathcal{E}$ with $\mathrm{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$ and $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{A}'$*
(b) *d-NCRs $\mathcal{E}'$ with $\mathrm{rig}_{\mathcal{A}'}(\mathcal{E}') \geq \mathrm{d}$*

If additionally $\mathcal{P}$ is n-cotilting with $d \geq n$, then the inclusion $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{A}'$ in (a) is always true.

Proof. The first part is more generally true: Let $\mathcal{A}$ be an exact category with enough projectives. Let $\mathcal{A}' \subseteq \mathcal{A}$ be a resolving subcategory. Let $\mathcal{E}$ be as in (a). The homologically exactness implies that $\mathcal{P}(\mathcal{E}) \subseteq \mathcal{A}'$ is d-rigid and since $\mathcal{A}'$ is deflation-closed we have $\mathrm{Res}_d^{\mathcal{A}'}(\mathcal{P}(\mathcal{E})) = \mathcal{A}'$. Clearly the map is injective since an exact substructure with enough projectives is determined by its category of projectives.
Now assume also that $\mathrm{id}_{\mathcal{A}} \mathcal{P} \leq n$ and $\mathbf{Gp}(\mathcal{A}) = {}^{\perp}\mathcal{P}$. If $d \geq n$ and $\mathcal{P}(\mathcal{E})$ is d-rigid, we have that $\mathcal{P} \subseteq \mathcal{P}(\mathcal{E})$, so $\mathrm{Ext}_{\mathcal{A}}^{1 \sim d}(\mathcal{P}(\mathcal{E}), \mathcal{P}) = 0$. Since $\mathrm{id}_{\mathcal{A}} \mathcal{P} \leq n$, it follows that $\mathcal{P}(\mathcal{E}) \subseteq {}^{\perp}\mathcal{P} = \mathbf{Gp}(\mathcal{A})$. □

Here are my open questions in this section:

(5.1) Given an exact category, does there always exist a non-trivial weak NCR?
(5.2) Exact substructures of an exact category form a complete lattice - do *maximal* elements exist in the subposet of regular exact substructures (maybe plus some rigidity...)?
(5.3) If $\mathcal{A}$ is an exact category with enough projectives. Is $\mathcal{A}$ regular if and only if $\mathrm{rdim}(\mathcal{A}) = \infty$?

6. NCRs from cluster tilting subcategories

Definition 6.1. Given two exact substructures $\mathcal{E}$ and $\mathcal{F}$ (with the same underlying additive category), we say that $\mathcal{F}$ is the **translate** of $\mathcal{E}$ (or $(\mathcal{E}, \mathcal{F})$ a **translated pair**) if $\mathcal{E}$ has enough projectives, $\mathcal{F}$ has enough injectives and $\mathcal{P}(\mathcal{E}) = \mathcal{I}(\mathcal{F})$.

Cf. Chapter 2, assume that the underlying additive category is weakly idempotent complete, then translated pairs are (via $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{P}(\mathcal{E})$) in bijection to functorially finite generator-cogenerators.

Example 6.2. We have that an exact substructure $\mathcal{E}$ is a Frobenius exact structure if and only if $(\mathcal{E}, \mathcal{E})$ is a translated pair.

The following example is the reason for the naming (translated stands for Auslander-Reiten translated).

Example 6.3. Let $\mathcal{A}$ be the category of finite-dimensional modules over a finite-dimensional algebra Λ. Let $G = \Lambda \oplus X$. We consider $\mathcal{E} = (\Lambda - \mathrm{mod}, F_G)$ the exact substructure with enough projectives given by $\mathrm{add}(G)$. By [15] we have that $\mathcal{E} = (\Lambda - \mathrm{mod}, F^H)$ is equal to the exact substructure with enough injectives given by $\mathrm{add}(H)$ with $H = \tau^- X \oplus \mathrm{D}\,\Lambda$. So, with short-hand notation $(F_{\tau^- G}, F_G)$ is a translated pair iff $\mathrm{add}(\tau^- G \oplus \Lambda) = \mathrm{add}(\tau^- G \oplus \mathrm{D}\,\Lambda)$ (i.e. Λ has to be self-injective or Λ of finite global dimension and G the Auslander generator)

Let us recall the following definition from e.g. [131]

Definition 6.4. Let $\mathcal{A}$ be an exact category. Let $\mathcal{M}$ be a full additively closed subcategory and $d \geq 0$ an integer. We $\mathcal{M}$ is $(d+1)$**-cluster tilting** if it is a functorially finite generator-cogenerator with

$$\begin{aligned}\mathcal{M} &= \{X \in \mathcal{E} \mid \operatorname{Ext}^{1\sim d}(X, M) = 0 \; \forall M \in \mathcal{M}\} \\ &= \{X \in \mathcal{E} \mid \operatorname{Ext}^{1\sim d}(M, X) = 0 \; \forall M \in \mathcal{M}\}\end{aligned}$$

Lemma 6.5. [68, Prop. 2.9] *Let $\mathcal{A}$ be an exact category and $\mathcal{M}$ a d-rigid, generating-cogenerating covariantly functorially finite subcategory. The following are equivalent*

(1) *$\mathcal{M}$ is $(d+1)$ cluster tilting*
(2) $\operatorname{Res}_d^{\mathcal{A}}(\mathcal{M}) = \mathcal{A}$

This has the the following corollary.

Corollary 6.6. *Let $\mathcal{A}$ be an exact category and $\mathcal{M}$ a full additively closed subcategory and $d \geq 0$ an integer. The following are equivalent*

(1) *$\mathcal{M}$ is $(d+1)$-cluster tilting in $\mathcal{A}$*
(2) *$\mathcal{M}$ is d-rigid and* $\operatorname{Res}_d^{\mathcal{A}}(\mathcal{M}) = \mathcal{A} = \operatorname{Cores}_d^{\mathcal{A}}(\mathcal{M})$

From this, we directly get:

Proposition 6.7. *Let $d \geq 1$ and $\mathcal{A}$ an exact category. The assignment $(\mathcal{E}, \mathcal{F}) \mapsto \mathcal{P}(\mathcal{E})$ gives a bijection between*

(1) *translated pairs $(\mathcal{E}, \mathcal{F})$ with* $\operatorname{gldim} \mathcal{E} \leq d$, $\operatorname{gldim} \mathcal{F} \leq d$ *and* $\operatorname{rig}_{\mathcal{A}}(\mathcal{E}) \geq \mathrm{d}$
(2) *$(d+1)$-cluster tilting subcategories in $\mathcal{A}$.*

In other words $(d+1)$-cluster tilting subcategories in $\mathcal{A}$ are the projectives in a d-NCR of $\mathcal{A}$ with $\mathcal{A}$-rigidity $\geq d$ and in a d-NCR of $\mathcal{A}^{op}$ with $\mathcal{A}^{op}$-rigidity $\geq d$.

Example 6.8. For $d = 0$, the split exact substructure $\mathcal{A}_0$ is the unique 0-NCR and it is a Frobenius exact category, so $(\mathcal{A}_0, \mathcal{A}_0)$ is a translated pair. It corresponds to the unique 1-cluster tilting subcategory in $\mathcal{A}$ given by $\mathcal{A}$ itself.

Then there are *geometrically* inspired examples.

Example 6.9. The first instances of noncommutative resolutions where found as algebraic analogues of algebraic geometric resolutions of very easy types of singularities (this is the reason for calling this concept 'noncommutative resolution').
The connection between cluster tilting and noncommutative resolutions of singularities is apparent in the following:

(1) For simple singularities: algebraic McKay correspondence (using an Auslander generator of an exact category of finite type, i.e. a 1-cluster tilting subcategory. This exact category is the Cohen-Macaulay modules of the local ring) [18], [134]
(2) For some non-isolated singularities in [107]

Furthermore, there are many more cluster tilting subcategories in Cohen-Macaulay modules over commutative noetherian local rings which are isolated singularities (i.e. geometrical examples) and more generally over (non-commutative) orders over isolated singularities found: [102],[109], [103], [51], [86], [5] (here: use [131] to pass from cluster tilting in the stable category to cluster tilting in the Frobenius exact category), also in graded Cohen-Macaulay module categories [105].

Corollary 6.10. *(also of* [68, Prop. 2.9]*) For $\mathcal{A}$ a Frobenius exact category, the projectives of a d-NCR with* $\operatorname{rig} \geq d$ *of $\mathcal{A}$ are already $(d+1)$-cluster tilting subcategory if and only if they are covariantly finite cogenerator (in $\mathcal{A}$).*

Using Prop. 5.17, the previous corollary amounts to: if $\mathcal{A}$ has enough projectives $\mathcal{P}$ and $\mathcal{P}$ is n-cotilting, then we have a bijection between

(1) d-NCRs $\mathcal{E}$ with rig $\geq d$ and the category $\mathcal{P}(\mathcal{E})$ is a covariantly finite cogenerator in ${}^{\perp}\mathcal{P}$
(2) $(d+1)$-cluster tilting subcategories in $\mathbf{Gp}(\mathcal{A})$

We conclude with: Let $d \geq 0$, by now, there is a remarkable list of examples of d-cluster tilting subcategories in exact categories already found, apart from geometrically inspired examples Ex. 6.9 we also have:

(1) Higher Auslander-Reiten theory is developed in cluster tilting subcategories (with many examples for artin algebras) [**106**], [**94**], [**93**],
(2) most instances of *cluster categories* are algebraic - this means their cluster tilting subcategories lift to cluster tilting subcategories in a Frobenius exact enhancement [**47**], [**46**], [**4**], [**121**], [**104**],...

We also have the following structural result of S. Kvamme.

THEOREM 6.11. *(*[**132**, Theorem A]*) Every weakly idempotent complete d-exact category is equivalent (as d-exact category) to a d-cluster tilting subcategory in a weakly idempotent complete exact category. Furthermore, the ambient exact category is unique up to exact equivalence.*

CHAPTER 12

The Yoneda category and effaceable functors

1. Synopsis

For an exact category we introduce its Yoneda category and the category of Yoneda effaceables. The category of Yoneda effaceables is a Frobenius category. We show that there is a triangle equivalence between the bounded derived category of the effaceable functors and the stable Yoneda effaceables. As an application, we show that the 2-functor assigning to an exact category its effaceable functors is preserving homological exactness.
What is new? The main result is new in this generality but known for finite-dimensional modules over finite-dimensional algebras.

2. Introduction

The category of effaceable functors is an abelian category which we can assign to every exact category. It is always an extension-closed subcategory in the category of all additive functors on $\mathcal{E}$. Similar to Auslander's correspondence for exact categories, cf. [**90**], the exact structure of $\mathcal{E}$ corresponds (by taking effaceable functors $\mathrm{eff}(\mathcal{E})$) to certain Serre subcategories in $\mathcal{P}^2(\mathcal{E})$, cf. [**72**] and Chapter 2. But unlike Auslander correspondence, many (non-equivalent) exact categories can have equivalent effaceable functor categories. In this case we say they are **stable equivalence** to each other.

Opposite to the Auslander categories effaceable functors still contain some residue of homological properties of an exact category. This was presumably also a motivation for Auslander and Reiten's series of papers [**20**], [**21**], [**22**],[**23**],[**24**] on stable equivalence of dualizing R-varieties (the category of finitely presented functors on the stable module category is the category of effaceable functors).

We show as a corollary of the second theorem:

Theorem 2.1. *(cf. Theorem 8.2) If $\mathcal{E} \to \mathcal{F}$ is a homologically exact functor between exact categories, then $\mathrm{eff}(\mathcal{E}) \to \mathrm{eff}(\mathcal{F})$ is also homologically exact.*

Furthermore, we proved the following: If an exact category $\mathcal{E}$ has enough projectives (resp. enough injectives) then so has $\mathrm{eff}(\mathcal{E})$. If $\mathcal{E}$ has enough injectives then $\operatorname{gldim} \mathrm{eff}(\mathcal{E}) \leq 3 \operatorname{gldim} \mathcal{E} - 1$, cf. Cor. 4.10. If $\mathcal{E}$ is a Frobenius category then so is $\mathrm{eff}(\mathcal{E})$.

Theorem 2.2. *(cf. Theorem 8.1) Let $\mathcal{E}$ be a weakly idempotent complete exact category. Then there is a triangle equivalence*

$$\mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}.$$

This result has been proven in [**104**] for $\mathcal{E} = kQ \,\mathrm{mod}$ with Q Dynkin quiver and k a field, for an arbitrary finite acyclic quiver Q in [**124**] and for $\mathcal{E} = \Lambda \,\mathrm{mod}$ with Λ a finite-dimensional algebra in [**85**].

What about the hereditary case? Using Neeman's result we would have a Verdier localization sequence

$$\mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \to \mathrm{K}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E})$$

Given two hereditary exact categories, when are they derived equivalent? In this case $\mathcal{Y}\text{eff} = \text{mod}_1 \, \mathrm{D}^b(\mathcal{E})$ and derived equivalence implies derived stable equivalence.

3. The Yoneda category

Definition 3.1. Let $\mathcal{E}$ be the full subcategory of $\mathrm{D}^b(\mathcal{E})$ given by the essential image of stalk complexes in degree 0. We define the following full subcategory of $\mathrm{D}^b(\mathcal{E})$

$$\mathcal{Y}(\mathcal{E}) := \text{add}\{E[n] \mid n \in \mathbb{Z}, E \in \mathcal{E}\}$$

as the **Yoneda category** of $\mathcal{E}$. More generally, for every admissible exact subcategory $\mathcal{C}$ in a triangulated category $\mathcal{T}$, we define $\mathcal{Y}_{\mathcal{T}}(\mathcal{C}) = \text{add}\{C[n] \mid n \in \mathbb{Z}, C \in \mathcal{C}\}$ as the Yoneda category of $\mathcal{C}$ in $\mathcal{T}$.

The Yoneda category is an additive category (with an autoequivalence). The extension-closure of $\mathcal{Y}(\mathcal{E})$ in $\mathrm{D}^b(\mathcal{E})$ is $\mathrm{D}^b(\mathcal{E})$.

Lemma 3.2. *Assume that $\mathcal{T}$ is triangulated category and that $n \geq 1$. For an admissible exact category $\mathcal{C}$ of a triangulated category $\mathcal{T}$ we consider:*

(1) $\text{Hom}(\mathcal{C}, \mathcal{C}[> n]) = 0$,
(2) $\mathcal{C}[n] * \mathcal{C} = \mathcal{C}[n] \oplus \mathcal{C}$,
(3) $\mathcal{C}[n] * \mathcal{C} \subseteq \mathcal{Y}_{\mathcal{T}}(\mathcal{C})$.

Then we have (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3), *and if $\mathcal{T}$ is also Krull-Schmidt then we also have* (3) $\Rightarrow$ (2).

Proof. The equivalence of (1) and (2) is trivially true and also the implication from (2) to (3). We just show that (3) implies (2). As $\mathcal{T}$ is Krull-Schmidt and $\text{Hom}(\mathcal{C}[n], \mathcal{C}) = 0$, it follows by [**109**], Prop.2.1 (1) that $\mathcal{C}[n] * \mathcal{C}$ is closed under summands. Assume (3), let $C[a] \in \mathcal{C}[n] * \mathcal{C}$ for some $a \in \mathbb{Z}$ and $C \in \mathcal{C}$. To show (2), we need to conclude that $a \in \{n, 0\}$. There exists a triangle $C_1[n] \to C[a] \to C_2 \to C_1[n+1]$ with $C_1, C_2 \in \mathcal{C}$. For $a > 0$ we have $\text{Hom}(C[a], C_2) = 0$ and then $a = n$ or $C = 0$. If $a < n$ then $\text{Hom}(C_1[n], C[a]) = 0$ and therefore $a = 0$ or $C = 0$.

□

We may also recall here the following result:

Theorem 3.3. *([**99**], Cor.1.2) Let $\mathcal{C}$ be an admissible exact subcategory in a triangulated category. The following are equivalent*

(1) *$\mathcal{C}$ is h-admissible hereditary abelian*
(2) $\mathcal{C}[1] * \mathcal{C} = \mathcal{C}[1] \oplus \mathcal{C} = \mathcal{C} * \mathcal{C}[1]$
(3) $\mathcal{Y}_{\mathcal{T}}(\mathcal{C}) = \mathcal{T}$.

Remark 3.4. In the situation of the corollary, in loc. cit. a realization functor is constructed without the assumption that $\mathcal{T}$ is algebraic.

We can look at $\mathbf{Y}$ the category of Yoneda categories (of small exact categories), where morphisms are given by additive functors which preserve degree i-objects ($i \in \mathbb{Z}$) and commute with the shift functor.
Let $\mathbf{Ex}$ be the category of small exact categories with morphisms given by exact functors. We leave it to the reader to formulate this for 2-categories.

Remark 3.5. We consider the assignment $\mathcal{E} \mapsto \mathcal{Y}(\mathcal{E})$ and an exact functor f is mapped to the induced functor $\mathcal{Y}(f)$ on Yoneda categories. This defines a functor

$$\mathcal{Y} \colon \mathbf{Ex} \to \mathbf{Y}$$

Furthermore:
An exact functor f is homologically exact if and only if $\mathcal{Y}(f)$ is fully faithful. An exact functor f is an exact equivalence if and only if $\mathcal{Y}(f)$ is an equivalence (If $\mathcal{Y}(f)$ is an equivalence f is homologically exact and an equivalence as we get an induced equivalence on degree 0 elements).

4. Effaceable functors

Effaceable functors (also called *category of defects*) appeared prominently in several places in the literature. We collect here where it appears, examples and properties of these categories. Let us start with the definition.

Definition 4.1. Let $\mathcal{E}$ be an exact category, we define $\operatorname{eff}(\mathcal{E})$ to be the full subcategory of $\operatorname{mod}_1 \mathcal{E}$ given by functors $F\colon \mathcal{E}^{op} \to (Ab)$ such that there exists an exact sequence $X \to Y \to Z$ in $\mathcal{E}$ such that

$$\operatorname{Hom}_{\mathcal{E}}(-,Y) \to \operatorname{Hom}_{\mathcal{E}}(-,Z) \to F \to 0$$

is exact.
Dually, we define $\mathcal{E} - \operatorname{eff} \subseteq \mathcal{E}\operatorname{Mod}(= \operatorname{Mod} \mathcal{E}^{op})$ as the full subcategory of functors $F\colon \mathcal{E} \to (Ab)$ such that there exists an exact sequence $X \to Y \to Z$ in $\mathcal{E}$ such that

$$\operatorname{Hom}_{\mathcal{E}}(Y,-) \to \operatorname{Hom}_{\mathcal{E}}(X,-) \to F \to 0$$

Observe that by definition $\mathcal{E} - \operatorname{eff} = \operatorname{eff}(\mathcal{E}^{op})$ as we would expect.

Let us denote by $i\colon \mathcal{E} \to \mathcal{E}^{ic}, X \mapsto (X,1)$ the idempotent completion of an exact category described in [**49**] - the functor i is fully faithful, exact and reflects exactness cf. loc. cit. The essential image of i is extension-closed, generating and cogenerating, so the induced derived functor $\mathrm{D}^b(\mathcal{E}) \to \mathrm{D}^b(\mathcal{E}^{ic})$ is fully faithful (cp. [**31**, Cor 2.12]).
Furthermore, one can show using [**33**, Lemma 21] that $\mathcal{E}$ is weakly idempotent complete if and only if the essential image of i is deflation-closed if and only if it is inflation-closed.

Lemma 4.2. *Let $\mathcal{E}$ be an exact category.*

(i) $\operatorname{eff}(\mathcal{E})$ *is extension-closed in* $\operatorname{mod}_1 \mathcal{E}$*.*
(ii) *If $\mathcal{E}$ is idempotent complete then* $\operatorname{eff}(\mathcal{E})$ *is idempotent complete.*
(iii) *We have* $\operatorname{eff}(\mathcal{E}) = \operatorname{eff}(\mathcal{E}^{ic})$ *is idempotent complete.*

Proof. (i) The category $\operatorname{eff}(\mathcal{E})$ is extension-closed in the so called category of admissible presentable functors by [**90**, Proposition 3.6], and the latter is extension-closed subcategory of $\operatorname{Mod}(\mathcal{E})$ by [**90**, Proposition 3.5].
(ii) Assume $\mathcal{E}$ is idempotent complete. By [**90**, Corollary 3.18] $\operatorname{eff}(\mathcal{E})$ is an additively closed subcategory (e.g. it is part of a torsion pair) of an idempotent complete additive category, so it is idempotent complete itself.
(iii) We see $\operatorname{eff}(\mathcal{E})$ as a full subcategory of $\operatorname{eff}(\mathcal{E}^{ic})$ (using the universal property of the idempotent completion of an additive category).
Given a functor $F \in \operatorname{eff}(\mathcal{E}^{ic})$ we can choose an $\mathcal{E}^{ic}$-exact sequence $(Z,1) \to (X,p) \xrightarrow{d} (Y,1)$ such $F = \operatorname{coker} \operatorname{Hom}(-,d)$ (because given a short exact sequence $(A,a) \to (B,b) \xrightarrow{\varphi} (C,c)$ with $(C,c) \oplus (C,1-c) = (C,1), (A,a) \oplus (A,1-a) = (A,1)$ we can just add the following sequences $0 \to (C,1-c) \xrightarrow{id} (C,1-c)$ and $(A,1-a) \xrightarrow{id} (A,1-a) \to 0$, this does not change the cokernel of $\operatorname{Hom}(-,\varphi)$). As $\mathcal{E}$ is extension-closed in $\mathcal{E}^{ic}$ it follows that $F \in \operatorname{eff}(\mathcal{E})$. Then by (ii) it follows that $\operatorname{eff}(\mathcal{E})$ is idempotent complete.

□

Theorem 4.3. *([**174**], Lemma 9) Let $\mathcal{E}$ be an exact category. Then* $\operatorname{eff}(\mathcal{E})$ *as fully exact subcategory of* $\operatorname{Mod} \mathcal{E}$*, is abelian.*

Remark 4.4. The previous result is proven only for idempotent complete exact categories but by Lemma 4.2, this implies it is true for all exact categories.

Lemma 4.5. *If $\phi\colon \mathcal{E} \to \mathcal{F}$ is an exact functor, there exists a well-defined exact functor*

$$\overline{\phi}\colon \operatorname{eff}(\mathcal{E}) \to \operatorname{eff}(\mathcal{F})$$

defined on objects via $\overline{\phi}(\operatorname{coker}\operatorname{Hom}_{\mathcal{E}}(-,d)) := \operatorname{coker}\operatorname{Hom}_{\mathcal{F}}(-,\phi(d))$ for $\mathcal{E}$-deflations d.

Proof. By [**90**], Thm 3.9 (2), there exists such an exact functor on the Auslander exact categories. As it restricts to the functor $\overline{\phi}\colon \operatorname{eff}(\mathcal{E}) \to \operatorname{eff}(\mathcal{F})$, it is automatically well-defined and exact. □

Definition 4.6. Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$. Then the **stable category** denoted by $\underline{\mathcal{E}}$ is defined as the ideal quotient category. For every two objects $X, Y \in \mathcal{E}$ let $\mathcal{P}(X,Y) \subseteq \operatorname{Hom}_{\mathcal{E}}(X,Y)$ to be the subgroup of all morphisms factoring through a projective object. This defines an ideal in the category $\mathcal{E}$. Then $\underline{\mathcal{E}}$ has the same objects as $\mathcal{E}$ but morphisms are defined as

$$\operatorname{Hom}_{\underline{\mathcal{E}}}(X,Y) := \underline{\operatorname{Hom}}_{\mathcal{E}}(X,Y) = \operatorname{Hom}_{\mathcal{E}}(X,Y)/\mathcal{P}(X,Y)$$

This defines an additive category. Dually if $\mathcal{E}$ has enough injectives then we define $\overline{\mathcal{E}} = (\underline{\mathcal{E}^{op}})^{op}$.

Observe that the stable category of $\mathcal{E}$ is an additive category with an endofunctor, given by taking syzygies Ω. If in addition $\mathcal{E}$ is a Frobenius category then its stable category has the structure of a triangulated category with Ω^{-1} being the suspension functor and the distinguished triangles induced by short exact sequences (cp. [**87**]).
In general, one can either study this as a pretriangulated category or use the Heller stabilization to obtain a triangulated category from the stable category.

Let us observe the following easy

Lemma 4.7. *Let $\mathcal{E}$ be an exact category.*

(1) *If $\mathcal{E}$ has enough projectives then for every morphism $\underline{g}$ in the category $\underline{\mathcal{E}}$ there exists an $\mathcal{E}$-deflation d with $\underline{d} = \underline{g}$.*
(2) *If $\mathcal{E}$ has enough injectives then for every morphism $\overline{g}$ in the category $\overline{\mathcal{E}}$ there exists an inflation i such that $\overline{i} = \overline{g}$*

Proof. If $g\colon X \to Y$ and (1) if $\mathcal{E}$ has enough projectives, take a deflation $p\colon P \to Y$ with P projective and then using the pullback of p along g we have an induced deflation $d = [g,p]\colon X \oplus P \to Y$ with $\overline{d} = \overline{g}$. (2) If $\mathcal{E}$ has enough injectives, take an inflation $j\colon X \to I$ with I injective and form the pushout to obtain an inflation $i = \binom{g}{j}\colon X \to Y \oplus I$ with $\overline{i} = \overline{g}$. □

For an additive category $\mathcal{P}$ we call $\operatorname{mod}_\infty \mathcal{P}$ to be the full subcategory of $\operatorname{Mod}\mathcal{P}$ of all additive functors $F\colon \mathcal{P}^{op} \to (Ab)$ such that there exists an exact sequence in

$$\operatorname{Hom}(-,P_n) \to \cdots \to \operatorname{Hom}(-,P_0) \to F \to 0$$

with $P_i \in \mathcal{P}$. This is a fully exact subcategory of $\operatorname{Mod}\mathcal{P}$ which has enough projectives and the Yoneda embedding $\mathcal{P}^{ic} \to \mathcal{P}(\operatorname{mod}_\infty \mathcal{P})$ induces an equivalence of additive categories.
Whenever we have an exact category $\mathcal{F}$ with enough projectives $\mathcal{P}$, then we have a functor

$$\mathbb{P}\colon \mathcal{F} \to \operatorname{mod}_\infty \mathcal{P}, \quad X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{P}^{op}}$$

which is homologically exact and induces an equivalence of $\mathcal{F}^{ic}$ to a resolving subcategory of $\operatorname{mod}_\infty \mathcal{P}$ but usually this is not essentially surjective.
Dually given an additive category $\mathcal{I}$ we define the category $\mathcal{I}\operatorname{mod}^\infty := (\operatorname{mod}_\infty \mathcal{I}^{op})^{op}$, this is an exact category with enough injectives and the Yoneda embedding

$\mathcal{I} \to (\operatorname{mod}_\infty \mathcal{I}^{op})^{op}, I \mapsto \operatorname{Hom}_{\mathcal{I}}(I,-)$ induces an equivalence $\mathcal{I}^{ic} \to \mathcal{I}((\operatorname{mod}_\infty \mathcal{I}^{op})^{op})$. Whenever an exact category $\mathcal{F}$ has enough injectives $\mathcal{I}$ then we consider

$$\mathbb{I}\colon \mathcal{F} \to (\operatorname{mod}_\infty \mathcal{I}^{op})^{op}, \quad X \mapsto \operatorname{Hom}(X,-)|_{\mathcal{I}}$$

this is homologically exact and induces an equivalence of $\mathcal{F}^{ic}$ to a coresolving subcategory of $(\operatorname{mod}_\infty \mathcal{I}^{op})^{op}$.

The following first part is [72], Lemma 2.13 (in the idempotent complete case)

Proposition 4.8. *Let $\mathcal{E}$ be an exact category.*

(1) *If $\mathcal{E}$ is an exact category with enough projectives, then $\operatorname{eff}(\mathcal{E})$ has enough projectives. The Yoneda functor $\underline{\mathcal{E}} \to \operatorname{mod}_1 \underline{\mathcal{E}}, X \mapsto \operatorname{Hom}_{\underline{\mathcal{E}}}(-,X)$ induces an equivalence of additive categories $(\underline{\mathcal{E}})^{ic} \to \mathcal{P}(\operatorname{eff}(\mathcal{E}))$. Furthermore, in this case, the functor $\mathbb{P}$ induces an equivalence*

$$\mathbb{P}\colon \operatorname{eff}(\mathcal{E}) \to \operatorname{mod}_\infty \underline{\mathcal{E}}$$

(2) *If $\mathcal{E}$ is an exact category with enough injectives, then $\operatorname{eff}(\mathcal{E})$ also has enough injectives. Furthermore, the functor $X \mapsto \operatorname{Ext}^1_{\mathcal{E}^{ic}}(-,X)$ gives an equivalence of additive categories $(\overline{\mathcal{E}})^{ic} \to \mathcal{I}(\operatorname{eff}(\mathcal{E}))$. Furthermore, we have an exact equivalence*

$$\mathbb{I}\colon \operatorname{eff}(\mathcal{E}) \to (\operatorname{mod}_\infty (\overline{\mathcal{E}})^{op})^{op}$$

Proof. (1) The proof in [72, Lemma 2.13] works also if $\mathcal{E}$ is not idempotent complete. For $\mathbb{P}$ essentially surjective, the main argument is just Lemma 4.7,(1).
(2) Again the essentially surjectivity of $\mathbb{I}$ follows from Lemma 4.7, (2). $\square$

Remark 4.9. In the light of the previous Proposition, it is sensible for arbitrary exact categories to define two exact categories as **stably equivalent** if there exists an equivalence ϕ between their effaceable functor categories (observe that additive equivalences between abelian categories are exact). In this case we would call ϕ the stable equivalence. It can be that a stable equivalence is induced by an exact functor as in Lemma 4.5 or it can also not be induced by a functor between the exact categories.
So continuing Auslander-Reiten's quest would mean: Try to classify/understand exact categories up to stable equivalence.

Corollary 4.10. *If $\mathcal{E}$ is an exact category with enough injectives and assume that $\operatorname{gldim} \mathcal{E} \leq n$, then we have*

$$\operatorname{gldim} \operatorname{eff}(\mathcal{E}) \leq 3n-1$$

This is an obvious generalization of [20, Prop. 10.2].

Proof. Let $F \in \operatorname{eff}(\mathcal{E})$, then there exists an exact sequence $A \to B \xrightarrow{g} C$ such that $F = \operatorname{coker} \operatorname{Hom}_{\mathcal{E}}(-,g)$. So by definition, the long exact sequence when applying a functor $\operatorname{Hom}(X,-)$ induces an exact sequence of functors on $\underline{\mathcal{E}}$

$$\begin{aligned} 0 \to F \to \operatorname{Ext}^1_{\mathcal{E}}(-,A) \to \operatorname{Ext}^1_{\mathcal{E}}(-,B) \quad &\to \operatorname{Ext}^1_{\mathcal{E}}(-,C) \\ \to \operatorname{Ext}^2_{\mathcal{E}}(-,A) \to \cdots \qquad\qquad &\to \operatorname{Ext}^n_{\mathcal{E}}(-,C) \to 0 \end{aligned}$$

as $\operatorname{Ext}^i_{\mathcal{E}}(-,X) \cong \operatorname{Ext}^1_{\mathcal{E}}(-,\Omega^{-(i-1)}X) \in \mathcal{I}(\operatorname{eff}(\mathcal{E}))$ by Prop. 4.8,(2), the claim follows. $\square$

Remark 4.11. If $\mathcal{E}$ is an exact category with enough projectives, the category of effaceables are just the category $\operatorname{mod}_\infty \underline{\mathcal{E}}$, so its global dimension can be determined by *higher* weak kernels in the additive category $\underline{\mathcal{E}}$ (in the sense of Enomoto).
In particular, if $\mathcal{E}$ has enough projectives the following are equivalent

(1) $\operatorname{gldim} \operatorname{eff}(\mathcal{E}) = 0$
(2) $\underline{\mathcal{E}}$ is abelian semi-simple
(3) Every non-isomorphism in $\mathcal{E}$ factors through a projective

In particular, we can find examples of exact categories of all global dimension (including ∞) such that the category of effaceable functors is semi-simple abelian. Take $\mathcal{E}$ an abelian Krull-Schmidt category such that every indecomposable is either projective or simple and all simples Hom-orthogonal (e.g. take a finite dimension Nakayama algebra Λ and pass to the quotient $\Lambda/\operatorname{rad}^2$). We look at $\Lambda_n = k(1 \to 2 \to \cdots \to n)/\operatorname{rad}^2$ (of global dimension n) and $\Lambda_\infty = k[X]/(X^2)$ (of infinite global dimension), then $\mathcal{E}_n = \Lambda_n \operatorname{mod}$ has semi-simple effaceable functors for all $n \leq \infty$.

Lemma 4.12. *If $\mathcal{E}$ is a Frobenius category then $\operatorname{eff}(\mathcal{E})$ is also a Frobenius category and $\operatorname{Hom}_{\underline{\mathcal{E}}}(-,\Omega^- X) \cong \operatorname{Ext}^1_{\mathcal{E}}(-,X)$ for all $X \in \mathcal{E}$.*

Proof. By Happel [87], $\underline{\mathcal{E}}$ is triangulated, then there is a general result that $\operatorname{mod}_\infty \underline{\mathcal{E}}$ is a Frobenius category by a Theorem of Freyd (cf. [77], Thm 1.7). The last statement is more generally proven in Lemma 6.1. □

5. Yoneda-effaceable functors

Definition 5.1. We define the category of **Yoneda-effaceables** $\mathcal{Y}\operatorname{eff}(\mathcal{E})$ to be the full subcategory of $\operatorname{mod}_\infty \mathcal{Y}(\mathcal{E})$ given by functors X such that there exists a triangle in $\mathrm{D}^b(\mathcal{E})$

$$A \to B \xrightarrow{f} C \to A[1]$$

with $A, B, C \in \mathcal{Y}(\mathcal{E})$ such that $X \cong \operatorname{Coker} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-,f)$, that is, X admits a presentation as

$$\operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-,B) \xrightarrow{\operatorname{Hom}(-,f)} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-,C) \to X \to 0$$

and we say that X is *presented* by f. In practice, we say that a Yoneda effaceable functor is presented by a map f and we implicitly assume that the domain, codomain and cone (in $\mathrm{D}^b(\mathcal{E})$) of f are in $\mathcal{Y}(\mathcal{E})$. In particular, any functor in $\mathcal{Y}\operatorname{eff}(\mathcal{E})$ is finitely presented (a.k.a. coherent).

We also have the following harmless looking result - which has a lengthy proof which is only completed in Lemma 5.13.

Proposition 5.2. *Let $f\colon X \to Y$ a morphism in $\mathcal{Y}(\mathcal{E})$ and $F = \operatorname{coker} \operatorname{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,f)$. Then: $F|_{\mathcal{Y}(\mathcal{E})} \in \mathcal{Y}\operatorname{eff}(\mathcal{E})$ if and only if $\operatorname{cone}(f) \in \mathcal{Y}(\mathcal{E})$.*

Example 5.3. If $\mathcal{E}$ is hereditary abelian, we know by Thm 3.3 that $\mathcal{Y}(\mathcal{E}) = \mathrm{D}^b(\mathcal{E})$, in particular for every morphism $f\colon X \to Y$ in $\mathcal{Y}(\mathcal{E})$ we have $\operatorname{cone}(f) \in \mathcal{Y}(\mathcal{E})$.
Therefore the category of Yoneda-effaceables $\mathcal{Y}\operatorname{eff}(\mathcal{E})$ coincides with the (Frobenius exact) category $\operatorname{mod}_1 \mathrm{D}^b(\mathcal{E})$ (of finitely presented functors $\mathrm{D}^b(\mathcal{E})^{op} \to (Ab)$).

Definition 5.4. Let $\mathcal{E}$ be an exact category and $\mathcal{M}$ a full subcategory. We say $\mathcal{M}$ is **generating** if for every $E \in \mathcal{E}$ there exists a deflation $d\colon M \to E$ with $M \in \mathcal{M}$. It is called **deflation-closed** if for every short exact sequence $X \to Y \to Z$ with $Y, Z \in \mathcal{M}$ also $X \in \mathcal{M}$ holds.
A **resolving subcategory** in an exact category is fully exact subcategory which is deflation-closed and generating. It is a **coresolving subcategory** if it is resolving in the opposite category. A **biresolving subcategory** in an exact category is a fully exact subcategory which is resolving and coresolving.

We often use the following:

Remark 5.5. (cf. [71, 2.6]) If $\mathcal{E}$ is an exact category and $\mathcal{F}$ is a fully exact subcategory closed under direct summands.
If $\mathcal{E}$ has enough projectives $\mathcal{P}$, $\mathcal{P} \subseteq \mathcal{F}$ and $\mathcal{F}$ closed under syzygies then $\mathcal{F}$ is resolving.
If $\mathcal{E}$ has enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ both contained in $\mathcal{F}$ and $\mathcal{F}$ is closed under syzygies and under cosyzygies, then $\mathcal{F}$ is biresolving.

Observe that biresolving subcategories in a Frobenius category are always again Frobenius categories (with the same projective-injectives).

Definition 5.6. Given an exact category $\mathcal{F}$ we denote by $\mathcal{P}(\mathcal{F})$ the full subcategory of projectives in $\mathcal{F}$. Let $\mathcal{C}$ be a fully exact subcategory. We call $\mathcal{C}$ **partially resolving** if $\mathcal{C}$ is deflation-closed, summand-closed and for every $C \in \mathcal{C}$ there exists an $\mathcal{F}$-deflation $d\colon P \to C$ with $P \in \mathcal{P}(\mathcal{F})$. Dually we define **partially coresolving** if $\mathcal{C}^{op}$ is partially resolving in $\mathcal{F}^{op}$.
We call $\mathcal{C}$ **partially biresolving** if it is partially resolving and partially coresolving.

Remark 5.7. We have the following (cf. Chapter 1)

(1) Let $\mathcal{C}$ be fully exact in an exact category $\mathcal{F}$ and closed under taking summands in $\mathcal{F}$. Then $\mathcal{C}$ is partially resolving if and only if for every $C \in \mathcal{C}$ there exists an $\mathcal{F}$-exact sequence $C' \rightarrowtail P \twoheadrightarrow C$ with $P \in \mathcal{P}(\mathcal{F}), C' \in \mathcal{C}$.
(2) If $\mathcal{C}$ is partially resolving then $\mathcal{C}$ has enough projectives with $\mathcal{P}(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{F})$. If $\mathcal{C}$ is partially biresolving then it has enough projectives and enough injectives - therefore it is a Frobenius category if and only if $\mathcal{P}(\mathcal{C}) = \mathcal{I}(\mathcal{C})$ holds.
(3) If $\mathcal{C}$ is partially resolving in $\mathcal{F}$ then it is homologically exact in $\mathcal{F}$.

Lemma 5.8. *Let $f\colon \mathcal{E} \to \mathcal{E}'$ be an exact functor which is homologically exact. Then we have a fully faithful embedding $\mathcal{Y}(\mathcal{E}) \subseteq \mathcal{Y}(\mathcal{E}')$. The full subcategory $\mathcal{C} = \{F \in \mathcal{Y}\mathrm{eff}(\mathcal{E}') \mid \exists f\colon X \to Y \text{ in } \mathcal{Y}(\mathcal{E}), \mathrm{cone}(f) \in \mathcal{Y}(\mathcal{E})\}$ is partially resolving in $\mathcal{Y}\mathrm{eff}(\mathcal{E}')$ and the restriction functor $\mathcal{C} \to \mathcal{Y}\mathrm{eff}(\mathcal{E}), F \mapsto F|_{\mathcal{Y}(\mathcal{E})}$ is an exact equivalence.*
In particular, the induced triangle functor

$$\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E}')}$$

is fully faithful

Proof. By the usual horseshoe argument $\mathcal{C}$ is extension-closed. By definition it has enough projectives and enough injectives equivalent to $\mathcal{Y}(\mathcal{E})$, therefore it partially biresolving. It is straight-forward to see that the restriction $\mathcal{C} \to \mathcal{Y}\mathrm{eff}(\mathcal{E}), F \mapsto F|_{\mathcal{Y}(\mathcal{E})}$ is an exact equivalence. As $\mathcal{C}$ is homologically exact in $\mathcal{Y}\mathrm{eff}(\mathcal{E}')$, we have induced isomorphisms on Ext-groups and by Lemma 6.1 these calculate the homomorphisms in the stable category, i.e. we have for all X, Y in $\mathcal{C}$ and $n \geq 1$

$$\mathrm{Hom}_{\underline{\mathcal{C}}}(X, \Omega^{-n}Y) \cong \mathrm{Ext}^n_{\mathcal{C}}(X,Y) = \mathrm{Ext}^n_{\mathcal{Y}\mathrm{eff}(\mathcal{E}')}(X,Y) \cong \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E}')}}(X, \Omega^{-n}Y)$$

as every object in $\mathcal{C}$ is a cosyzygy for some $n \geq 1$, the claim follows. □

Lemma 5.9. *We have $\mathcal{Y}\mathrm{eff}(\mathcal{E}) \subseteq \mathbf{Gp}(\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E}))$ is extension-closed. Furthermore, as fully exact category, $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is biresolving and therefore as a fully exact subcategory, it is a Frobenius exact category.*

Proof. Let $\mathcal{Y}$ denote $\mathcal{Y}(\mathcal{E})$. Let X be a Yoneda-effaceable functor presented by the presented by the map f fitting in the triangle $A \to B \xrightarrow{f} C \to A[1]$ in $\mathrm{D}^b(\mathcal{E})$ with A, B and C in $\mathcal{Y}$. Then the complex of finitely generated projective $\mathcal{Y}$–modules

$$\ldots \to \mathcal{Y}(-,A) \to \mathcal{Y}(-,B) \to \mathcal{Y}(-,C) \xrightarrow{d_0} \mathcal{Y}(-,A[-1]) \to \ldots$$

is totally acyclic and X is the image of d_0. Therefore X is in $\mathrm{GP}(\mathrm{mod}_\infty(\mathcal{Y}))$. Note that $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is extension closed. Indeed, consider and short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathbf{Gp}(\mathrm{mod}_\infty \mathcal{Y}(\mathcal{E}))$ with X and Z in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ and chose a presenting maps $f\colon B \to C$ and $f'\colon B' \to C'$ of X and Z, respectively. Use the horseshoe lemma in $\mathrm{Mod}\,\mathcal{Y}(\mathcal{E})$ to produce a complex of the form

$$\ldots \to (-, \mathrm{A}'') \to (-, B \oplus B') \xrightarrow{(f'')^*} (-, C \oplus C') \to Y \to 0$$

where f'' is the matrix $\begin{pmatrix} f & 0 \\ g & f' \end{pmatrix}$ for some $g\colon B \to C'$. We claim that f'' is a presenting map for Y. Let D denote the cocone of f'' in $\mathrm{D}^b(\mathcal{E})$. Then comparing the long exact sequence obtained from the triangle induced by f'' and the previous complex obtained from the horseshoe lemma, we obtain that $(-, A'') \cong (-, D)$ as functors from $\mathcal{Y}(\mathcal{E})$ to (Ab). Then Lemma 5.10 will give us that $A'' \cong D$ in $\mathrm{D}^b(\mathcal{E})$. But A'' is in $\mathcal{Y}(\mathcal{E})$. This completes the claim.

It remains to show that $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is Frobenius exact as fully exact subcategory. In fact we need to see that it contains the projective-injectives and is closed under syzygies and cosyzygies but all three claims are clear when looking at the long exact sequence above. □

Lemma 5.10. *Let $L, M \in \mathrm{D}^b(\mathcal{E})$ and $M \in \mathcal{Y}(\mathcal{E})$.*
If $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,L)|_{\mathcal{Y}(\mathcal{E})} \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,M)|_{\mathcal{Y}(\mathcal{E})}$ then $L \cong M$ as objects in $\mathrm{D}^b(\mathcal{E})$.

Proof. As $M \in \mathcal{Y}(\mathcal{E})$, there exists a morphism $f\colon M \to L$ (in $\mathcal{Y}(\mathcal{E})$) which corresponds to $\mathrm{id}_M\colon M \to M$ under the assumed natural isomorphism of functors. This induces a natural transformation $f^*\colon \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,M) \to \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,L)$ which is an equivalence when restricted to $\mathcal{Y}(\mathcal{E})$. The extension-closure of $\mathcal{Y}(\mathcal{E})$ in $\mathrm{D}^b(\mathcal{E})$ is $\mathrm{D}^b(\mathcal{E})$ and using the long exact sequences obtained when applying $\mathrm{Hom}(-,M)$ and $\mathrm{Hom}(-,L)$ we conclude that f^* is an isomorphism of functors. Then by the Yoneda-embedding, a quasi-inverse functor is given by an inverse morphism for f and f has to be an isomorphism in $\mathrm{D}^b(\mathcal{E})$. □

The shift functor [1] in $\mathcal{D} := \mathrm{D}^b(\mathcal{E})$ induces by precomposition an autoequivalence on $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ which maps representable (i.e. projectives) to projectives, therefore we have induced quasi-inverse autoequivalences

$$[1]_{\mathcal{D}}\colon \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \leftrightarrow \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}\colon [-1]_{\mathcal{D}}.$$

As $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ is a Frobenius exact category we also have the quasi-inverse equivalences

$$\Sigma := \Omega^-\colon \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} \leftrightarrow \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}\colon \Omega =: \Sigma^-$$

given by taking cosyzygies and syzygies (they are the suspension and cosuspension of the triangulated structure discussed before, therefore we will rename them as $\Sigma^{\pm}$).

Then the following corrollary is immediate from the previous lemma.

Corollary 5.11. *We have a natural isomorphism of functors $\Omega^3 = [1]_{\mathcal{D}}$ on $\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$. Furthermore, we have for all $F, G \in \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ there exists an $n = n_{F,G} << 0$ such that*

$$\mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(F, \Sigma^{<n} G) = \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(\Sigma^{>(-n)} F, G) = 0$$

Proof. The statement is obvious. □

We make the following auxilliary definition.

Observe that $\mathcal{F} = \mathrm{mod}_1\, \mathrm{D}^b(\mathcal{E}) = \mathrm{mod}_\infty\, \mathrm{D}^b(\mathcal{E}) = \mathbf{Gp}(\mathrm{mod}_\infty\, \mathrm{D}^b(\mathcal{E})$ as every projective presentation of a functor can be extended to a complete projective resolution by taking the associated completion of a morphism to a distinguished triangle. This is a Frobenius category.

Definition 5.12. We define $\widetilde{\mathcal{Y}\mathrm{eff}}$ to be the full subcategory of $\mathrm{mod}_1\, \mathrm{D}^b(\mathcal{E})$ given by all F such that there exists an $f\colon X \to Y$ in $\mathcal{Y}(\mathcal{E})$ with $\mathrm{cone}(f) \in \mathcal{Y}(\mathcal{E})$ such that $F = \mathrm{coker}\, \mathrm{Hom}(-,f)$.

By the horseshoe lemma it is obvious that $\widetilde{\mathcal{Y}\mathrm{eff}}$ is extension-closed in $\mathcal{F}$.

Lemma 5.13. *Let $\mathcal{E}$ be idempotent complete.*

(1) *Then $\widetilde{\mathcal{Y}\mathrm{eff}}$ is partially biresolving in $\mathcal{F} = \mathrm{mod}_1\, \mathrm{D}^b(\mathcal{E})$. Furthermore, it is a Frobenius category.*
(2) *A morphism $f \in X \to Y$ in $\mathcal{Y}(\mathcal{E})$ the following are equivalent:*
   (a) $\mathrm{cone}_{\mathrm{D}^b(\mathcal{E})}(f) \in \mathcal{Y}(\mathcal{E})$
   (b) $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,f)$ *is* $\widetilde{\mathcal{Y}\mathrm{eff}}$*-admissible*
   (c) $\mathrm{coker}\, \mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,f) \in \widetilde{\mathcal{Y}\mathrm{eff}}$
(3) *The restriction functor $\widetilde{\mathcal{Y}\mathrm{eff}} \to \mathcal{Y}\mathrm{eff}(\mathcal{E})$, $F \mapsto F|_{\mathcal{Y}(\mathcal{E})}$ is an exact equivalence.*

We just state (2) in the previous Lemma, to combine it with the equivalence in (3) - then it implies Prop. 5.2.

Proof. To see (1) use the same argument as before, (2) follows by definition. The equivalence in (3) has been considered in bigger generality in Chapter 3. □

Definition 5.14. Now we define $\widetilde{\mathrm{eff}} \subseteq \mathcal{Y}\mathrm{eff}(\mathcal{E})$ to be the full subcategory given by functors X such that there exists a triangle $A \to B \xrightarrow{g} C \to A[1]$ in $\mathrm{D}^b(\mathcal{E})$ with A, B, C in $\mathcal{E}$ such that $X \cong \operatorname{coker} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, g)$.

The category $\widetilde{\mathrm{eff}}$ is extension-closed in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ (using the same horseshoe argument as in the Lemma above). But $\widetilde{\mathrm{eff}}$ does not contain any projectives.

Lemma 5.15. *The restriction functor* $\widetilde{\mathrm{eff}} \to \mathrm{eff}(\mathcal{E}), F \mapsto F|_{\mathcal{E}^{op}}$ *is an exact equivalence.*

Proof. As restriction functors on functor categories are exact, also their restriction to fully exact subcategories are exact functors.
By definition is this functor is essentially surjective and using the definition it is also straight-forward to see that for an additive functor $G\colon \mathcal{Y}(\mathcal{E})^{op} \to (Ab)$, and for X such that there exists an exact sequence $A \to B \xrightarrow{g} C$ in $\mathcal{E}$ such that $X = \operatorname{coker} \operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-, g)$ we have isomorphisms

$$\operatorname{Hom}(X, G) = \ker_{(Ab)}(G(C) \xrightarrow{G(g)} G(B)) = \operatorname{Hom}(X|_{\mathcal{E}^{op}}, G|_{\mathcal{E}^{op}})$$

therefore the functor is an equivalence of categories. As every fully faithful exact functor it induces a monomorphism on Ext^1-groups. We need to see it is surjective.

Let $0 \to G \to H \to F \to 0$ be a complex in $\widetilde{\mathrm{eff}}$ such that when evaluated at objects of $\mathcal{E} (\subseteq \mathcal{Y}(\mathcal{E}))$ this yields an exact sequence of abelian groups. We need to see that $0 \to G \to H \to F \to 0$ evaluated at $E[i]$ with $E \in \mathcal{E}, i \in \mathbb{Z}, i \neq 0$ still gives an exact sequence of abelian groups. But this follows from the next Lemma 5.16. □

Lemma 5.16. *Given a two composable morphisms of distinuished triangles* $X_* \xrightarrow{f_*} Y_* \xrightarrow{g_*} Z_*$ *which is degree-wise split exact in a triangulated category* $\mathcal{T}$ *with suspension* $[1]$, *i.e. we have commuting diagrams*

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{a_1} & X_2 & \xrightarrow{a_2} & X_3 & \xrightarrow{a_3} & X_1[1] \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_1[1]} \\
Y_1 & \xrightarrow{b_1} & Y_2 & \xrightarrow{b_2} & Y_3 & \xrightarrow{b_3} & X_1[1] \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_1[1]} \\
Z_1 & \xrightarrow{c_1} & Z_2 & \xrightarrow{c_2} & Z_3 & \xrightarrow{c_3} & X_1[1]
\end{array}$$

with (f_i, g_i) *is a split exact sequence for all* $i \in \{1,2,3\}$. *Let* A *be an object in* $\mathcal{T}$ *and apply* $(A, -) := \operatorname{Hom}_{\mathcal{T}}(A, -)$ *to obtain two morphisms of long exact sequences. Assume that* $0 \to \operatorname{coker}(A, a_2) \to \operatorname{coker}(A, b_2) \to \operatorname{coker}(A, c_2) \to 0$ *is an exact sequence of abelian groups, then we have that also* $0 \to \operatorname{coker}(A, a_i) \to \operatorname{coker}(A, b_i) \to \operatorname{coker}(A[i], c_i) \to 0$ *is an exact sequence of abelian groups for* $i \in \{1,2,3\}$. *In particular, also* $0 \to \operatorname{coker}(A[i], a_2) \to \operatorname{coker}(A[i], b_2) \to \operatorname{coker}(A[i], c_2) \to 0$ *is an exact sequence of abelian groups for every* $i \in \mathbb{Z}$.

Proof. Apply the snake lemma in the category of abelian groups. □

Definition 5.17. Let $\mathcal{T}$ be a triangulated category (we will usually denote the suspension by [1]) and $\mathcal{C} \subseteq \mathcal{T}$ be a full additively closed subcategory. Then we say $\mathcal{C}$ is **admissible exact** in $\mathcal{T}$ if it is extension-closed and non-negative (i.e. $\operatorname{Hom}_{\mathcal{T}}(C, C'[-n]) = 0$ for all $n > 0$, $C, C' \in \mathcal{C}$).

Lemma 5.18. *The composition* $\text{eff}(\mathcal{E}) \cong \widetilde{\text{eff}} \to \mathcal{Y}\text{eff}(\mathcal{E}) \to \underline{\mathcal{Y}\text{eff}(\mathcal{E})}$ *is fully faithful, furthermore its essential image is an admissible exact category.*

Proof. First we proof that $\widetilde{\text{eff}} \to \underline{\mathcal{Y}\text{eff}(\mathcal{E})}$ is fully faithful: Given a morphism $\phi\colon X \to Y$ in $\widetilde{\text{eff}}$ which factors in $\mathcal{Y}\text{eff}(\mathcal{E})$ as a composition $X \xrightarrow{a} \text{Hom}_{\mathcal{Y}(\mathcal{E})}(-, E[t]) \xrightarrow{b} Y$ with $E \in \mathcal{E}, t \in \mathbb{Z}$. We claim that $\phi = 0$ holds in $\widetilde{\text{eff}}$. Using the definition of objects in $\widetilde{\text{eff}}$, it is easy to see that $\text{Hom}(X, E[t]) = 0$ for $t < 0$ and $\text{Hom}(E[t], Y) = 0$ for $t > 0$. For $t = 0$, we show that $\text{Hom}_{\mathcal{Y}\text{eff}(\mathcal{E})}(\widetilde{\text{eff}}, \text{Hom}_{\mathcal{Y}(\mathcal{E})}(-, E)) = 0$ for all $E \in \mathcal{E}$, so take a projective resolution

$$\text{Hom}_{\mathcal{Y}}(-, X) \to \text{Hom}_{\mathcal{Y}}(-, Y) \to \text{Hom}_{\mathcal{Y}}(-, Z) \to F \to 0$$

with $\sigma\colon X \to Y \to Z$ a short exact sequence in $\mathcal{E}$. When we apply $\text{Hom}_{\mathcal{Y}}(-, E)$ with $E \in \mathcal{E}$, then the conclude that $\text{Hom}_{\mathcal{Y}}(F, \text{Hom}_{\mathcal{Y}}(-, E)) = 0$ as it has to be the zero to start the long exact sequence associated to $\text{Hom}_{\mathcal{E}}(\sigma, E)$.
Next, we are going to see that the essential image of this functor is non-negative, i.e. we will show

$$\text{Hom}_{\underline{\mathcal{Y}\text{eff}(\mathcal{E})}}(F_1, \Sigma^{<0} F_2) = 0$$

for all $F_1, F_2 \in \widetilde{\text{eff}}$. By definition of the ideal quotient (Hom-sets) it is enough to show that $\text{Hom}_{\mathcal{Y}\text{eff}(\mathcal{E})}(F_1, \Omega^t F_2) = 0$ for all $t \geq 1$. For $t \geq 3$ this follows directly from lifting a morphism to projective resolutions and using that $\text{Hom}_{\mathcal{Y}(\mathcal{E})}(E, E'[< 0]) = 0$. For $t = 1, 2$, let $X_i \xrightarrow{a_i} Y_i \xrightarrow{b_i} Z_i$ be the short exact sequences in $\mathcal{E}$ such that $F_i = \text{coker}\,\text{Hom}_{\mathcal{Y}(\mathcal{E})}(-, b_i)$. For $t \in \{1, 2\}$: By definition, we have a monomorphism $\Omega^t F_2 \to \text{Hom}_{\mathcal{Y}(\mathcal{E})}(-, A)$ where $A = Z_2$ for $t = 2$ and $A = Y_2$ for $t = 2$, now by the previous discussion, we have that $\text{Hom}(F_1, \text{Hom}_{\mathcal{Y}}(-, A)) = 0$. This implies also $\text{Hom}(F_1, \Omega^t F_2) = 0$ for $t = 1, 2$.
Lastly, we still have to see that the essential image is extension closed. But this follows from the next Lemma (as $\widetilde{\text{eff}}$ is extension-closed in the Frobenius exact category $\mathcal{Y}\text{eff}(\mathcal{E})$). □

Lemma 5.19. *Let* $\mathcal{F}$ *be a Frobenius exact category and let* $q\colon \mathcal{F} \to \underline{\mathcal{F}}$ *be the ideal quotient functor to its stable category. If* $\mathcal{C}$ *an extension closed full subcategory in* $\mathcal{F}$*, then the essential image of* $\mathcal{C}$ *is extension closed in* $\underline{\mathcal{F}}$.

Proof. Given a standard triangle $X \to Y \to Z \to X[1]$ in $\underline{\mathcal{F}}$ with $X, Z \in q(\mathcal{C})$. We may assume that there exist injective-projective objects P and Q in $\mathcal{F}$ such that $X = C \oplus P, Z = C' \oplus Q$ with $C, C' \in \mathcal{C}$. By the construction of the triangulated structure on $\underline{\mathcal{F}}$, we have that $Y \to Z$ is the pushout of an inflation $X \to I$ into a projective-injective object in $\mathcal{F}$ along the morphism $X \to Y$. By [**49**, Proposition 3.1], this implies that there is a short exact sequence $C \oplus P \to Y \oplus I \to C' \oplus Q$. By [**49**, Proposition 2.12], the short exact sequence splits into a direct sum of of short exact sequences $P \to P \to 0$, $0 \to Q \to Q$ and $C \to \tilde{Y} \to C'$. Since $\mathcal{C}$ is extensions closed, it follows that $q(Y) \cong q(\tilde{Y})$ lies in the essential image of $\mathcal{C}$. □

Remark 5.20. Every extension-closed subcategory $\mathcal{C}$ in a triangulated category $\mathcal{T}$ can be equipped with the structure of an extriangulated structure by restricting the triangles to this category.
This extriangulated structure is an exact structure if and only if $\text{Hom}_{\mathcal{T}}(C, C'[-1]) = 0$ for all C, C' in $\mathcal{C}$. In particular, every admissible exact subcategory has an exact structure given by all triangles $A \to B \to C \to A[1]$ in $\mathcal{T}$ such that A, B, C in $\mathcal{C}$. We will always equip an admissible exact subcategory with this exact structure.

We recall the following result:

Theorem 5.21. *([**133**] or Chapter 9) For every admissible exact subcategory* $\mathcal{C}$ *in an algebraic triangulated category* $\mathcal{T}$ *there exists a triangle functor*

$$\text{D}^b(\mathcal{C}) \to \mathcal{T}$$

which extends the inclusion $\mathcal{C} \subseteq \mathcal{T}$*. It is called a* realization functor *of* $\mathcal{C}$.

We will call a subcategory in a triangulated category **admissible abelian** if it is admissible exact and the induced exact structure from the triangles is abelian.

6. Fully faithfulness of the realization functor

We now want to see that $\mathrm{eff}(\mathcal{E})$ is h-admissible exact in $\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$. This means we need to see

$$\mathrm{Ext}^t_{\widetilde{\mathrm{eff}}}(F,G) \to \mathrm{Hom}_{\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}}(F, \Omega^{-t}_{\mathcal{Y}\mathrm{eff}(\mathcal{E})} G) \quad \forall t \geq 1$$

is an isomorphism for all $F, G \in \widetilde{\mathrm{eff}}$.

Lemma 6.1. *Let $\mathcal{F}$ be a Frobenius category and $X, Y \in \mathcal{F}$, then we have natural isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{F}}(X,Y) \to \underline{\mathrm{Hom}}(X, \Omega^{-n}Y) \quad \forall n \geq 1$$

where $\underline{\mathrm{Hom}} := \mathrm{Hom}_{\underline{\mathcal{F}}}$

Proof. We look at short exact sequences $\Omega^{-n}Y \to I^n \to \Omega^{-(n+1)}Y$, $n \geq 0$ with I^n projective-injective and apply $\mathrm{Hom}_{\mathcal{F}}(X,-)$. We have induced an exact sequence of abelian groups $\mathrm{Hom}(X, I^n) \to \mathrm{Hom}(X, \Omega^{-n}Y) \to \underline{\mathrm{Hom}}(X, \Omega^{-(n+1)}Y)$. Comparism with the long exact sequence gives an induced isomorphism $\mathrm{Ext}^1_{\mathcal{F}}(X, \Omega^{-n}Y) \to \underline{\mathrm{Hom}}(X, \Omega^{-(n+1)}Y)$. Now, the usual dimension shift argument using cosyzygies gives $\mathrm{Ext}^{n+1}_{\mathcal{F}}(X,Y) \cong \mathrm{Ext}^1_{\mathcal{F}}(X, \Omega^{-n}Y)$. □

So, we are actually asking when $\widetilde{\mathrm{eff}}$ is an homologically exact subcategory of $\mathcal{Y}\mathrm{eff}(\mathcal{E})$.

Proposition 6.2. $\widetilde{\mathrm{eff}}$ *is an homologically exact subcategory of* $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ *(or equivalently:* $\widetilde{\mathrm{eff}}$ *is h-admissible exact in* $\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$*).*

Proof. We proceed by first making two general remarks in (1) and (2) before we proceed inductively in (3).
(1) We first remark that for every morphism $g\colon X[m] \to Y[m+1]$ in $\mathcal{Y}(\mathcal{E})$ with $X, Y \in \mathcal{E}$, $m \in \mathbb{Z}$ the following holds $\mathrm{Im}\,\mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-g) \in \Omega^{3m}_{\mathcal{Y}\mathrm{eff}(\mathcal{E})}\widetilde{\mathrm{eff}}$.
(2) We secondly remark that in $\mathcal{Y}(\mathcal{E})$ every morphism $f\colon X \to Y[n]$ with $X, Y \in \mathcal{E}, n \geq 1$ can be written as a composition $X = X_0 \xrightarrow{f_1} X_1[1] \xrightarrow{f_2} X_2[2] \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n[n] = Y[n]$ with $X_i \in \mathcal{E}$, $0 \leq i \leq n$. (3) Now, we claim the following: For every morphism $h\colon F \to \Omega^{-t}G$ in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$ with $F, G \in \widetilde{\mathrm{eff}}$, $t \geq 2$ there exists an $s \in \mathbb{N}$, $h^s\colon \Omega^s F \to \Omega^{s-t}G$ with $\Omega^{-s}h_s = h$ in $\underline{\mathcal{Y}\mathrm{eff}}(\mathcal{E})$ such that h^s is a composition $\Omega^s F = \Omega^s F_0 \to \Omega^{s-1}F_1 \to \Omega^{s-2}F_2 \to \cdots \to \Omega^{s-t}F_t = \Omega^{s-t}G$.
We fix short exact sequences $X'' \xrightarrow{i} X \xrightarrow{p} X'$ and $Y'' \xrightarrow{j} Y \xrightarrow{q} Y'$ with $F = \mathrm{coker}\,\mathrm{Hom}(-,p)$, $G = \mathrm{coker}\,\mathrm{Hom}(-,q)$. The morphism $h\colon F \to \Omega^{-t}G$ induces morphisms between the long exact sequences of representable functors which induces morphisms $h_s\colon \Omega^s F \to \Omega^{s-t}G$ for all $s \in \mathbb{Z}$. Now we study the morphisms of long exact sequences to find the factorization, to shorten notation, we use $(-,?) := \mathrm{Hom}_{\mathcal{Y}(\mathcal{E})}(-,?)$.
If $t = 2$, we look at the commutative diagram

$$\begin{array}{ccccccc}
(-,X'') & \xrightarrow{(-,i)} & (-,X) & \xrightarrow{(-,p)} & (-X') & \longrightarrow & F \\
\downarrow{\scriptstyle (-,a'')} & & \downarrow{\scriptstyle (-,a)} & & \downarrow{\scriptstyle (-,a')} & & \downarrow{\scriptstyle h} \\
(-,Y') & \xrightarrow[\sigma]{} & (-,Y''[1]) & \xrightarrow[(-,j[1])]{} & (-,Y[1]) & \longrightarrow & \Omega^{-2}G
\end{array}$$

Then set $F_1 = \mathrm{Im}(-,j[1]) \circ (-,a) \in \widetilde{\mathrm{eff}}$ (by (1)) and using the exactness of the rows we conclude that h_1 factors as $\Omega F \to F_1 \to \Omega^{-1}G$.

If $t=3$, we look at the commutative diagram

$$\begin{array}{ccccccc}
(-,X'') & \xrightarrow{(-,i)} & (-,X) & \xrightarrow{(-,p)} & (-X') & \longrightarrow & F \\
\downarrow{\scriptstyle (-,a'')} & & \downarrow{\scriptstyle (-,a)} & & \downarrow{\scriptstyle (-,a')} & & \downarrow{\scriptstyle h} \\
(-,Y''[1]) & \xrightarrow[(-,j[1])]{} & (-,Y[1]) & \xrightarrow[(-,q[1])]{} & (-,Y'[1]) & \longrightarrow & \Omega^{-3}G
\end{array}$$

then with $F_1 = \operatorname{Im}(-,q[1])\circ(-,a)$ we get a factorization of h_1 as $\Omega F \to F_1 \to \Omega^{-2}$.
For $t \geq 4$ we proceed inductively and find a factorization of the form $\Omega F \to F_1 \to \Omega^{-t+1}G$ with $F_1 \in \widetilde{\text{eff}}$ as follows; Consider the commutative diagram

$$\begin{array}{ccccccc}
(-,X'') & \xrightarrow{(-,i)} & (-,X) & \xrightarrow{(-,p)} & (-X') & \longrightarrow & F \\
\downarrow{\scriptstyle (-,a'')} & & \downarrow{\scriptstyle (-,a)} & & \downarrow{\scriptstyle (-,a')} & & \downarrow{\scriptstyle h} \\
(-,Y_1[n_1]) & \xrightarrow[(-,\ell)]{} & (-,Y_2[n_2]) & \xrightarrow[(-,m)]{} & (-,Y_3[n_3]) & \longrightarrow & \Omega^{-t}G
\end{array}$$

with the second row is induced by the suitably-number rotated triangle, we have $\{Y_1,Y_2,Y_3\}=\{Y'',Y,Y'\}$ and certain $n_i \in \mathbb{N}_{\geq 1}$, $n_3 \geq 2$. By (2), the morphism $(-,a')$ factors as $(-,X') \xrightarrow{f_1'} (-,X_1[1]) \to (-,Y_3[n_3])$. We precompose with $(-,p)$ to obtain a factorization of $(-,m)\circ(-,a)\colon (-,X)\to(-,Y_3[n_3])$ as $(-,X) \xrightarrow{f_1} (-,X_1[1]) \to (-,Y_3[n_3])$. We define $F_1 := \operatorname{Im} f_1 \in \widetilde{\text{eff}}$. As the second row is exact, we find an induced morphism $F_1 \to \Omega^{-t+1}$. By definition, we have $f_1\circ(-,i)=0$, this induces a morphism $\Omega F \to F_1$, this gives the factorization of h_1. The previous claim (3) implies that the maps $\operatorname{Ext}^t_{\widetilde{\text{eff}}}(F,G) \to \operatorname{Hom}_{\underline{\mathcal{Y}\text{eff}(\mathcal{E})}}(F, \Omega^{-t}_{\underline{\mathcal{Y}\text{eff}(\mathcal{E})}}G)$ are surjective for all $t\geq 1$ (as they are isomorphisms for $t=1$). Then it is a standard argument that this implies that they are isomorphisms for all $t \geq 2$. $\square$

7. The realization functor is essentially surjective

Is the inclusion $\operatorname{Thick}_\Delta(\widetilde{\text{eff}}) \subseteq \underline{\mathcal{Y}\text{eff}(\mathcal{E})}$ an equality? We start with the following duality:

7.1. The Auslander-Bridger transpose.

$$\operatorname{Tr}\colon (\underline{\operatorname{mod}}_1 \mathcal{Y}(\mathcal{E}))^{op} \to \underline{\operatorname{mod}}_1 \mathcal{Y}(\mathcal{E}^{op})$$

maps $\operatorname{coker}\operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(-,f)$ to $\operatorname{coker}\operatorname{Hom}_{\mathcal{Y}(\mathcal{E})}(f,-)$, compare [**90**, section 5.2]. It restricts to a duality, i.e. a functor

$$\operatorname{Tr}\colon \underline{\mathcal{Y}\text{eff}(\mathcal{E})}^{op} \to \underline{\mathcal{Y}\text{eff}(\mathcal{E}^{op})}$$

the quasi-inverse is given by the same transpose defined for $\mathcal{E}^{op}$ and by definition $\operatorname{Tr}(\Omega F) \cong \Omega^- \operatorname{Tr} F$. It restricts to a duality $\Omega^- \circ \operatorname{Tr}\colon \widetilde{\text{eff}}(\mathcal{E}) \to \widetilde{\text{eff}}(\mathcal{E}^{op})$.

Definition 7.1. Let $F\colon \mathcal{Y}(\mathcal{E}) \to (Ab)$ be a covariant additive functor, we define the graded support of F as

$$\operatorname{supp}(F) := \{i \in \mathbb{Z} \mid \exists E \in \mathcal{E} \mid F(E[i]) \neq 0\} \quad \subseteq \mathbb{Z}$$

If F is contravariant, we take the same definition but we write supp^{op} instead of supp.
Let $X \in \mathrm{D}^b(\mathcal{E})$ we have a covariant functor $F_X = \operatorname{Hom}_{\mathrm{D}^b(\mathcal{E})}(X,-)|_{\mathcal{Y}(\mathcal{E})}$ and a contravariant functor $F^X = \operatorname{Hom}_{\mathrm{D}^b(\mathcal{E})}(-,X)|_{\mathcal{Y}(\mathcal{E})}$. We call $\operatorname{supp}(F_X)$ resp. $\operatorname{supp}^{op}(F^X)$ the covariant resp. contravariant **graded support** of X.
We define the two **Yoneda degrees** of X via

(*) $\mathcal{Y}\deg(X) = n$ if $n \in \operatorname{supp}(F_X) \subseteq [n,\infty)$.
(*) $\mathcal{Y}\deg^{op}(X) = n$ if $n \in \operatorname{supp}^{op}(F^X) \subseteq (-\infty, n]$.

Remark 7.2. For $X \in \mathrm{D}^b(\mathcal{E})$.
If $\mathcal{Y}\mathrm{deg}(X) = n \in \mathbb{Z}$ then $\mathcal{Y}\mathrm{deg}(X[-n]) = 0$.
If $\mathcal{Y}\mathrm{deg}^{op}(X) = m$ then $\mathcal{Y}\mathrm{deg}^{op} X[-m] = 0$.
Given $X = A_i \oplus A_{i+1} \oplus \cdots \oplus A_j$, $i \leq j$ with $A_t \in \mathcal{E}[t]$ for all t and $A_i \neq 0$, $A_j \neq 0$, then we have $\mathcal{Y}\mathrm{deg}(X) = i$, $\mathcal{Y}\mathrm{deg}^{op}(X) = j$. Conversely, every $X \in \mathcal{Y}(\mathcal{E})$ with $\mathcal{Y}\mathrm{deg}(X) = i$, $\mathcal{Y}\mathrm{deg}^{op}(X) = j$ can be written in this way. In particular, for $X \in \mathcal{Y}(\mathcal{E})$, we have $X \in \mathcal{E}$ if and only if $\mathcal{Y}\mathrm{deg}(X) = 0 = \mathcal{Y}\mathrm{deg}^{op}(X)$.
Now, on $\mathrm{D}^b(\mathcal{E})$, for $n, m \in \mathbb{N}$, the conditions $\mathcal{Y}\mathrm{deg} \geq n$ and $\mathcal{Y}\mathrm{deg}^{op} \leq m$ are extension-closed. This implies that the Yoneda degrees are well-defined for all $X \in \mathrm{D}^b(\mathcal{E})$ because this is the extension-closure of $\mathcal{Y}(\mathcal{E})$.

Remark 7.3. We have a problem when we want to extend this definitions to the stable category of Yoneda-effaceables as then the support is no longer a well-defined invariant of the isomorphism class (e.g. the zero functor is isomorphic to every projective - but their supports vary).

To overcome this issue, we define these degree functions first for triangles.

Definition 7.4. Given a triangle without a split summand $\Delta\colon A \to B \to C \xrightarrow{+1}$, $A, B, C \in \mathcal{Y}(\mathcal{E})$. We number the objects as follows $A[n] =: D^{3n-2}, B[n] =: D^{3n-1}, C[n] =: D^{3n}$, $n \in \mathbb{Z}$. We define

$$\mathcal{Y}\mathrm{deg}(\Delta) := (\inf\{n \in \mathbb{Z} \mid \mathcal{Y}\mathrm{deg}(D^n) > 0\}) - 1$$

If $\mathcal{Y}\mathrm{deg}(\Delta) = t$ and Δ has no split triangles as summands, then we have for the suitably times rotated triangle $\Delta'\colon D^{t-2} \to D^{t-1} \to D^t \xrightarrow{+1}$ the following property

(*) $\mathcal{Y}\mathrm{deg}(D^t) = \mathcal{Y}\mathrm{deg}(D^{t-1}) = \mathcal{Y}\mathrm{deg}(D^{t-2}) = 0$.
To see this, by definition $\mathcal{Y}\mathrm{deg}(D^{t+1}) \geq 1$, so $\mathcal{Y}\mathrm{deg}(D^{t-2}) \geq 0$. But by definition $\mathcal{Y}\mathrm{deg}(D^{t-2})$ can not be > 0, so it has to be $= 0$. Also by definition $\mathcal{Y}\mathrm{deg}(D^t) \leq 0, \mathcal{Y}\mathrm{deg}(D^{t-1} \leq 0$. Assume that $\mathcal{Y}\mathrm{deg}(D^t) = (-n) < 0$, take $X[-n] \in \mathcal{E}[-n|$, then $(D^t, X[-n]) \cong (D^{t-1}, X[-n])$. Then one can get a contradiction to the assumption that Δ has no split summand. Therefore we have $\mathcal{Y}\mathrm{deg}(D^t) = 0$ and as $\mathcal{Y}\mathrm{deg} \geq 0$ is extension-closed, we conclude that $\mathcal{Y}\mathrm{deg}(D^{t-1}) = 0$

Furthermore, $\mathcal{Y}\mathrm{deg}(\Delta)$ only depends on the homotopy equivalence class of the complex $\mathrm{Hom}_{\mathrm{D}^b(\mathcal{E})}(\Delta, -) \in K(\mathcal{P}(\mathrm{Mod}\,\mathcal{Y}(\mathcal{E}^{ic})^{op}))$. In particular, if $F \in \mathcal{Y}\mathrm{eff}(\mathcal{E})$ is represented by Δ, then $\mathcal{Y}\mathrm{deg}(\Delta)$ is well-defined for $\mathrm{Tr}\,\underline{F} \in \underline{\mathcal{Y}\mathrm{eff}}$ and Tr is a duality on $\underline{\mathcal{Y}\mathrm{eff}}$, so

$$\underline{\mathrm{deg}}(\underline{F}) := \mathcal{Y}\mathrm{deg}(\Delta) \quad \text{ for } \underline{F} \in \underline{\mathcal{Y}\mathrm{eff}}(\mathcal{E})$$

is a well-defined integer.
Dually, we may define for a triangle as before

$$\mathcal{Y}\mathrm{deg}^{op}(\Delta) = (\sup\{n \in \mathbb{Z} \mid \mathcal{Y}\mathrm{deg}^{op}(D^n) < 0\}) + 1$$

If $\mathcal{Y}\mathrm{deg}^{op}(\Delta) = s$, then for the suitable rotated triangle $\Delta''\colon D^s \to D^{s+1} \to D^{s+2} \xrightarrow{+1}$ the following property holds

$(*)^{op}$ $\mathcal{Y}\mathrm{deg}^{op}(D^s) = \mathcal{Y}\mathrm{deg}^{op}(D^{s+1}) = \mathcal{Y}\mathrm{deg}^{op}(D^{s+2}) = 0$

In this case, also $\mathrm{deg}^{op}(\underline{F}) := \mathcal{Y}\mathrm{deg}^{op}(\Delta)$ is well-defined for F represented by Δ in $\mathcal{Y}\mathrm{eff}(\mathcal{E})$.

Example 7.5. Let $\underline{F}$ be in $\Omega^t\widetilde{\mathrm{eff}}$, then $\underline{\mathrm{deg}}(\underline{F}) = t$, $\underline{\mathrm{deg}}^{op}(\underline{F}) = t - 2$.

Lemma 7.6. *Given a Yoneda-effaceable functor F and $t = \underline{\mathrm{deg}}(\underline{F})$. Then $\underline{\mathrm{deg}}(\underline{F}) \geq \underline{\mathrm{deg}}^{op}(\underline{F}) - 2$ and it is $=$ if and only if $F \in \Omega^t\widetilde{\mathrm{eff}}$.*

Proof. The inequality follows by definition. Equality means that the triangles Δ' and Δ'' coincide. But this means that $\mathcal{Y}\mathrm{deg}^{op} = \mathcal{Y}\mathrm{deg} = 0$ for D^{t-2}, D^{t-1}, D^t, i.e. we have a distinguished triangle with three consecutive terms in $\mathcal{E}$. Then the first two maps in such a triangle are given by a short exact sequence in $\mathcal{E}$ and the claim follows. □

We think of the number $d_F = \underline{\deg}(\underline{F}) - \underline{\deg}^{op}(\underline{F}) + 2$ as the distance of F being a (co)syzygy of an effaceable. Now, the strategy is the following: Show that every $F \in \mathcal{Y}\text{eff}(\mathcal{E})$ fits into a short exact sequence $G \rightarrowtail F \twoheadrightarrow \Omega^t E$ for some t such that $d_G < d_F$.

Remark 7.7. Observe that mod $_\infty\mathcal{Y}(\mathcal{E})$ is deflation-closed in $\operatorname{Mod}\mathcal{Y}(\mathcal{E})$ and $\mathbf{Gp}(\operatorname{mod}_\infty \mathcal{Y}(\mathcal{E}))$ is deflation-closed in $\operatorname{mod}_\infty \mathcal{Y}(\mathcal{E})$ and $\mathcal{Y}\text{eff}(\mathcal{E})$ is deflation-closed in $\mathbf{Gp}(\operatorname{mod}_\infty \mathcal{Y}(\mathcal{E}))$. Therefore we have that arbitrary kernels of epimorphisms between Yoneda effaceable functors are again Yoneda effaceable.

Lemma 7.8. *Let $\mathcal{E}$ be weakly idempotent complete. Given a triangle $\Delta\colon Z \to Y \to X \xrightarrow{+1}, X, Y, Z \in \mathcal{Y}(\mathcal{E})$ without split summands of $\mathcal{Y}\deg(\Delta) = 0$. Then we have*

$$Z_{>0} \oplus Z_0 \xrightarrow{\begin{pmatrix} c & d \\ 0 & f \end{pmatrix}} Y = Y_{>0} \oplus Y_0 \xrightarrow{\begin{pmatrix} a & b \\ 0 & p \end{pmatrix}} X_{>0} \oplus X_0 \xrightarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} Z_{>0}[1] \oplus Z_0[1]$$

where $Z_{>0}, X_{>0}, Y_{>0} \in \bigvee_{i>0} \mathcal{E}[i]$, $X_0, Y_0, Z_0 \in \mathcal{E}$ and $p\colon Y_0 \to X_0$ an deflation.

Proof. That we can write it in this form follows from (*). Now, $\delta\colon X_0 \to Z_0[1]$ corresponds to a short exact sequence, say this is $Z_0 \overset{i}{\rightarrowtail} V_0 \overset{q}{\twoheadrightarrow} X_0$. Then, $\begin{pmatrix} 0 \\ q \end{pmatrix} : V_0 \to X_{>0} \oplus X_0$ satisfies $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore there exists a morphism $\begin{pmatrix} g \\ h \end{pmatrix} : V_0 \to Y_{>0} \oplus Y_0$ such that $\begin{pmatrix} a & b \\ 0 & p \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix}$, this implies $p \circ g = q$. Now, as $\mathcal{E}$ is weakly idempotent complete it follows from the obscure axiom [**49**, Prop. 7.6] that p is a deflation. □

Remark 7.9. If we could show that p is a deflation, then the proof can be completed: let $u\colon Y \to Z$, $F = \operatorname{coker}(-, u)|_{\mathcal{Y}(\mathcal{E})}$ be as before, take $E = \operatorname{coker}\operatorname{Hom}(-, p)|_{\mathcal{Y}(\mathcal{E})}$ then the kernel can be described as $G = \operatorname{coker}(-, a)|_{\mathcal{Y}(\mathcal{E})}$. As we know that G is Yoneda eaffaceable, it follows from Prop. 5.2 that a has a cone in $\mathcal{Y}(\mathcal{E})$ and can be completed to a complete projective resolution of G. Then $\mathcal{Y}\deg(G) < 0 = \mathcal{Y}\deg(F)$ and $\mathcal{Y}\deg^{op}(G) = \mathcal{Y}\deg^{op}(F)$, so the induction would work.

Proposition 7.10. *Let $\mathcal{E}$ be weakly idempotent complete. Assume $F \in \mathcal{Y}\text{eff}(\mathcal{E})$ is represented by a triangle Δ without split summands and $\mathcal{Y}\deg(\Delta) = 0$, then there exists a short exact sequence in $\mathcal{Y}\text{eff}(\mathcal{E})$*

$$\Omega G \rightarrowtail \Omega F \twoheadrightarrow \Omega E$$

with E in $\widetilde{\mathcal{E}}$ and if F is not in $\widetilde{\text{eff}}$ then $d_G < d_F$.

Proof. We take for Δ the notation of the previous Lemma and set $u := \begin{pmatrix} a & b \\ 0 & p \end{pmatrix}, v = \begin{pmatrix} c & d \\ 0 & f \end{pmatrix}$, i.e. $F = \operatorname{coker}(-, u)|_{\mathcal{Y}(\mathcal{E})}$. By the previous Lemma, p is a deflation, say with $\mathcal{E}$-kernel $K_0 \xrightarrow{j} Y_0$, then $E = \operatorname{coker}(-, p)|_{\mathcal{Y}(\mathcal{E})}$ is in $\widetilde{\text{eff}}$. Furthermore, we define $G = \operatorname{coker}(-, a)|_{\mathcal{Y}(\mathcal{E})}$. By the 3×3-Lemma for traingulated categories we find an object C and morphisms such that all rows and columns are distinguished triangles in the following diagram

$$\begin{array}{ccccccc}
X_0[-1] & \longrightarrow & K_0[-1] & \longrightarrow & Y_0[-1] & \longrightarrow & X_0[-1] \\
\downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
X_{>0}[-1] & \longrightarrow & C & \longrightarrow & Y_{>0} & \xrightarrow{a} & X_{>0} \\
\downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle \binom{1}{0}} & & \downarrow{\scriptstyle \binom{1}{0}} \\
X[-1] & \longrightarrow & Z & \xrightarrow{v} & Y & \xrightarrow{u} & X \\
\downarrow{\scriptstyle (0,1)} & & \downarrow & & \downarrow{\scriptstyle (0,1)} & & \downarrow{\scriptstyle (0,1)} \\
X_0[-1] & \longrightarrow & K_0 & \xrightarrow{j} & Y_0 & \xrightarrow{p} & X_0
\end{array}$$

Observe that by definition $\Omega G = \mathrm{Im}(-,a)|_{\mathcal{Y}(\mathcal{E})}$, $\Omega F = \mathrm{Im}(-,u)|_{\mathcal{Y}(\mathcal{E})}$, $\Omega E = \mathrm{Im}(-,p)|_{\mathcal{Y}(\mathcal{E})}$. We now look at the induced diagramm of all representable functors $(-,A)|_{\mathcal{Y}}$ restricted to the Yoneda category $\mathcal{Y} = \mathcal{Y}(\mathcal{E})$.

$$\begin{array}{ccccccc}
(-,X_{>0}[-1])|_{\mathcal{Y}} & \longrightarrow & (-,C)|_{\mathcal{Y}} & \longrightarrow & (-,Y_{>0})|_{\mathcal{Y}} & \xrightarrow{a} & (-,X_{>0})|_{\mathcal{Y}} \\
\downarrow {\scriptstyle \binom{1}{0}} & & \downarrow {\scriptstyle (-,z)} & & \downarrow {\scriptstyle \binom{1}{0}} & & \downarrow {\scriptstyle \binom{1}{0}} \\
(-,X[-1])|_{\mathcal{Y}} & \longrightarrow & (-,Z)|_{\mathcal{Y}} & \xrightarrow{v} & (-,Y)|_{\mathcal{Y}} & \xrightarrow{u} & (-,X)|_{\mathcal{Y}} \\
\downarrow {\scriptstyle (0,1)} & & \downarrow & & \downarrow {\scriptstyle (0,1)} & & \downarrow {\scriptstyle (0,1)} \\
(-,X_0[-1])|_{\mathcal{Y}} & \longrightarrow & \mathrm{coker}(-,z)|_{\mathcal{Y}} & \xrightarrow{j} & (-,Y_0)|_{\mathcal{Y}} & \xrightarrow{p} & (-,X_0)|_{\mathcal{Y}}
\end{array}$$

Now, use the snake lemma (in $\mathrm{Mod}\,\mathcal{Y}(\mathcal{E})$) to obtain an exact sequence $\Omega G \rightarrowtail \Omega F \twoheadrightarrow \Omega E$. We conclude that G is also Yoneda effaceable and by Prop. 5.2 it follows C in $\mathcal{Y}(\mathcal{E})$. As $\mathcal{Y}\mathrm{deg}() > 0$ is extension-closed it follows $\mathcal{Y}\mathrm{deg}(C[1]) > 0$, so, let us denote Δ' the distinguished triangle $C \to Y_{>0} \to X_{>0} \xrightarrow{+1}$. By definition, if F is not in $\widetilde{\mathrm{eff}}$, then we have $\mathcal{Y}\mathrm{deg}(\Delta') \leq (-2)$ and we have $\mathcal{Y}\mathrm{deg}^{op}(\Delta') \geq \mathcal{Y}\mathrm{deg}^{op}(\Delta)$ (use the columns in the 3×3 diagram and the definition to see this). □

Then just use the distinguished triangles induced by a the short exact sequence from the previous Proposition and an induction on d_F, to see the following corollary.

Corollary 7.11. *If $\mathcal{E}$ is weakly idempotent complete:* $\mathrm{Thick}(\widetilde{\mathrm{eff}}) = \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$

Remark 7.12. We conjecture $\mathcal{Y}(\mathcal{E}) = \mathcal{Y}(\mathcal{E}^{ic})$, this would imply that we can leave out the assumption $\mathcal{E}$ weakly idempotent complete in the corollary 7.11 is obsolete.

8. Main results

From the previous two sections we conclude the following theorem which is our main result

Theorem 8.1. *If $\mathcal{E}$ is a weakly idempotent complete exact category. Then the realization functor for the admissible exact category* $\mathrm{eff}(\mathcal{E}) \cong \widetilde{\mathrm{eff}} \subseteq \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}$ *is a triangle equivalence*

$$\mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \to \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}.$$

Theorem 8.2. *If $\mathcal{E} \to \mathcal{E}'$ is a homologically exact functor. Then the induced functor* $\mathrm{eff}(\mathcal{E}) \to \mathrm{eff}(\mathcal{E}')$ *is homologically exact.*

Proof. We get a commutative diagram

$$\begin{array}{ccc}
\mathrm{D}^b(\mathrm{eff}(\mathcal{E})) & \longrightarrow & \mathrm{D}^b(\mathrm{eff}(\mathcal{E}')) \\
\downarrow & & \downarrow \\
\underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})} & \longrightarrow & \underline{\mathcal{Y}\mathrm{eff}(\mathcal{E})}
\end{array}$$

with the vertical arrows are fully faithful triangle equivalence and the lower one is fully faithful by Lemma 5.8. This implies the upper triangle functor is also fully faithful. □

9. Some special situations

Definition 9.1. For an exact category we define a Frobenius pair (in the sense of Schlichting) by

$$\mathrm{eff}(\mathbf{Gp}(\mathcal{E})) \subseteq \mathbf{Gp}(\mathrm{eff}(\mathcal{E}))$$

The associated Verdier quotient

$$\underline{\mathbf{Gp}(\mathrm{eff}(\mathcal{E}))}/\underline{\mathrm{eff}(\mathbf{Gp}(\mathcal{E}))}$$

will be called the **Frobenius gap** of $\mathcal{E}$.

Open question: Is $\mathcal{E}$ a Frobenius category if and only if its Frobenius gap is zero? (This seems to be true for exact categories with enough projectives...)

Similary, if $\mathcal{E}$ is an exact category then $\mathcal{P}^{<\infty}(\mathcal{E}) = \{X \in \mathcal{E} \mid \mathrm{pd}_{\mathcal{E}} X < \infty\}$ is a thick subcategory We say $\mathcal{E}$ is **regular** if $\mathcal{E} = \mathcal{P}^{<\infty}(\mathcal{E})$. If we assume that $\mathcal{E}$ has enough projectives, then $\mathcal{P}^{<\infty}(\mathcal{E})$ is a resolving subcategory of $\mathcal{E}$, so in particular it is homologically exact. Then, we have a chain of homological exact functors $\mathrm{eff}(\mathcal{P}^{\infty}(\mathcal{E})) \subseteq \mathcal{P}^{<\infty}(\mathrm{eff}(\mathcal{E})) \subseteq \mathrm{eff}(\mathcal{E})$ this induces three short exact sequences of triangulated categories

$$\mathrm{D}^b(\mathrm{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))) \to \mathrm{D}^b(\mathcal{P}^{<\infty}(\mathrm{eff}(\mathcal{E}))) \to \mathrm{D}^b(\mathcal{P}^{<\infty}(\mathrm{eff}(\mathcal{E})))/\mathrm{D}^b(\mathrm{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))$$

$$\mathrm{D}^b(\mathrm{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))) \to \mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \to \mathrm{D}^b(\mathrm{eff}(\mathcal{E}))/\mathrm{D}^b(\mathrm{eff}(\mathcal{P}^{<\infty}(\mathcal{E}))$$

$$\mathrm{D}^b(\mathcal{P}^{<\infty}(\mathrm{eff}(\mathcal{E}))) \to \mathrm{D}^b(\mathrm{eff}(\mathcal{E})) \to D_{sg}(\mathrm{eff}(\mathcal{E}))$$

and we get an induced fourth exact sequence of exact categories

$$\mathrm{D}^b(\mathcal{P}^{<\infty}(\mathrm{eff}(\mathcal{E})))/\mathrm{D}^b(\mathrm{eff}(\mathcal{P}^{<\infty}(\mathcal{E})) \to \mathrm{D}^b(\mathrm{eff}(\mathcal{E}))/\mathrm{D}^b(\mathrm{eff}(\mathcal{P}^{<\infty}(\mathcal{E})) \to D_{sg}(\mathrm{eff}(\mathcal{E}))$$

If $\mathcal{E}$ is regular then $\mathrm{eff}(\mathcal{E})$ is regular and all three triangulated categories are zero.

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