Universität Bielefeld

Faculty of Mathematics
Prof. Dr. Barbara Gentz
Diana Kämpfe, Christian Wiesel WS 2014/2015

# Probability Theory III - Homework Assignment 3

Due date: **Friday, November 7, 12:00 h**

*Solutions to the assigned homework problems must be deposited in Diana Kämpfe's drop box 84 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.*

**Exercise 3.I** [4 pts]

Show that for each $d \geq 2$, the Bessel family with dimension $d$ is a strong Markov family (where we modify the definition of a strong Markov family to account for the state space $[0, \infty)$).

**Exercise 3.II** [4 pts] (Example of a local martingale which is *not* a martingale)

Let $(R_t)_{t\geq 0}$ be a Bessel process with dimension $d \geq 3$, starting at $r = 0$. Show that $(M_t)_{t\geq 1}$ given by $M_t := 1/R_t^{d-2}$ for all $1 \leq t < \infty$

a) is a local martingale;

b) satisfies $\sup_{1\leq t<\infty} \mathbb{E}(M_t^p) < \infty$ for every $0 < p < d/(d-2)$ (and is thus uniformly integrable);

c) is *not* a martingale.

**Exercise 3.III** [4 pts] (Exponential martingales)

Let $(U_t)_{t\geq 0}$ be a *bounded*, one-dimensional stochastic process and $(W_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion. Assume that both processes are adapted to a given filtration $(\mathcal{F}_t)_{t\geq 0}$.

a) Show that the process $(M_t)_{t\geq 0}$ defined by

$$M_t := \exp\left\{-\frac{1}{2}\int_0^t U_s^2 ds - \int_0^t U_s dW_s\right\} \tag{1}$$

satisfies the (stochastic differential) equation

$$dM_t = -M_t U_t \, dW_t,$$

i.e.

$$M_t = M_0 - \int_0^t M_s U_s \, dW_s,$$

and conclude that $(M_t)_t$ is an $(\mathcal{F}_t)_t$-martingale.

b) Consider the process

$$dX_t = U_t dt + dW_t,$$

i.e.

$$X_t := X_0 + \int_0^t U_s ds + W_t.$$

It can be shown that $(X_t)_{t\geq 0}$ is *not* an $(\mathcal{F}_t)_t$-martingale, unless $U_s(\omega) = 0$ for almost all $(s,\omega) \in [0,\infty) \times \Omega$. However it turns out that the martingale property can be recaptured by multiplying $(X_t)_t$ by a suitable exponential martingale. More precisely define

$$Y_t := X_t M_t,$$

where $(M_t)_t$ is defined as in equation $(1)$. Show that $(Y_t)_{t\geq 0}$ is indeed an $(\mathcal{F}_t)_t$-martingale.

*Remark:*
This result is a special case of the *Girsanov theorem*, which in this context roughly states that one can 'transform away' the 'drift term' $U_t dt$ (which we would like to get rid of, since it destroyed the martingale property) by changing the underlying probability measure from $\mathbb{P}$ into a 'martingale measure' $Q$ defined for every $T > 0$ on $\mathcal{F}_T$ by $dQ := M_T d\mathbb{P}$.

**Exercise 3.IV**

Let $(R_t)_{t\geq 0}$ be a Bessel process with dimension $d \geq 2$ starting at $r \geq 0$. Show that for $d = 2$ the limit $\lim_{t\to\infty} R_t$ does $\mathbb{P}$-a.s. *not* exist, while for $d \geq 3$ we have $\mathbb{P}[\lim_{t\to\infty} R_t = \infty] = 1$.
*(This is an oral exercise, to be prepared for a mini-presentation on Wednesday, November 12)*