5.17 Problem. Let X, Y be in $\mathscr{M}^{c,\mathrm{loc}}$. Then there is a unique (up to indistinguishability) adapted, continuous process of bounded variation $\langle X, Y\rangle$ satisfying $\langle X, Y\rangle_0 = 0$ a.s. P, such that $XY - \langle X, Y\rangle \in \mathscr{M}^{c,\mathrm{loc}}$. If $X = Y$, we write $\langle X\rangle = \langle X, X\rangle$, and this process is nondecreasing.

5.18 Definition. We call the process $\langle X, Y\rangle$ of Problem 5.17 the *cross-variation* of X and Y, in accordance with Definition 5.5. We call $\langle X\rangle$ the *quadratic variation* of X.

Proof of 5.17

There are sequences $\{S_n\}$, $\{T_n\}$ of stopping times such that $S_n \uparrow \infty$, $T_n \uparrow \infty$, and $X_t^{(n)} \triangleq X_{t\wedge S_n}$, $Y_t^{(n)} \triangleq Y_{t\wedge T_n}$ are $\{\mathscr{F}_t\}$-martingales. Define

$$R_n \triangleq S_n \wedge T_n \wedge \inf\{t \geq 0: |X_t| = n \quad \text{or} \quad |Y_t| = n\},$$

and set $\tilde{X}_t^{(n)} = X_{t\wedge R_n}$, $\tilde{Y}_t^{(n)} = Y_{t\wedge R_n}$. Note that $R_n \uparrow \infty$ a.s. Since $\tilde{X}_t^{(n)} = X_{t\wedge R_n}^{(n)}$, and likewise for $\tilde{Y}^{(n)}$, these processes are also $\{\mathscr{F}_t\}$-martingales (Problem 3.24), and are in $\mathscr{M}_2^c$ because they are bounded. For $m > n$, $\tilde{X}_t^{(n)} = \tilde{X}_{t\wedge R_n}^{(m)}$ and so

$$(\tilde{X}_t^{(n)})^2 - \langle \tilde{X}^{(m)}\rangle_{t\wedge R_n} = (\tilde{X}_{t\wedge R_n}^{(m)})^2 - \langle \tilde{X}^{(m)}\rangle_{t\wedge R_n}$$

is a martingale. This implies $\langle \tilde{X}^{(n)}\rangle_t = \langle \tilde{X}^{(m)}\rangle_{t\wedge R_n}$. We can thus decree $\langle X\rangle_t \triangleq \langle \tilde{X}^{(n)}\rangle_t$ whenever $t \leq R_n$ and be assured that $\langle X\rangle$ is well defined. The process $\langle X\rangle$ is adapted, continuous, and nondecreasing and satisfies $\langle X\rangle_0 = 0$ a.s. Furthermore,

$$X_{t\wedge R_n}^2 - \langle X\rangle_{t\wedge R_n} = (\tilde{X}_t^{(n)})^2 - \langle \tilde{X}^{(n)}\rangle_t$$

is a martingale for each n, so $X^2 - \langle X\rangle \in \mathscr{M}^{c,\mathrm{loc}}$. As in Theorem 5.13, we may now take $\langle X, Y\rangle = \frac{1}{4}[\langle X + Y\rangle - \langle X - Y\rangle]$.

As for the question of uniqueness, suppose both A and B satisfy the conditions required of $\langle X, Y\rangle$. Then $M \triangleq XY - A$ and $N \triangleq XY - B$ are in $\mathscr{M}^{c,\mathrm{loc}}$, so just as before we can construct a sequence $\{R_n\}$ of stopping times with $R_n \uparrow \infty$ such that $M_t^{(n)} \triangleq M_{t\wedge R_n}$ and $N_t^{(n)} \triangleq N_{t\wedge R_n}$ are in $\mathscr{M}_2^c$. Consequently $M_t^{(n)} - N_t^{(n)} = B_{t\wedge R_n} - A_{t\wedge R_n} \in \mathscr{M}_2^c$, and being of bounded variation this process must be identically zero (see the proof of Theorem 5.13). It follows that $A = B$.