Universität Bielefeld

Faculty of Mathematics
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Probability Theory III - Homework Assignment 5

Due date: **Friday, November 21, 12:00 h**

Solutions to the assigned homework problems must be deposited in Diana Kämpfe's drop box 84 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 5.I [6 pts]

a) We cannot expect to be able to define the stochastic integral $\int_0^1 X_s \,\mathrm{d}W_s$ with respect to Brownian motion $(W_t)_{t\geq 0}$ for measurable adapted processes $(X_t)_{t\geq 0}$ which are not square integrable almost surely. Indeed, show that if

$$\mathbb{P}\left\{\int_0^t X_s^2 \,\mathrm{d}s < \infty\right\} = 1, \quad \text{for } 0 \leq t < 1 \quad \text{and} \quad E := \left\{\int_0^1 X_s^2 \,\mathrm{d}s = \infty\right\},$$

then

$$\limsup_{t\uparrow 1} \int_0^t X_s \,\mathrm{d}W_s = -\liminf_{t\uparrow 1} \int_0^t X_s \,\mathrm{d}W_s = +\infty, \quad \text{a.s. on } E.$$

b) Consider the semimartingale $X_t = x + M_t + C_t$ with $x \in \mathbb{R}$, $(M_t)_{t\geq 0} \in \mathcal{M}^{c,loc}$, $(C_t)_{t\geq 0}$ a continuous process of bounded variation, and assume that there exists a constant $\varrho > 0$ such that $|C_t| + \langle M\rangle_t \leq \varrho t$, for all $t \geq 0$ is valid almost surely. Show that for fixed $T > 0$ and sufficiently large $n \geq 1$, we have

$$\mathbb{P}\left\{\max_{0\leq t\leq T} |X_t| \geq n\right\} \leq \exp\left\{\frac{-n^2}{18\varrho T}\right\}.$$

Exercise 5.II [3 pts]

Assume the hypotheses of Theorem 3.5.1 (Grisanov's theorem) and suppose $Y = \{Y_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a measurable adapted process satisfying $\mathbb{P}\{\int_0^t Y_s^2 \,\mathrm{d}s < \infty\} = 1$, for $0 \leq T < \infty$. Under $\mathbb{P}$ we may define the Itô integral $\int_0^t Y_s \,\mathrm{d}W_s^{(i)}$, whereas under $\tilde{\mathbb{P}}_T$ we may define the Itô integral $\int_0^t Y_s \,\mathrm{d}\tilde{W}_s^{(i)}$ for $0 \leq T < \infty$. Show that for $1 \leq i \leq d$, we have

$$\int_0^t Y_s \,\mathrm{d}\tilde{W}_s^{(i)} = \int_0^t Y_s \,\mathrm{d}W_s^{(i)} - \int_0^t Y_s X_s^{(i)} \,\mathrm{d}s, \quad \text{for } 0 \leq t \leq T \quad \mathbb{P}\text{-a.s. and } \tilde{\mathbb{P}}_T\text{-a.s..}$$

Hint: Use Proposition 3.2.24.

Exercise 5.III [3 pts]

Let τ be a stopping time on the filtration $\{\mathcal{F}_t^W\}$ with $\mathbb{P}\{\tau < \infty\} = 1$. Show that a necessary and sufficient condition for the validity of the *Wald identity*

$$\mathbb{E}\left[\exp\{\mu W_\tau - \tfrac{1}{2}\mu^2\tau\}\right] = 1,$$

where μ is a given real number, is that

$$\mathbb{P}^{(\mu)}\{\tau < \infty\} = 1.$$

Show also, in particular, if $b \in \mathbb{R}$ and $\mu b < 0$, then this condition holds for the stopping time

$$\sigma_b := \inf\left\{t \geq 0 \mid W_t - \mu t = b\right\}.$$

Exercise 5.IV

(*This is an oral exercise, to be prepared for a mini-presentation on Wednesday, November 26*)

Let $S := \langle M\rangle_\infty = \lim_{t\to\infty}\langle M\rangle_t$. Show that if $\mathbb{P}\left\{S < \infty\right\} > 0$, it is still possible to define a Brownian motion $(B_t)_{t\geq 0}$ for which $M_t = B_{\langle M\rangle_t}$ holds.

Hint: The time-change $\tau(s)$ is now given as in Problem 3.4.5; assume, as you may, that the probability space has been suitably extended to support an independent Brownian motion.