Why exact categories?

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The heart of homological algebra. Exact categories were introduced by Quillen (1972) in [38] to define algebraic K-theory of an exact category. He axiomatized properties of extension-closed subcategories in an abelian category. So, an exact category consists of an additive category together with a class of kernel-cokernel pairs called short exact sequences (or conflations). These have to satisfy axioms which ensure that equivalence classes of short exact sequences define an additive bifunctor $\mathrm{Ext}^1_{\mathcal{E}}$ called the extension functor. Using longer exact sequences one can define higher Ext-functors and show that every short exact sequence (observe the different arrows marking the short exact sequence)

$$X \rightarrowtail Y \twoheadrightarrow Z$$

gives rise to a long exact sequence in abelian groups

$$0 \to \mathrm{Hom}(V, X) \to \mathrm{Hom}(V, Y) \to \mathrm{Hom}(V, Z) \to \mathrm{Ext}^1(V, X) \to \mathrm{Ext}^1(V, Y) \to \cdots$$

For the author, this is at the heart of everything named **homological algebra**. Homological invariants are conditions on Ext-groups (e.g. certain ext-vanishing). Exact categories provide precisely the minimal set of axioms such that all classical concepts of homological algebra are defined. Many cohomology groups of interest are instances of Ext-groups in exact categories (sheaf cohomology, group cohomology, Hochschild cohomology, singular cohomology). Bühler [13] showed that every well-known diagram lemma generalizes from abelian to exact categories.

Ubiquity. Exact categories are studied in many *algebraic* contexts, for example

(*) in **algebra** as subcategories of (graded) module categories over a (graded) ring: as filtered modules over a filtered ring [35], as torsion modules [14], flat modules, Gorenstein projective modules and relative exact structures [18], [28], as almost modules [21], for more specific rings: as Cohen Macaulay modules and lattices over orders [41], [40], as modules filtered by standard modules over a quasi-hereditary algebra [16], as perpendicular categories for categories of quiver representations [22], as monomorphism categories, as semistable modules [32], as modules of finite projective dimension, as Auslander-Solberg exact structures on finitely generated modules over an artin algebra [5], [3], [4], as selforthogonal subcategories in finitely generated module categories of artin algebra (in homological conjectures [20] or in tilting theory [2]),

(*) also as representations of groups, representations of posets, representations of bocses and differential biquivers [12],

(*) in **algebraic geometry** as subcategories of quasi-coherent categories of sheaves on a scheme: as coherent sheaves, vector bundles, torsion sheaves, supported on a closed subscheme [24], also other categories of sheaves: Sheaves on sites, coherent sheaves on complex analytic spaces, flabby sheaves on a topological space [30], [31],

(*) in **functional analysis** as subcategories of locally convex spaces [15]: Banach spaces, barrelled spaces, Schwartz spaces, Frechet spaces, Montel spaces [36], [39].

Flexibility. Constructive methods for exact categories are very flexible (which admittedly makes it difficult to be systematic about it), for example:
You can filter with respect to objects, take perpendicular categories (wrt. to Ext- or Hom-functors) or intersections of exact subcategories. You can pass to exact substructures, e.g. look at an exact

substructure making a left exact functor exact. You can look at the category of all short exact sequences in an exact category, or the category of complexes in an exact category. You can look at (many) functor categories with extra properties. You can Ind-complete your exact category (completion with respect to arbitrary small filtered colimits). You can look at recollements of exact categories. And much more.

The main open problem... We know many ways of finding exact categories but then we know hardly anything about their homological algebra. When you pass to an exact substructure or an exact subcategory, everything can happen to your homological invariants. One way to prevent this is to study if the inclusion functor is *homologically exact* (i.e. induces isomorphisms on Ext-groups) - this only applies to exact subcategories. For exact substructures new ideas are needed. Or another open question: when is the global dimension preserved under Ind-completion (cf. [**37**])?

Derived/stable and singular equivalences. Despite of this ubiquity of exact categories, they are surprisingly rarely studied generally from a homological algebra point of view. The primary invariant here is of course the bounded derived category of an exact category. There are very few authors considering them, mostly because of the (sometimes) missing t-structures. Secondly the singularity category is a homological invariant, it 'annihilates' all objects of finite projective dimension in the derived category. Also Auslander's effaceable functor categories reflect some homological properties of an exact category (see next paragraph).
Therefore we ask to study **derived / singular** and **stable equivalence** for exact categories. Of course, this means first that we investigate the three associated categories. When it comes to the desired equivalences, of course, even for finite-dimensional modules over finite-dimensional algebras we only know in some special situations answers to these questions.

Auslander's ideas work for exact categories. As a vague approximation to his ideas, Auslander promotes to study module categories of artin algebras through their categories of finitely presented functors. The most famous is the category of functors represented by deflations, called effaceable functors (or Auslander defect category). Together with Idun Reiten he developed the theory of almost split exact sequences (which correspond to simple effaceable functors). Enomoto carried these ideas into the generality of exact categories [**19**].
Auslander correspondence (and Auslander formula) are telling us that the category of all finitely presented functors (on a small abelian category) can recover the abelian category. The same ideas work for small exact categories, cf. [**25**], [**17**], [**23**].
Also, in a series of papers [**6**], [**7**], [**8**], [**9**], [**10**] Auslander-Reiten started to study stable equivalence of artin algebras (and more general dualizing varieties). They investigated homological properties of effaceable functors. The interesting observation is that homological properties of an exact category are reflected in its effaceable functor category.
We think all of Auslander's work should be generalized to exact categories because it is useful for an understanding of homological properties of exact categories.

Tilting and support τ-tilting. Ideas from representation theory of finite dimensional algebras which are purely based on homological algebra generalize trivially to exact categories. This includes tilting theory and support tau tilting theory. But to understand induced derived equivalence we first need to see that we have to *replace* the 'endomorphism of a tilting module' with a certain functor category (functors represented by admissible morphisms - cf. previous paragraph).
Support τ-tilting subcategories were advertised as a mutation-completion of tilting subcategories [**1**]. For exact categories this is no longer true. The challenge is here: Find a new construction to mutation-complete support τ-tilting subcategories.

Homological conjectures, tame-wild dichotomy. Once we are restricting the study of exact categories to Krull-Schmidt categories (and possibly assuming more properties), it becomes also reasonable to ask if *Homological conjectures* for modules over artin algebras are true in these classes of exact categories (for example: Auslander and Solberg [**4**] showed that the finitistic dimension conjecture for artin algebras is equivalent to the same conjecture for Auslander-Solberg exact substructures).

By the Krull-Schmidt assumption, *finite type* means only finitely many indecomposable objects. But how *wild* can the infinite types look like? In ongoing work, Schlegel defines *finitely definable* subcategories in modules categories over an artin algebra and then conjectures a dichotomy of 'finite' or 'strongly unbounded' type (generalizing the second Brauer Thrall conjecture).
Enomoto investigated properties of exact categories of finite representation-type in [19].

Why not more general? So, why not directly study any of the following generalizations of exact categories

(1) one-sided or weak exact categories, e.g. [27], [26] or [11] (leave out some axioms)
Then we have no ext-bifunctor. Proto-exact categories - drop the assumption that the underlying category is additive. Again the ext-functor is not valued in abelian groups.
(2) extriangulated categories [34] (axiomatizing extension-closed in triangulated)
Then we do not know a sensible definition of a derived category. The axioms are long. We inherit all known problems with the axioms for triangulated categories (which have lead to a big number of different enhancements of triangulated categories). Often, the only known examples are just exact or triangulated categories.
(3) dg exact categories, model structures or other enhanced situations
(such as A_∞, or ∞-categories). This becomes quickly homotopical category theory. Even though these generalizations may encompass much of the theory of exact categories, we are loosing the simplicity and flexibility that make exact categories so appealing.
(4) n-exact categories [29] (higher homological algebra) - these make sense to be studied together with exact categories - they are exclusively occurring as cluster tilting subcategories ([33]) and then we are back in the realm of representation theoretic ideas.

But why should we leave an easy algebraic theory behind when it is so little studied and so central in homological algebra? Exact categories have the very tempting balance of impressive potential for creating abstract theory but still being simplistic enough to provide explicit examples. Even for seemingly well-studied categories (e.g. take the category of finitely generated abelian groups, do you know all its exact substructures?).

Quick overview of the project. Every chapter starts with a synopsis explaining its content and my contribution.

(I) Subcategories and functor categories
This part is on constructive methods. First, we look at the lattices of exact substructures (Chapter 2) and the much bigger lattice of exact subcategories (Chapter 3).
We study categories of functors represented by (certain) morphisms (Chapter 4) and faithfully balancedness for (the usually considered) functor categories (Chapter 5).
(II) Derived methods
We start with the definition and existence of the derived category (Chapter 6), then we look at tilting subcategories in an exact category (Chapter 7) and have a closer look at tilting subcategories for infinite quivers (Chapter 8). Then we discuss how one can find derived equivalences more generally (Chapter 9,10).
(III) Singular and stable equivalence
This part is not complete (so far we have not really addressed the title): We introduce the singularity category and the concept of a non-commutative resolution with exact substructures in Chapter 11. Then we only start the study of effaceable functors in Chapter 12.

We do not cover the following topics (but may do so in future):
Support τ-tilting, homological conjectures, tame-wild dichotomy, recollements of exact categories (a more thorough treatment of derived functors of additive functors between exact categories is required for this).

Notation and conventions for exact categories

$\mathcal{E} = (\mathcal{A}, \mathcal{S})$ being an exact category means $\mathcal{A}$ is an additive category and $\mathcal{S}$ is a collection of kernel-cokernel pairs in $\mathcal{A}$ satisfying the axioms below. We call elements in $\mathcal{S}$ **short exact sequences** (in the literature these are usually called **conflations**)[1] Given a short exact sequence $X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$, we call $i\colon X \rightarrowtail Y$ an **inflation** and $p\colon Y \twoheadrightarrow Z$ a **deflation**. The axioms of an exact category are:

(E0) all split exact sequences are in $\mathcal{S}$,
(E1) deflations are closed under composition and inflations too,
(E2) Pull backs of deflations along arbitrary morphisms exist and are again deflations. Push outs of inflations along arbitrary morphisms exist and are again inflations.

An **admissible morphism** f is one that factors as $f = j \circ p$ with p a deflation and j an inflation. Given a(n integer interval indexed) sequence of composable morphisms f_n, $n \in I$ (for an interval $I \subseteq \mathbb{Z}$ with at least two elements)

$$\cdots \to X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} X_{n+2} \to \cdots$$

we call it **exact** (or **acyclic**) at X_{n+1} if the morphisms factor as $f_n = j_n p_n$ with j_n and inflation and p_n a deflation (i.e. are admissible) and (j_n, p_{n+1}) is a short exact sequence. If we call such a sequence exact, it means exact at every inner object (here: 'inner' means not at the boundary of the interval).

$\operatorname{Ext}^1_{\mathcal{E}}(X, Y)$ is the class of all short exact sequences $Y \rightarrowtail Z \twoheadrightarrow X$ up to isomorphism of short exact sequences fixing the end terms.
For $n > 1$: $\operatorname{Ext}^n_{\mathcal{E}}(X, Y)$ is the class of all exact sequences $Y \rightarrowtail Z_1 \to \cdots \to Z_n \twoheadrightarrow X$ up to the equivalence relation generated by morphisms of n-exact sequences fixing the end terms.
$\mathcal{P}(\mathcal{E})$ (resp. $\mathcal{I}(\mathcal{E})$) are the full subcategories of projectives (resp. injectives)
$\operatorname{pd}_{\mathcal{E}} X \leq n$ means $\operatorname{Ext}^{n+1}_{\mathcal{E}}(X, -) = 0$
$\mathcal{P}^{\leq n}(\mathcal{E})$ (resp. $\mathcal{I}^{\leq n}(\mathcal{E})$) denotes the subcategory of objects of projective (resp. injective) dimension at most n
$\mathcal{P}^n(\mathcal{A}) := \mathcal{P}^{\leq n}(\operatorname{mod}_\infty \mathcal{A})$ is a special case of the former, see below for the functor category $\operatorname{mod}_\infty \mathcal{A}$
$\mathcal{P}^{<\infty}(\mathcal{E}) = \bigcup_n \mathcal{P}^{\leq n}(\mathcal{E})$ (resp. $\mathcal{I}^{<\infty}(\mathcal{E}) = \bigcup_n \mathcal{I}^{\leq n}(\mathcal{E})$)
Most common properties of subcategories: Given a short exact sequence $X \rightarrowtail Y \twoheadrightarrow Z$ in an exact category

(*) extension-closed: if X, Z are inside the subcategory then Y too
(*) inflation-closed: if X, Y are inside the subcategory then Z too
(*) deflation-closed: if Y, Z are inside the subcategory then X too

thick subcategory[2] means all 2-out-of-3-properties (see above) and closed under summands.
Serre subcategory means extension-closed and if a middle term is contained then both outer terms are as well.
A subcategory $\mathcal{G}$ is a **generator** in an exact category if for every object X, there exists a deflation $d\colon G \twoheadrightarrow X$ with G in $\mathcal{G}$.
Resolving means extension-closed, deflation-closed, summand-closed and a generator.

[1] Be aware, e.g. in [**42**], short exact sequence is a synonym for kernel cokernel pair in an additive category.
[2] we often add: 'in the exact category'- do not confuse this with thick in the triangulated sense.

Functor categories. Let $\mathcal{A}$ be a small additive category, (Ab) the category of abelian groups. For A in $\mathcal{A}$, we define $P_A \colon \mathcal{A}^{ob} \to (Ab), P_A(X) := \mathrm{Hom}_{\mathcal{A}}(X, A)$ and

$$\begin{aligned}
\mathrm{Mod}\,\mathcal{A} &= \text{ category of all additive functors } \mathcal{A}^{op} \to (Ab) \\
\mathrm{mod}\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists\, P_A \twoheadrightarrow A\} \\
\mathrm{mod}_1\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists \text{ exact seq. } P_{A_1} \to P_{A_0} \twoheadrightarrow F\} \\
\mathrm{mod}_n\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists \text{ exact seq. } P_{A_n} \to \cdots \to P_{A_0} \twoheadrightarrow F\} \\
\mathrm{mod}_\infty\,\mathcal{A} &= \{F \in \mathrm{Mod}\,\mathcal{A} \mid \exists \text{ exact seq. } \cdots \to P_{A_n} \to \cdots \to P_{A_0} \twoheadrightarrow F\}
\end{aligned}$$

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