Universität Bielefeld
Faculty of Mathematics
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Probability Theory III - Homework Assignment 12

Due date: **Friday, January 30, 12:00 h**

Solutions to the assigned homework problems must be deposited in Christian Wiesel's drop box 55 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 12.I (Doob's h-transform) [4 pts]

Let $(B_t)_{t\geq 0}$ be a d-dim. Brownian motion, $\mathcal{D}\subset\mathbb{R}^d$ a bounded open set and $\mathcal{C}^2(\mathcal{D})\ni h>0$ a harmonic function on $\mathcal{D}$ (i.e. $\Delta h=0$ in $\mathcal{D}$). Let $(X_t)_{t\geq 0}$ be the solution of the stochastic differential equation

$$\mathrm{d}X_t=\nabla(\ln h)(X_t)\,\mathrm{d}t+\mathrm{d}B_t,$$

defined in the following way: Choose an increasing sequence $(\mathcal{D}_k)_{k\in\mathbb{N}}$ of open subsets of $\mathcal{D}$ such that $\overline{\mathcal{D}}_k\subset\mathcal{D}$ and $\bigcup_{k=1}^\infty\mathcal{D}_k=\mathcal{D}$. Then for each k the equation above can be solved (strongly) for $t<\tau_{\mathcal{D}_k}$. This gives in a natural way a solution for $t<\tau_{\mathcal{D}}:=\lim_{k\to\infty}\tau_{\mathcal{D}_k}$.

a) Show that the characteristic generator $\mathcal{A}$ of X_t satisfies

$$\mathcal{A}f=\frac{\Delta(hf)}{2h},\qquad \text{for } f\in\mathcal{C}_0^2(\mathcal{D}).$$

In particular, if $f=1/h$ then $\mathcal{A}f=0$ holds.

b) Use a) to show that if there exists $x_0\in\partial\mathcal{D}$ such that

$$\lim_{x\to y\in\partial\mathcal{D}}h(x)=\begin{cases}0 & \text{if } y\neq x_0,\\ \infty & \text{if } y=x_0,\end{cases}\qquad \text{(i.e. } h \text{ is a \textit{kernel fucntion})},$$

then $\lim\limits_{t\to\tau}X_t=x_0$ a.s.

Hint: Consider $\mathbb{E}^x[f(X_T)]$ for suitable stopping times T and with $f=1/h$.

In other words, we have imposed a drift on B_t which causes the process to exit from $\mathcal{D}$ at the point x_0 only. This can also be formulated as follows: X_t is obtained by *conditioning B_t to exit from $\mathcal{D}$ at x_0*. See Doob (1984).

Exercise 12.II [2 pts]

Let α, β be some constants. In each of the cases below find an Itô diffusion whose generator coincides with L on $\mathcal{C}_0^2(\mathcal{D})$:

a) $Lf(t,x)=\alpha\dfrac{\partial f(t,x)}{\partial t}+\dfrac{\beta^2}{2}\dfrac{\partial^2 f(t,x)}{\partial x^2}$, for $\mathcal{D}=\mathbb{R}^2$;

b) $Lf(x_1,x_2)=\alpha\dfrac{\partial f(x_1,x_2)}{\partial x_1}+\beta\dfrac{\partial f(x_1,x_2)}{\partial x_2}+\dfrac{1}{2}\left(\dfrac{\partial^2 f(x_1,x_2)}{\partial x_1^2}+\dfrac{\partial^2 f(x_1,x_2)}{\partial x_2^2}\right)$, for $\mathcal{D}=\mathbb{R}^2$;

c) $Lf(x) = \alpha x f'(x) + \frac{\beta^2}{2} f''(x)$, for $\mathcal{D} = \mathbb{R}$;

d) $Lf(x) = \alpha f'(x) + \frac{\beta^2}{2} x^2 f''(x)$, for $\mathcal{D} = \mathbb{R}$.

Exercise 12.III [4 pts]

Given the following generator

$$Lu(x) := rxu'(x) + \frac{1}{2}\alpha^2 x^2 u''(x), \qquad \text{for } x \in \mathbb{R},\ u \in \mathcal{C}_0^2(\mathbb{R}),$$

where r, α are given constants with $r \geq \alpha^2/2$.

a) Find an Itô diffusion $(X_t)_{t\geq 0}$ whose characteristic generator $\mathcal{A}$ coincides with L on $\mathcal{C}_0^2(\mathbb{R})$.

b) Solve the SDE from a), i.e. find a explicit representation for X_t.
Hint: Use the Itô formula for the function $g(x) = \ln(x)$.

c) Use the result of the lecture for the regular Dirichlet problem in order to find a solution to the following *boundary valued problem* on $\mathcal{D} = (0, x_0)$

$$\begin{cases} Lu(x) = 0, & 0 < x < x_0, \\ u(x_0) = x_0^2, \end{cases}$$

for a given $x_0 > 0$.

Exercise 12.IV [2 pts]

Define the following generator

$$Lw(t,x) := \frac{\partial w(t,x)}{\partial t} + \frac{\partial^2 w(t,x)}{\partial x^2}, \qquad \text{for } (t,x) \in \mathbb{R}^2,\ w \in \mathcal{C}_0^2(\mathbb{R}^2).$$

a) Find an Itô diffusion whose characteristic generator $\mathcal{A}$ coincides with L on $\mathcal{C}_0^2(\mathbb{R}^2)$.

b) Define $\mathcal{D} = (0,T) \times \mathbb{R}$ and use the result of the lecture for the regular combined Dirichlet-Poisson problem in order to find a solution to the following *boundary valued problem* on

$$\begin{cases} Lw(t,x) = -e^{\varrho t} g(x), & 0 < t < T,\ x \in \mathbb{R}, \\ w(T,x) = \phi(x), & x \in \mathbb{R}, \end{cases}$$

where g, ϕ are given bounded, continuous functions and ϱ is an arbitrary constant.

Exercise 12.V

Prepare a mini-presentation for the tutorial on Wednesday, February 4, on the proof of Lemma 7.4 in Chapter 5.7 of the book *Karatzas, I. & Shreve, S. E. (2010)* Brownian Motion and Stochastic Calculus. *2nd edition, Springer, New York*.