Universität Bielefeld
Faculty of Mathematics
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Probability Theory III - Homework Assignment 4

Due date: **Friday, November 14, 12:00 h**

Solutions to the assigned homework problems must be deposited in Diana Kämpfe's drop box 84 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 4.I [4 pts]

Let $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous local martingale, i.e. $M \in \mathcal{M}^{c,loc}$, and assume that its quadratic variation process $\langle M \rangle$ is integrable: $\mathbb{E}(\langle M \rangle_\infty) < \infty$. Show that

a) $M \in \mathcal{M}^2$;

b) M and the submartingale M^2 are both uniformly integrable; in particular, $M_\infty := \lim_{t\to\infty} M_t$ exists $\mathbb{P}$-a.s. and $\mathbb{E}(M_\infty^2) = \mathbb{E}(\langle M \rangle_\infty)$;

c) there is a right-continuous modification of $Z_t := \mathbb{E}(M_\infty^2 \mid \mathcal{F}_t) - M_t^2$; $t \ge 0$, which is a *potential*, i.e. a nonnegative supermartingale with $\lim_{t\to\infty} \mathbb{E}(Z_t) = 0$.

Exercise 4.II (Brownian motion on the unit circle)[4 pts]

For a given *one-dimensional* Brownian motion $(B_t)_{t\ge0}$ define the $\mathbb{R}^2$-valued stochastic process $(Y_t)_{t\ge0}$ by

$$Y_t := \begin{pmatrix} \cos(B_t) \\ \sin(B_t) \end{pmatrix}.$$

a) Show that $(Y_t)_t$ satisfies the (stochastic differential) equation

$$\begin{aligned} dY_t^{(1)} &= -\tfrac{1}{2} Y_t^{(1)} dt - Y_t^{(2)} dB_t \\ dY_t^{(2)} &= -\tfrac{1}{2} Y_t^{(2)} dt + Y_t^{(1)} dB_t \end{aligned}.$$

b) Let the martingale $(M_t)_{t\ge0}$ be defined by

$$\begin{aligned} M_t^{(1)} &:= -\textstyle\int_0^t Y_s^{(2)} dB_s = -\int_0^t \sin(B_s) dB_s \\ M_t^{(2)} &:= +\textstyle\int_0^t Y_s^{(1)} dB_s = +\int_0^t \cos(B_s) dB_s \end{aligned}.$$

Following the steps outlined in the proof of theorem 4.2. determine explicitly the matrix-valued processes $(Z_t)_{t\ge0}$, $(Q_t)_{t\ge0}$ and $(\Lambda_t)_{t\ge0}$ associated to our given martingale $(M_t)_t$. Verify that $(Q_t)_t$ and $(\Lambda_t)_t$ are indeed progressively measurable.

c) Again following the proof of theorem 4.2. calculate the 'rotated' process $(N_t)_{t\ge0}$ as well as the Brownian motion $(W_t)_{t\ge0}$. Check that the representation of M in terms of N and W given in this proof reproduces the correct result.

d) Argue that remark 4.3. applies to our setup.

Exercise 4.III [4 pts]

Let $\left\{M = (M_t^{(1)}, ..., M_t^{(d)}), \mathcal{F}_t; 0 \le t < \infty\right\}$ be a d-dim vector of continuous local martingales, and define

$$A^{(i,j)} := \left\langle M^{(i)}, M^{(j)} \right\rangle \quad \text{for } 1 \le i, j \le d,$$

$$A_t(\omega) := \sum_{i=1}^{d} \sum_{j=1}^{d} \tilde{A}_t^{(i,j)}(\omega),$$

where $\tilde{A}_t^{(i,j)}(\omega)$ denotes the total variation of $A^{(i,j)}(\omega)$ on $[0, t]$. Let $T_s(\omega)$ be the inverse of the function $A_t(\omega) + t$, i.e. $A_{T_s(\omega)} + T_s(\omega) = s$ for all $0 \le s < \infty$.

a) Show that for each s, T_s is a stopping time of $\{\mathcal{F}_t\}$.

b) Define $\mathcal{G}_s := \mathcal{F}_{T_s}; 0 \le s < \infty$. Show that if $\{\mathcal{F}_t\}$ satisfies the usual conditions, then $\{\mathcal{G}_s\}$ does also.

c) Define

$$N_s^{(i)} := M_{T_s}^{(i)} \quad 1 \le i \le d;\ 0 \le s < \infty.$$

Show that for each $1 \le i \le d$ we have $N^{(i)} \in \mathcal{M}^{c,loc}$, and the cross-variation $\left\langle N^{(i)}, N^{(j)} \right\rangle_s$ is $\mathbb{P}$-a.s. an absolutely continuous function of s.

Exercise 4.IV

Prepare a mini-presentation for the tutorial on Wednesday, November 19, on the topic of the *Martingale Moment Inequalities* (proposition 3.26.). It suffices to consider *case 3* within the proof, i.e. to provide the *lower bound* in the case of $1/2 < m \le 1$.