Universität Bielefeld

Faculty of Mathematics
Prof. Dr. Barbara Gentz
Diana Kämpfe, Christian Wiesel WS 2014/2015

Probability Theory III - Homework Assignment 2

Due date: **Friday, October 31, 12:00 h**

Solutions to the assigned homework problems must be deposited in Diana Kämpfe's drop box 84 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Let $W = (W_t)_{t\geq 0}$ be a d-dimensional Brownian motion starting $\mathbb{P}^x$-a.s. in $x \in \mathbb{R}^d$ and $\|x\|$ the Euclidean norm in $\mathbb{R}^d$.

Exercise 2.I [6 pts] (Recurrence and transience of Brownian motion)

a) Apply the multidimensional Itô rule in order to show that for $f \in C^2(\mathbb{R}^d, \mathbb{R})$ and any initial point $x \in \mathbb{R}^d$ we have:

$$f(W_t) = f(x) + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(W_s) dW_s^{(i)} + \frac{1}{2} \int_0^t \Delta f(W_s) ds, \ \mathbb{P}^x\text{-a.s.} \tag{1}$$

where $\Delta := \sum_{i=1}^d \partial^2/\partial x_i^2$ denotes the d-dimensional Laplace operator.

b) Given the previous setup and the additional assumption of f having a *compact support* ($f \in C_0^2(\mathbb{R}^d, \mathbb{R})$), conclude that for any integrable stopping time τ, i.e. $\mathbb{E}^x(\tau) < \infty$, we have

$$\mathbb{E}^x \left(f(W_\tau) \right) = f(x) + \frac{1}{2} \mathbb{E}^x \left(\int_0^\tau \Delta f(W_s) ds \right). \tag{2}$$

Hint: First look at the *finite* stopping times $\tau^k := \tau \wedge k$ and afterward employ the Itô isometry in order to show convergence in the limit of $k \to \infty$.

c) Argue that for the special case of an integrable stopping time τ satisfying $\tau \leq \tau_D$, with $\tau_D := \inf\{t > 0 \mid W_t \notin D\}$ being the first exit time of $(W_t)_t$ from a *bounded* set D, equation (2) holds true even if f does not have a compact support.

d) Given a fixed radius $\|x\| < R < \infty$, we want to study $\tau_R := \inf\{t > 0 \mid W_t \notin K_R\}$, the first *exit* time of $(W_t)_t$ from the ball $K_R := \left\{ y \in \mathbb{R}^d : \|y\| < R \right\}$. Show that

$$\mathbb{E}^x (\tau_R) = \frac{1}{d} (R^2 - \|x\|^2) < \infty. \tag{3}$$

Hint: Make use of equation (2) with $f(x) := \|x\|^2$ and $\sigma_R^{(k)} := \tau_R \wedge k$ and first deduce that $\mathbb{E}\left(\sigma_R^{(k)}\right) \leq \frac{1}{d}(R^2 - \|x\|^2)$, before then taking the limit $k \to \infty$.

e) Given a fixed radius $0 < R < \|x\|$, we want to estimate $T_R := \inf\{t > 0 \mid W_t \in K_R\}$, the first *hitting* time, and show that

$$\mathbb{P}^x \{T_R < \infty\} = \begin{cases} 1 & \text{if } d \leq 2 \\ \left(\frac{\|x\|}{R}\right)^{2-d} & \text{if } d > 2 \end{cases}, \tag{4}$$

from which it follows that Brownian motion is *recurrent* in dimension $d \in \{1, 2\}$ and *transient* in all higher dimensions $d > 2$.
In order to prove statement (4), look at the first *exit* times $\alpha_k := \inf\{t > 0 \mid W_t \notin A_k\}$ from the annulus $A_k := \left\{y \in \mathbb{R}^d : R < \|y\| < 2^k R\right\}$ and employ equation (2), using a *harmonic* function f, defined (depending on the dimension d) by

$$f(x) := \begin{cases} x & \text{if } d = 1 \\ -\ln(\|x\|) & \text{if } d = 2 \\ \|x\|^{2-d} & \text{if } d > 2 \end{cases}.$$

Deduce from the resulting equation the value of $\lim_{k\to\infty} p_k$, where $p_k := \mathbb{P}^x\{\|W_{\alpha_k}\| = R\}$ is the probability that the Brownian motion leaves the annulus at the 'inner' boundary, i.e. by hitting K_R.

f) Generalize equation (4) to the case of first hitting a ball $K_R(a) := \{y \in \mathbb{R} : \|y - a\| < R\}$ centered at a point $a \in \mathbb{R}^d$.

Exercise 2.II [6 pts] (Harmonic measure of BM on a sphere and the mean-value property)
Recall that by definition of the shift-operator $\theta_s : \Omega \to \Omega$, we have $X_{t+s}(\omega) = X_t(\theta_s\omega)$. For a finite random time σ we define a *random shift* θ_σ, by setting $\theta_\sigma := \theta_s$ on $\{\sigma = s\}$.

a) Show that for any $F \in \mathcal{F}_\infty^W$, any optional time σ and all $x \in \mathbb{R}^d$ we have

$$\mathbb{P}^x\left[\theta_\sigma^{-1}F \mid \mathcal{F}_{\sigma+}\right] = \mathbb{P}^{W_\sigma}(F),\ \mathbb{P}^x\text{-a.s. on } \{\sigma < \infty\}.$$

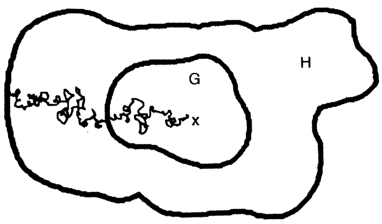
b) Conclude that for any bounded, $\mathcal{F}_\infty^W$-measurable random variable Y, for all $x \in \mathbb{R}^d$ and any optional time σ we have

$$\mathbb{E}^x\left[Y \circ \theta_\sigma \mid \mathcal{F}_{\sigma+}\right] = \mathbb{E}^{W_\sigma}[Y],\ \mathbb{P}^x\text{-a.s. on } \{\sigma < \infty\}.$$

c) Deduce that for any bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$, any finite optional time τ, for all $x \in \mathbb{R}^d$ and any optional time σ we have

$$\mathbb{E}^x\left[f(W_\tau) \circ \theta_\sigma \mid \mathcal{F}_{\sigma+}\right] = \mathbb{E}^{W_\sigma}\left[f(W_\tau)\right],\ \mathbb{P}^x\text{-a.s. on } \{\sigma < \infty\}.$$

Given $x \in G \subset H \subset \mathbb{R}^d$, where G and H are bounded, open sets, we now consider the special case of $\tau := \tau_H$, $\sigma := \tau_G$, where $\tau_H := \inf\{t > 0 \mid W_t \notin H\}$ denotes the first exit time from the set H and τ_G analogously the first exit time from G.



Sample path of Brownian motion exiting first G and afterward H

i) Show that for any bounded and measurable $f : \mathbb{R}^d \to \mathbb{R}$ we have $f(W_{\tau_H}) \circ \theta_{\tau_G} = f(W_{\tau_H})$ $\mathbb{P}^x$-a.s. and conclude that

$$\phi(x) := \mathbb{E}^x \left[f(W_{\tau_H}) \right] = \mathbb{E}^x \left[\mathbb{E}^{W_{\tau_G}} \left[f(W_{\tau_H}) \right] \right] \tag{5}$$

ii) The *harmonic measure* μ_G^x on the boundary ∂G is defined by $\mu_G^x(F) := \mathbb{P}^x \{W_{\tau_G} \in F\}$ for $F \subset \partial G$ measurable. Show that the function $\phi : \mathbb{R}^d \to \mathbb{R}$ defined in equation (5) satisfies the *mean value property* with respect to the harmonic measure

$$\phi(x) = \int_{\partial G} \phi(y) \mu_G^x(dy).$$

iii) We finally specialize to the case of $G := K_R(x) := \{y \in \mathbb{R} \mid \|y - a\| < R\} \subset W$ being a ball centered at the starting point x. Use the rotational invariance of Brownian motion (cf. exercise 2.III below) to show that the harmonic measure μ_G^x now coincides with the *normalized surface measure* σ on ∂G. We thus recover the classical mean-value property

$$\phi(x) = \int_{\partial K_R(x)} \phi(y) \sigma(dy).$$

Exercise 2.III (Rotational invariance of Brownian motion)
Let $(W_t)_t$ be a d-dimensional Brownian motion starting at $x = 0$ and let $Q \in \mathbb{R}^{d \times d}$ be an orthogonal matrix $(Q^T = Q^{-1})$. Show that $\tilde{W}_t := QW_t$ is also a d-dimensional Brownian motion.
(This is an oral exercise, to be prepared for a mini-presentation on Wednesday, November 5)