QUASI-PROJECTIVE VARIETIES ARE QUIVER GRASSMANNIANS FOR EXACT CATEGORIES

Projective varieties are quiver Grassmannians in several ways as shown in Reineke [2] with a precursor already in [1] and further variants in [3], [4]. We slightly modify Reineke's construction to show that every quasi-projective variety (i.e. finite union of principal opens) is a quiver Grassmannian for an exact category.
We start recalling Reineke's isomorphism and first look at a single principle open subset of a projective variety. Let K be an algebraically closed field. We consider $X = \mathrm{Proj(R)}$ with $R = K[T_0, \dots, T_n]/(f_1, \dots, f_t)$, f_i homogeneous polynomials of degree $d_i > 0$, $1 \leq i \leq t$. Let f be another homogeneous polynomial such that $\overline{f} \in R$ has a positive degree $d_f > 0$ - in particular f is not a scalar multiple of any f_i, $1 \leq i \leq t$. Then the principal open subset $D_+(\overline{f}) \subset X$ equals $\{x \in \mathbb{P}^n(K) \mid f_i(x) = 0, 1 \leq i \leq t, f(x) \neq 0\}$.

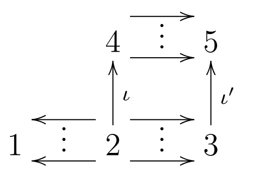
(1) We may assume that $d := d_i = d_j = d_f$ for all $i \neq j$. Otherwise, take $\ell = \mathrm{lcm(d_1, \dots, d_t, d_f)}$ and replace f_i by $f_i^{\frac{\ell}{d_i}}$ and f by $f^{\frac{\ell}{d_f}}$. This does not change the variety, nor the open subset.

(2) We use the d-uple embedding of $\mathbb{P}^n$. More precisely: Let $M_{n,d} := \{(m_0, \dots, m_n) \in \mathbb{N}_0^{n+1} \mid \sum_{j=0}^n m_j = d\}$, $M := |M_{n,d}|$ and $N := |M_{n,d-1}|$ the cardinalities. Let

$$j\colon \mathbb{P}^n \to \mathbb{P}^{M-1}, \quad [x_0\colon \dots \colon x_n] \mapsto [\cdots \colon x_m \colon \cdots]_{m \in M_{n,d}},$$
$$\text{where} \quad x_m = x_0^{m_0} \cdots x_n^{m_n}, \text{ for } m = (m_0, \dots, m_n) \in M_{n,d}$$

This is a closed embedding called the d-uple embedding.
Now, we consider the quiver $Q = (Q_0, Q_1)$ with three vertices $1, 2, 3$ and $n + 1$ arrows from 2 to 3 and t arrows from 2 to 1. We define the following Q-representations: We denote by $v_m, m \in M_{n,d}$ a vector space basis for $K^{M_{n,d}}$ (which is the set of all maps from $M_{n,d} \to K$, given a vector space structure with the pointwise addition and scalar multiplication). Let $f_j = \sum_{m \in M_{n,d}} a_m^{(j)} T^m$, $f = \sum_{m \in M_{n,d}} a_m T^m$, we denote by φ_j resp. φ the linear map $K^{M_{n,d}} \to K$ sending $v_m \mapsto a_m^{(j)}$ resp to a_m.
Let V be the Q-representation with $V_1 = K$, $V_2 = K^{M_{n,d}}$, $V_3 = K^{M_{n,d-1}}$ and linear maps $\varphi_j\colon V_2 \to V_1, 1 \leq j \leq t$, $g_i\colon V_2 \to V_3$, $0 \leq j \leq n$ defined by $g_i(v_m) := v_{m-e_i}$ if $m_i > 0$ and $g_i(v_m) = 0$ if $m_i = 0$.
Then [2] it has been shown that $j(X)$ is isomorphic to $\mathrm{Gr}_Q(V, (0, 1, 1))$.

We are going to extend this quiver to a quiver Σ with vertices $\{1, 2, 3, 4, 5\}$



with t arrows from 2 to 1 one arrow $\iota\colon 2 \to 4$ and one $\iota'\colon 3 \to 5$, $n+1$ arrows $\alpha_0, \dots, \alpha_n$ from 2 to 3 and $n + 1$ arrows $\beta_0, \dots, \beta_n$ from 4 to 5. Additionally we impose the commutativity relations $I = (\alpha_u \circ \iota - \iota' \circ \beta_u, 0 \leq u \leq n)$. Let $\Lambda = K\Sigma/I$. We extend the Q-representation V to a Λ-module $\mathbb{V}$ by imposing that at arrow ι and arrow ι' have to be the identity map, then we have $\underline{\dim}\mathbb{V} = (1, M, N, M, N)$ (here dimension vector $\underline{\dim}L = (d_1, \dots, d_5)$ means $d_i = \dim_K L_i$). Observe that every $KQ \otimes K\mathbb{A}_2$-module, i.e. a KQ-module morphism $a\colon U \to V$, restricts to a Λ-module $e(a)$ when we leave out the vector space V_1 and the maps to it. Now let $d = (0, 1, 1, M, N)$ be a dimension vector, then it is straight forward to see that we get an isomorphism of varieties:

$$\mathrm{Gr}_Q(V, (0,1,1)) \to \mathrm{Gr}_\Lambda(\mathbb{V}, d)$$

which sends $i\colon U \subset V$ to $e(i) \subset \mathbb{V}$ and conversely, just restricts the inclusions to the vertices $1, 2, 3$, call this $i\colon U \to V$. Then the inclusion i_2 of i at vertex 2 has the same image as $e(i) \subset \mathbb{V}$ at vertex 2. This is a very silly map but it ensures that different point $x \neq x'$ in X correspond to non-isomorphic Λ-modules U_x and $U_{x'}$.
Now let $\mathbb{V}_f$ be the Λ-module with underlying vector space as $\mathbb{V}$ and the restriction to the subquiver given by $2, 3, 4, 5$ is the same as for $\mathbb{V}$ but with all the linear maps $V_2 \to V_1$ are all chosen to be φ.
We claim:

Lemma 0.1. (i) $\mathrm{Hom}_\Lambda(\mathbb{V}, \mathbb{V}_f) = 0 = \mathrm{Hom}_\Lambda(\mathbb{V}_f, \mathbb{V})$ *and* $\mathrm{End}_\Lambda(\mathbb{V}_f) = K = \mathrm{End}_\Lambda(\mathbb{V})$.

(ii) *If* $U_x = e(i) \in \mathrm{Gr}_\Lambda(\mathbb{V}, d)$ *such that* $i\colon U \subset V$ *corresponds under Reinekes isomorphism to* $x \in X \subset \mathbb{P}^n(K)$ *(i.e.* $j(x) = \mathrm{Im}\, i_2$*), then:*
$\mathrm{Hom}_\Lambda(U_x, \mathbb{V}_f) = 0$ *iff* $f(x) \neq 0$.

Proof. (i) It is clear that we may restrict to the full subquiver Q at vertices $1, 2, 3$ since for these modules at arrow i and j we have the identity. We look at the full subquiver Q' given by the vertices $\{2, 3\}$. The restricted representation V' onto this subquiver is for $\mathbb{V}$ and $\mathbb{V}_f$ the same. In the last paragraph of loc. cit it is shown that $\mathrm{End}_{KQ'}(V') = K$, more precisely every endomorphism consists of a pair of linear maps $\psi_2\colon V_2 \to V_2, \psi_3\colon V_3 \to V_3$ with $\psi_2 = C\,\mathrm{id}_{V_2}$ for a $C \in K$ and ψ_3 is determined by ψ_2. Now, since f is not a scalar multiple of any of the f_i, the rest of the claim is clear from the definitions.

(ii) $\mathrm{Hom}_\Lambda(U_x, \mathbb{V}_f) \neq 0$ is equivalent to $\mathrm{Hom}_\Lambda(U_x, \mathbb{V}_f) = 0$ (using [2], last paragraph). This is equivalent to that there is a monomorphism $U_x \to \mathbb{V}_f$ and this is equivalent to $U_x \subset \mathbb{V}_f$. The last statement is by the argument used for Reineke's isomorphism equivalent to $f(x) = 0$.

□

Now let $\mathcal{E} \subset \Lambda - \mathrm{mod}$ be the extension-closed subcategory consisting of all modules L such that $\mathrm{Hom}_\Lambda(L, \mathbb{V}_f) = 0$. Observe that given a short exact sequence of Λ-modules $0 \to U \to V \to W \to 0$, with U, V in $\mathcal{E}$, it follows that W is in $\mathcal{E}$.
From the previous Lemma it follows directly:

Theorem 0.2. *The isomorphism* $\mathrm{Gr}_\Lambda(\mathbb{V}, d) \to j(X) \cong X, (i\colon U \subset \mathbb{V}) \mapsto \mathrm{Im}\, i_2$*, restricts to an isomorphism of open subsets* $\mathrm{Gr}_{\mathcal{E}}(\mathbb{V}, d) \to D_+(\overline{f})$ *where* $\mathcal{E}$ *is the exact category defined before.*

In this second part, we look at a finite union of principal open subsets $D_+(\overline{h}_1) \cup \cdots \cup D_+(\overline{h}_s)$ inside the projective variety $X = V_+(f_1, \ldots, f_t)$. Let $\mathcal{E}_i$ be the full subcategory of $\Lambda - \mathrm{mod}$ of objects L such that $\mathrm{Hom}(L, \mathbb{V}_{h_i}) = 0$. In this situation, we define $\mathcal{E}$ to be the full subcategory of $\Lambda - \mathrm{mod}$ all objects L such that there exists a filtration

$$0 = L_0 \subset L_1 \subset \cdots \subset L_r = L$$

such that $L_i/L_{i-1} \in \mathcal{E}_{j_i}$ for some $j_i \in \{1, \ldots, s\}$ for all $1 \leq i \leq r$.

Lemma 0.3. *$\mathcal{E}$ is an extension-closed subcategory of $\Lambda - \mathrm{mod}$.*

Proof. Denote by $\mathrm{Filt}^{\mathrm{r}}$ the subcategory of modules L admitting a filtration as above with $L_r = L$. Given a short exact sequence $0 \to A \to B \to L \to 0$ with $A \in \mathcal{E}$ and $L \in \mathrm{Filt}^{\mathrm{r}}$ one pulls back the short exact sequence along $L_{r-1} \to L$ to a short exact sequence $0 \to A \to B' \to L_{r-1} \to 0$. Inductively, one concludes that $B' \in \mathcal{E}$. On the other hand the pullback induces an exact sequence $0 \to B' \to B \to L/L_{r-1} \to 0$ and therefore $B \in \mathcal{E}$. □

Lemma 0.4. $\mathrm{Gr}_{\mathcal{E}}(\mathbb{V}, d) = \mathrm{Gr}_{\mathcal{E}_1}(\mathbb{V}, d) \cup \cdots \cup \mathrm{Gr}_{\mathcal{E}_s}(\mathbb{V}, d)$. *In particular, this is an open subscheme of* $\mathrm{Gr}_\Lambda(\mathbb{V}, d)$.

Proof. Clearly we have the right hand side is a subset of $\mathrm{Gr}_{\mathcal{E}}(\mathbb{V}, d)$. For the other inclusion we look at a submodule U of $\mathbb{V}$ of dimension vector d in $\mathcal{E}$. If U is in none of the $\mathcal{E}_i$, then there exists non-zero morphisms $U \to \mathbb{V}_{h_i}$ for every i. In this case, since $\mathrm{Hom}_\Lambda(U, \mathbb{V}_{h_i}) = K$ it follows by the same argument used before that there is a monomorphism $U \to \mathbb{V}_{h_i}$ for every i. This implies that for every i every non-zero submodule of U is also not in $\mathcal{E}_i$ contradicting $U \in \mathcal{E}$. □

As a corollary we obtain.

Theorem 0.5. *The isomorphism* $\mathrm{Gr}_\Lambda(\mathbb{V}, d) \to j(X) \cong X, (i\colon U \subset \mathbb{V}) \mapsto \mathrm{Im}\, i_2$*, restricts to an isomorphism of open subsets* $\mathrm{Gr}_{\mathcal{E}}(\mathbb{V}, d) \to \bigcup_{a=1}^{s} D_+(\overline{h}_a)$ *where $\mathcal{E}$ is the exact category as before. In particular, every quasi-projective variety is a quiver Grassmannian for an exact category.*

But of course there are deep open questions: For which exact categories and dimension vectors and objects is a quiver Grassmannian(-functor) representable by a scheme (or variety)? For example, looking at the exact category $\mathcal{E}$ as before, do quiver Grassmannian for all dimension vectors and objects exist?

References

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