Probability Theory III - Homework Assignment 1

Due date: **Friday, October 24, 12:00 h**

Solutions to the assigned homework problems must be deposited in Diana Kämpfe's drop box 84 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion.

Exercise 1.I [2 pts] (Restarting of Brownian motion at a random time)

Give an example of a random time σ with $\mathbb{P}\{0 \leq \sigma < \infty\} = 1$, s.t. with $W_t := B_{t+\sigma} - B_\sigma$, the process $W = \left\{W_t, \mathcal{F}_t^W; t \geq 0\right\}$ is *not* a Brownian motion.

Exercise 1.II [5 pts] (Tanaka's formula and local time)

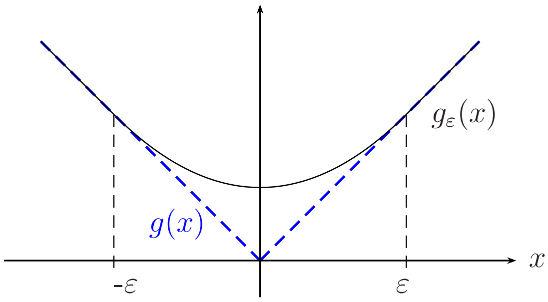
Observe that for $g \in C^2(\mathbb{R})$ it follows from Itô's formula that

$$g(B_t) = g(B_0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds, \tag{1}$$

and it can be shown by approximation arguments, that this result remains true for a larger class $\mathcal{K}$ of functions $g \in C^1(\mathbb{R})$, which are twice continuously differentiable outside of finitely many points $z_1, \dots, z_n$, with $|g''(x)| \leq M$ for some finite $M \geq 0$ and all $x \notin \{z_1, \dots, z_n\}$.

We would now like to apply Itô's formula to $g(x) = |x|$. As $g \notin C^2(\mathbb{R})$, we approximate g by functions g_ε, $\varepsilon > 0$, given by

$$g_\varepsilon(x) = \begin{cases} |x| & \text{if } |x| \geq \varepsilon \\ \frac{1}{2}\left(\varepsilon + \frac{x^2}{\varepsilon}\right) & \text{if } |x| < \varepsilon \end{cases}.$$



a) Show that $g_\varepsilon \in \mathcal{K}$ and make use of equation (1) in order to show that

$$g_\varepsilon\big(B_t\big) = g_\varepsilon\big(B_0\big) + \int_0^t g_\varepsilon'\big(B_s\big)dB_s + \frac{1}{2\varepsilon}\lambda\big\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\big\},$$

where λ denotes the Lebesgue measure.

b) Prove that

$$\int_0^t g_\varepsilon'\big(B_s\big)\mathbb{1}_{\{B_s \in (-\varepsilon,\varepsilon)\}}dB_s = \int_0^t \frac{B_s}{\varepsilon}\mathbb{1}_{\{B_s\in(-\varepsilon,\varepsilon)\}}dB_s \xrightarrow{\varepsilon\to 0} 0 \quad \text{(limit in } \mathcal{L}^2(\mathbb{P})\text{)}$$

Hint: Apply the Itô isometry to $\mathbb{E}\Big[\Big(\int_0^t \frac{B_s}{\varepsilon}\mathbb{1}_{\{B_s\in(-\varepsilon,\varepsilon)\}}dB_s\Big)^2\Big]$.

c) By letting $\varepsilon \to 0$ prove that

$$|B_t| = |B_0| + \int_0^t \operatorname{sign}\Big(B_s\Big) dB_s + L_t, \qquad (\star)$$

where $L_t : \Omega \to \mathbb{R}$, defined by

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \lambda \Big\{ s \in [0,t] \mid B_s \in (-\varepsilon, \varepsilon) \Big\} \quad \text{(limit in } \mathcal{L}^2(\mathbb{P})\text{)}.$$

and

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x < 0 \end{cases}.$$

L_t is called the *local time* for Brownian motion at 0 and $(\star)$ is called the *Tanaka formula* (for Brownian motion).

Exercise 1.III [5 pts] (Sign change and $\mathbb{P}$-a.s. non-differentiability of Brownian motion)
For a given, fixed $k > 0$, define the stopping time $\tau^{(k)} := \inf \Big\{ t \geq 0 \mid B_t > k \cdot \sqrt{t} \Big\}$.

a) Show that $\mathbb{P}\Big\{\tau^{(k)} = 0\Big\} \geq \inf_{t>0} \mathbb{P}\Big\{B_t > k \cdot \sqrt{t}\Big\}$.

b) Recall that for any positive constant c, $(B_{cs}/\sqrt{c})_{s \geq 0}$ again defines a Brownian motion, employ this rescaling property in order to express the right-hand side of the previous inequality in terms of B_1 and finally conclude that $\mathbb{P}\Big\{\tau^{(k)} = 0\Big\} > 0$.

c) Make use of Blumenthal's 0-1 law, to show that we must have $\mathbb{P}\Big\{\tau^{(k)} = 0\Big\} = 1$.

d) Choose a sequence $k_n \to \infty$ and use the previous result to prove that $\limsup_{t \to 0} B_t/\sqrt{t} = \infty$.

e) Deduce that $\liminf_{t \to 0} B_t/\sqrt{t} = -\infty$ and conclude that with probability one, a standard, one-dimensional Brownian motion changes sign infinitely many times in any given time-interval $[0, \varepsilon], \varepsilon > 0$.

f) Show that $t \mapsto B_t(\omega)$ is $\mathbb{P}$-a.s. *not* differentiable in $t = 0$.

g) Show that for any given $t_0 \geq 0$, $t \mapsto B_t(\omega)$ is $\mathbb{P}$-a.s. *not* differentiable in $t = t_0$.

Exercise 1.IV (Time-inversion of Brownian motion)
Let the process $Y = \Big\{Y_t, \mathcal{F}_t^Y; 0 \leq t < \infty\Big\}$ be given by

$$Y_t = \begin{cases} t \cdot B_{1/t} & \text{if } 0 < t < \infty \\ 0 & \text{if } t = 0 \end{cases}.$$

Show that $(Y_t)_t$ again defines a Brownian motion.
Hint: Apart from the distribution properties, one in particular has to show the $\mathbb{P}$-a.s. continuity of samples paths in $t = 0$, i.e. that $\lim_{t \to 0} t \cdot B_{1/t} = \lim_{s \to \infty} B_s/s = 0$.
(This is an oral exercise, to be prepared for a mini-presentation on Wednesday, October 29)