Universität Bielefeld

Faculty of Mathematics
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Probability Theory III - Homework Assignment 10

Due date: **Friday, January 16, 12:00 h**

Solutions to the assigned homework problems must be deposited in Christian Wiesel's drop box 55 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 10.I [4 pts]

Let $b_i(y,t)$ and $\sigma_{ij}(y,t)$, for $1 \le i \le d$, $1 \le j \le r$, be progressively measurable functionals on $\mathcal{C}([0,\infty)^d \times [0,\infty), \mathbb{R})$. The components of the diffusion matrix $a(y,t)$ are given by

$$a_{ik}(y,t) := \sum_{j=1}^{r} \sigma_{ij}(y,t)\sigma_{kj}(y,t), \qquad \text{for } 0 \le t < \infty,\ y \in \mathcal{C}([0,\infty),\mathbb{R}^d).$$

Suppose that (X,W), $(\Omega,\mathcal{F},\mathbb{P})$, $\{\mathcal{F}_t\}_t$ is a weak solution to the functional stochastic differential equation

$$\mathrm{d}X_t = b(X,t)\,\mathrm{d}t + \sigma(X,t)\,\mathrm{d}W_t, \quad \text{for } 0 \le t < \infty, \tag{1}$$

and set as the corresponding functional-valued second-order differential operator

$$\big(\mathcal{A}'_t u\big)(y) = \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(y,t)\,\frac{\partial^2 u(x)}{\partial x_i \partial x_k}\bigg|_{x=y(t)} + \sum_{i=1}^{d} b_i(y,t)\,\frac{\partial u(x)}{\partial x_i}\bigg|_{x=y(t)}, \tag{2}$$

$$\text{for } 0 \le t < \infty,\ y \in \mathcal{C}([0,\infty),\mathbb{R}^d),\ u \in \mathcal{C}^2(\mathbb{R}^d,\mathbb{R}).$$

Show that, for any functions $f, g \in \mathcal{C}^{2,1}(\mathbb{R}^d \times [0,\infty), \mathbb{R})$,

a) the stochastic process $(M_t^f)_t$, defined by

$$M_t^f := f(X_t,t) - f(X_0,0) - \int_0^t \left[\frac{\partial f}{\partial s}(X_s,s) + \mathcal{A}'_s f(X,s)\right] \mathrm{d}s, \quad \text{for } 0 \le t < \infty,$$

is in $\mathcal{M}^{\mathrm{loc},c}$;

b) the covariation process $(\langle M^f, M^g\rangle_t)_t$ is given by

$$\langle M^f, M^g\rangle_t = \sum_{i=1}^{d}\sum_{k=1}^{d}\int_0^t a_{ik}(X,s)\,\frac{\partial f(x,s)}{\partial x_i}\bigg|_{x=X_s}\frac{\partial g(x,s)}{\partial x_k}\bigg|_{x=X_s}\mathrm{d}s, \quad \text{for } 0 \le t < \infty;$$

c) if in addition, the first derivatives of f are bounded, and for each $0 < T < \infty$

$$\|\sigma(y,t)\| \le K_T, \quad \text{for } 0 \le t \le T,\ y \in \mathcal{C}([0,\infty),\mathbb{R}^d), \tag{3}$$

holds, where K_T is a constant depending on T, then $(M_t^f)_t \in \mathcal{M}_2^c$.

Exercise 10.II [4 pts]

A continuous, $(\mathcal{F}_t)_t$-adapted process $(W_t)_t$ is a d-dim. Brownian motion if and only if $(N_t^f)_t$, defined by

$$N_t^f := f(W_t) - f(W_0) - \frac{1}{2}\int_0^t \Delta f(W_s)\,\mathrm{d}s, \quad \text{for } 0 \le t < \infty,$$

is in $\mathcal{M}^{\mathsf{loc},c}$ for every $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$.

Exercise 10.III [4 pts]

Assume either

(A.1)′ For each $0 < T < \infty$, $\|b(y,t)\| + \|\sigma_{ij}(y,t)\| \le K_T$ for $0 \le t \le T,\ y \in \mathcal{C}([0,\infty), \mathbb{R}^d)$, where K_T is a constant depending on T, or

(A.2)′ Each $b_i(y,t)$ and $\sigma_{ij}(y,t)$ are of the form $b_i(y,t) = \hat{b}_i(y(t),t)$, $\sigma_{ij}(y,t) = \hat{\sigma}_{ij}(y(t),t)$, where the Borel-measurable functions $\hat{b}_i, \hat{\sigma}_{ij}(y,t)\colon\ \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ are bounded on compact sets.

Let $\mathbb{P}$ be a probability measure on $(\mathcal{C}([0,\infty), \mathbb{R}^d), \mathscr{B}(\ldots))$; and choose $(\mathcal{F}_t)_t$ as in the definition of *solution to the local martingale problem*. Assume $f \in \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R})$. Show that if $(M_t^f)_t$ is a $(\mathscr{B}_t)_t$-martingale, then is $(M_t^f)_t$ a $(\mathcal{F}_t)_t$-martingale.

Exercise 10.IV

Prepare a mini-presentation for the tutorial on Wednesday, January 21, on the proof of Proposition 5.4.11:

Given progressively measurable functionals $b_i(y,t)$ and $\sigma_{ij}(y,t)$, for $1 \le i \le d$, $1 \le j \le r$, the associated family of operators $\{\mathcal{A}_t'\}$ (given by (2)), and a probability measure μ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, we can consider the following three conditions:

(A) There exists a *weak solution* to the functional stochastic differential equation (1) with initial distribution μ.

(B) There exists a solution $\mathbb{P}$ to the *local martingale problem* associated with $\{\hat{\mathcal{A}}_t\}$, and $\mathbb{P}\{y(0) \in \Gamma\} = \mu(\Gamma)$ for all $\Gamma \in \mathscr{B}(\mathbb{R}^d)$.

(C) There exists a solution $\mathbb{P}$ to the *martingale problem* associated with $\{\hat{\mathcal{A}}_t\}$, and $\mathbb{P}\{y(0) \in \Gamma\} = \mu(\Gamma)$ for all $\Gamma \in \mathscr{B}(\mathbb{R}^d)$.

Proposition 5.4.11: Conditions (A) and (B) are equivalent and are implied by (C). Furthermore, (A) implies (C) under either of the additional assumptions:

(A.1) For each $0 < T < \infty$, condition (3) holds.

(A.2) Each $\sigma_{ij}(y,t)$ is of the form $\sigma_{ij}(y,t) = \hat{\sigma}_{ij}(y(t),t)$, where the Borel-measurable functions $\hat{\sigma}_{ij}(y,t)\colon\ \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ are bounded on compact sets.