Universität Bielefeld

Faculty of Mathematics
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Probability Theory III - Homework Assignment 11

Due date: **Friday, January 23, 12:00 h**

Solutions to the assigned homework problems must be deposited in Christian Wiesel's drop box 55 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 11.I [4 pts]

Assume that the coefficients $b_i\colon \mathbb{R}^d \to \mathbb{R}$, $\sigma_{ij}\colon \mathbb{R}^d \to \mathbb{R}$, for $1 \le i \le d$, $1 \le j \le r$, are measurable and bounded on compact subsets of $\mathbb{R}^d$, and let $\mathcal{A}$ be the associated operator

$$\big(\mathcal{A}f\big)(x) = \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(x)\frac{\partial^2 f(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{d} b_i(x)\frac{\partial f(x)}{\partial x_i}, \quad \text{for } x \in \mathbb{R}^d,\ f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}).$$

Let $(X_t)_{t\ge 0}$ be a continuous, $(\mathcal{F}_t)_{t\ge 0}$-adapted stochastic process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that $(\mathcal{F}_t)_t$ satisfies the usual conditions. For $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ and $\alpha \in \mathbb{R}$, introduce the $(\mathcal{F}_t)_{t\ge 0}$-adapted stochastic processes $(M_t)_{t\ge 0}$ and $(\Lambda_t)_{t\ge 0}$ given by

$$M_t := f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\,\mathrm{d}s, \quad \text{for } 0 \le t < \infty,$$
$$\Lambda_t := e^{-\alpha t} f(X_t) - f(X_0) + \int_0^t e^{-\alpha s}\big(\alpha f - \mathcal{A}f\big)(X_s)\,\mathrm{d}s, \quad \text{for } 0 \le t < \infty.$$

Show the following statements:

a) $(M_t)_t \in \mathcal{M}^{\text{loc},c}$ if and only if $(\Lambda_t)_t \in \mathcal{M}^{\text{loc},c}$.

b) If moreover, f is bounded away from zero on compact sets. Then the two conditions of a) are also equivalent to: $(N_t)_t \in \mathcal{M}^{\text{loc},c}$, where $(N_t)_{t\ge 0}$ is a $(\mathcal{F}_t)_{t\ge 0}$-adapted stochastic process given by

$$N_t := f(X_t)\exp\left\{-\int_0^t \frac{\mathcal{A}f(X_s)}{f(X_s)}\,\mathrm{d}s\right\} - f(X_0), \quad \text{for } 0 \le t < \infty.$$

Hint: Recall from Lemma 3.3.12 (Partial Integration of continuous Semimartingales) that if $(M_t)_t \in \mathcal{M}^{loc,c}$ and $(C_t)_{t\ge 0}$ is a continuous stochastic process of bounded variation, the $C_t M_t - \int_0^t M_s\,\mathrm{d}C_s = \int_0^t C_s\,\mathrm{d}M_s$ is in $\mathcal{M}^{loc,c}$.

Exercise 11.II [4 pts]

Let the coefficients b, σ be bounded on compact subsets of $\mathbb{R}^d$, and assume that for each $x \in \mathbb{R}^d$, the time-homogeneous martingale problem has a solution $\mathbb{P}^x$ satisfying

$$\mathbb{P}^x\Big(y \in \mathcal{C}([0,\infty), \mathbb{R}^d); y(0) = x\Big) = 1.$$

Suppose that there exists a function $f\colon \mathbb{R}^d \to [0,\infty)$ of class $\mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ such that

$$\mathcal{A}f(x) + \lambda f(x) \le c, \quad \text{for all } x \in \mathbb{R}^d$$

holds for some $\lambda > 0$, $c \geq 0$. Show that then

$$\mathbb{E}^x f(y(t)) \leq f(x)e^{-\lambda t} + \frac{c}{\lambda}(1 - e^{-\lambda t}), \quad \text{for } 0 \leq t < \infty,\ x \in \mathbb{R}^d.$$

Exercise 11.III (Komogorov's forward equation) [4 pts]
Let $(X_t)_{t\geq 0}$ be an Itô diffusion in $\mathbb{R}^d$ with (infinitesimal) generator

$$Af(y) = \sum_{i=1}^d b_i(y)\frac{\partial}{\partial y_i}f(y) + \frac{1}{2}\sum_{i,j=1}^d a_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j}f(y), \quad \text{for } f \in \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R}),$$

and assume that the transition measure of $(X_t)_t$ has *density* $p_t(x, y)$, i.e. that

$$\mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} f(y)p_t(x, y)\,\mathrm{d}y, \quad \text{for } f \in \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R}). \tag{1}$$

Assume that $y \mapsto p_t(x, y)$ is smooth for each $0 \leq t < \infty$, $x \in \mathbb{R}^d$. Prove that $p_t(x, y)$ satisfies the *Kolomogorov forward equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(x, y) = A_y^\star p_t(x, y) \quad \text{for all } x, y \in \mathbb{R}^d,$$

where $A_y^\star$ operates on the variable y and is given by

$$A_y^\star \phi(y) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial y_i \partial y_j}\Big(a_{ij}(y)\phi(y)\Big) - \sum_{i=1}^d \frac{\partial}{\partial y_i}\Big(b_i(y)\phi(y)\Big), \quad \text{for } \phi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}),$$

(i.e. $A_y^\star$ is the *adjoint* of A_y.)
Hint: By (**??**) and Dynkin's formula we have

$$\int_{\mathbb{R}^d} f(y)p_t(x, y)\,\mathrm{d}y = f(x) + \int_0^t \int_{\mathbb{R}^d} A_y f(y)p_s(x, y)\,\mathrm{d}y\,\mathrm{d}s, \quad \text{for } f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}).$$

Now differentiate w.r.t. t and use that

$$\langle A\phi, \psi\rangle = \langle \phi, A^\star \psi\rangle, \quad \text{for } \phi \in \mathcal{C}_0^2(\mathbb{R}^d, \mathbb{R}),\ \psi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ denotes inner product in $L^2(\mathrm{d}y)$.

Exercise 11.IV
Prepare a mini-presentation for the tutorial on Wednesday, January 28, on d-dim. stochastic differential equation in which the solution process enters linearly. :

In this exercise we consider the following *linear* stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t &= [\mathbb{A}(t)X_t + a(t)]\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t, \quad \text{for } 0 \leq t < \infty \\ X_0 &= \xi, \quad \text{a.s.,} \end{aligned} \tag{2}$$

where $(W_t)_t$ is an r-dim. Brownian motion independent of the d-dim. vector ξ, and the matrices $\mathbb{A}(t) \in \mathbb{R}^{d\times d}$, $a(t) \in \mathbb{R}^{d\times 1}$ and $\sigma(t) \in \mathbb{R}^{d\times r}$ are nonrandom, measurable and locally bounded. The deterministic equation corresponding to (**??**) is

$$\dot{\xi}(t) = \mathbb{A}(t)\xi(t) + a(t), \quad \xi(0) = \xi. \tag{3}$$

Standard existence ans uniqueness results[1] imply that for every initial condition $\xi \in \mathbb{R}^d$, (**??**) has an absolutely continuous solution $\xi(t)$ defined for $0 \le t < \infty$. Likewise, the matrix differential equation

$$\dot{\mathbb{U}}(t) = \mathbb{A}(t)\mathbb{U}(t), \quad \mathbb{U}(0) =_{d\times d},$$

has a unique (absolutely continuous) solution defined for $0 \le t < \infty$. (Here $_{d\times d}$ is the identity matrix in $\mathbb{R}^{d\times d}$.) This matrix function $\mathbb{U}$ is called *fundamental solution* to the homogeneous equation

$$\dot{\xi}(t) = \mathbb{A}(t)\xi(t).$$

In terms of $\mathbb{U}$, the solution of (**??**) is[2] $\xi(t) = \mathbb{U}(t)\left\{\xi(0) + \int_0^t \mathbb{U}^{-1}(s)a(s)\,\mathrm{d}s\right\}$.

a) Show that
   i) for each $t \le 0$, the matrix $\mathbb{U}(t)$ is nonsingular;
   ii) $X_t = \mathbb{U}(t)\left\{X_0 + \int_0^t \mathbb{U}^{-1}(s)a(s)\,\mathrm{d}s + \int_0^t \mathbb{U}^{-1}(s)\sigma(s)\,\mathrm{d}W_s\right\}$ solves (**??**);
   iii) pathwise uniqueness for equation (**??**) holds.
b) Suppose that $\mathbb{E}\,\|X_0\|^2 < \infty$, and introduce the *mean vector* and the *covariance matrix* functions

$$m(t) := \mathbb{E}\,X_t, \quad \varrho(s,t) := \mathbb{E}[(X_s - m(s))(X_t - m(t))^T], \quad V(t) := \varrho(t,t).$$

Show that

$$m(t) = \mathbb{U}(t)\left\{m(0) + \int_0^t \mathbb{U}^{-1}(s)a(s)\,\mathrm{d}s\right\},$$
$$\varrho(s,t) = \mathbb{U}(s)\left\{V(0) + \int_0^{s\wedge t} \mathbb{U}^{-1}(u)\sigma(u)[\mathbb{U}^{-1}(u)\sigma(u)]^T\,\mathrm{d}u\right\}\mathbb{U}^T(t),$$

hold for every $0 \le s,t < \infty$. In particular, $m(t)$ and $V(t)$ solve the linear equations

$$\dot{m}(t) = \mathbb{A}(t)m(t) + a(t),$$
$$\dot{V}(t) = \mathbb{A}(t)V(t) + V(t)\mathbb{A}^T(t) + \sigma(t)\sigma^T(t).$$

[1] Hale, J. (1969) *Ordinary Differential Equations.* J. Wiley&Sons/Interscience, New York; Section I.5
[2] Hale, J. (1969) *Ordinary Differential Equations.* J. Wiley&Sons/Interscience, New York; Chapter 3