Universität Bielefeld
Faculty of Mathematics
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Probability Theory III - Homework Assignment 6

Due date: **Friday, November 28, 12:00 h**

Solutions to the assigned homework problems must be deposited in Diana Kämpfe's drop box 84 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 6.I [4 pts]

Denote by

$$h(t;b,\mu) := \frac{|b|}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(b-\mu t)^2}{2t}\right\}; \qquad t>0, b\neq 0, \mu\in\mathbb{R}$$

the (possibly defective) density of the passage time T_b to the level $b\neq 0$ under the measure $\mathbb{P}^{(\mu)}$ of a Brownian motion with drift μ. Show that

$$h(\ \cdot\ ; b_1+b_2,\mu) = h(\ \cdot\ ;b_1,\mu) * h(\ \cdot\ ;b_2,\mu); \qquad b_1,b_2>0, \mu\in\mathbb{R},$$

where $*$ denotes convolution.

Exercise 6.II [4 pts]

The following exercise provides an example in which $Z(X)$ is not a martingale.
With $W=\{W_t,\mathcal{F}_t; 0\le t\le 1\}$ a Brownian motion, we define $T := \inf\{0\le t\le 1 \mid t+W_t^2=1\}$ and for $0\le t<1$ $X_t := -(2/(1-t)^2)\cdot W_t\cdot 1_{\{t\le T, t<1\}}$.

a) Prove that $\mathbb{P}[T<1]=1$, and therefore $\int_0^1 X_t^2 dt<\infty$ a.s.

b) Apply Itô's rule to the process $\{(W_t/(1-t))^2; 0\le t<1\}$ to conclude that

$$\int_0^1 X_t dW_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T \left[\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3}\right] W_t^2 dt \le -1.$$

c) Show that the exponential supermartingale $\{Z_t(X),\mathcal{F}_t; 0\le t\le 1\}$, defined by

$$Z_t(X) := \exp\left[\int_0^t X_s dW_s - \frac{1}{2}\int_0^t |X_s|^2 ds\right]$$

is not a martingale.

d) Show that for each $n\le 1$ and $\sigma_n := 1-(1/\sqrt{n})$, $\left\{Z_{t\wedge\sigma_n(X)},\mathcal{F}_t; 0\le t\le 1\right\}$ is a martingale.

Exercise 6.III [4 pts]

Suppose that $\{L_t,\mathcal{F}_t; 0\le t<\infty\}\in\mathcal{M}^{c,loc}$ is such that $Z_t := \exp\left[L_t - 1/2\left\langle L\right\rangle_t\right]$ is a martingale under $\mathbb{P}$, and define the new probability measure $\tilde{\mathbb{P}}_T(A) := \mathbb{E}(1_A \cdot Z_T); A \in \mathcal{F}_T$. Establish the following generalization of Proposition 5.4 and of the Girsanov theorem: if $M \in \mathcal{M}^{c,loc}$, then

$$\tilde{M}_t := M_t - \langle L, M \rangle_t = M_t - \int_0^t \frac{1}{Z_s} d\langle Z, M \rangle_s, \quad \mathcal{F}_t; \quad 0 \le t \le T$$

is in $\tilde{M}^{c,loc}$.
Hint: Imitate the proof of Proposition 5.4.

Exercise 6.IV

Prepare a mini-presentation for the tutorial on Wednesday, December 3, on the Remark following Corollary 5.2.