Universität Bielefeld
Faculty of Mathematics
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Probability Theory III - Homework Assignment 7

Due date: **Friday, December 5, 12:00 h**

Solutions to the assigned homework problems must be deposited in Christian Wiesel's drop box 55 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 7.I [5 pts]

i) Verify that the given processes solve the given corresponding stochastic differential equations

a) $X_t = X_0 \exp\left\{\int_0^t a(s)\,\mathrm{d}s + k(W_t - \frac{k}{2}t)\right\}$ solves $\mathrm{d}X_t = a(t)X_t\,\mathrm{d}t + kX_t\,\mathrm{d}W_t$, where $(W_t)_{t\geq 0}$ is a 1-dim. Brownian motion, k a real constant and $a\colon \mathbb{R} \to \mathbb{R}$ a continuous function with bounded variation.

b) $\mathbb{X}_t = e^{\mathbb{A}t}\mathbb{X}_0 + \int_0^t e^{\mathbb{A}(t-s)}\mathbb{K}\,\mathrm{d}\mathbb{W}_s$ solves $\mathrm{d}\mathbb{X}_t = \mathbb{A}\mathbb{X}_t\,\mathrm{d}t + \mathbb{K}\,\mathrm{d}\mathbb{W}_t$, where $(\mathbb{W}_t)_{t\geq 0}$ is a d-dim. Brownian motion, $\mathbb{K}$ and $\mathbb{A}$ are constant $(d \times d)$ matrices.

ii) Solve the following stochastic differential equations

a) $\begin{pmatrix} \mathrm{d}X_t^{(1)} \\ \mathrm{d}X_t^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 1 & 0 \\ 0 & X_t^{(1)} \end{pmatrix} \begin{pmatrix} \mathrm{d}W_t^{(1)} \\ \mathrm{d}W_t^{(2)} \end{pmatrix}$, where $(W_t^{(1)}, W_t^{(2)})_{t\geq 0}$ is a 2-dim. Brownian motion.

b) $\mathrm{d}X_t = X_t\,\mathrm{d}t + e^{-t}\,\mathrm{d}W_t$, where $(W_t)_{t\geq 0}$ is a 1-dim. Brownian motion.

c) $\mathrm{d}X_t = r\,\mathrm{d}t + \alpha X_t\,\mathrm{d}W_t$, where $(W_t)_{t\geq 0}$ is a 1-dim. Brownian motion and r, α are real constants. *Hint: Multiply the equation by* $F_t = e^{-\alpha W_t + \alpha^2 t/2}$.

Exercise 7.II [2 pts]

Let $(W_t)_{t\geq 0}$ be a 1-dim. Brownian motion. Show that there is a unique strong solution X_t, $t \geq 0$, of the 1-dim. stochastic differential equation

$$\mathrm{d}X_t = \ln(1 + X_t^2)\,\mathrm{d}t + \mathbb{1}_{\{X_t > 0\}}\, X_t\,\mathrm{d}W_t.$$

Exercise 7.III [5 pts]

For fixed $a, b \in \mathbb{R}$ consider following 1-dim. stochastic differential equation

$$\mathrm{d}Y_t = \frac{b - Y_t}{1 - t}\,\mathrm{d}t + \mathrm{d}W_t, \quad 0 \leq t < 1, \quad Y_0 = a.$$

Verify that

$$Y_t = a(1 - t) + bt + (1 - t)\int_0^t \frac{1}{1 - s}\,\mathrm{d}W_s, \quad 0 \leq t < 1,$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s. The process Y_t is called the *Brownian bridge* (from a to b.)

Hint: Use the Borel-Cantelli Lemma for the events

$$A_n = \left\{ \omega \in \Omega : \sup_{t \in [1-2^{-n}, 1-2^{-n-1}]} (1-t) \left| \int_0^t \frac{1}{1-s} \, \mathrm{d}W_s \right| > 2^{-n/4} \right\}, \quad n \in \mathbb{N}\,.$$

to show that for a.a. $\omega \in \Omega$ *there is* $N(\omega) < \infty$ *such that for* $n > N(\omega)$ *follows* $\omega \notin A_n$*.*

Exercise 7.IV

Make use of ideas from the proofs for Theorem 2.5 and Problem 2.10 to show that for fixed $T > 0$, a stochastic process X_t with $(d \times r)$ dispersion matrix $\sigma(t, x)$ and r-dim. Brownian motion $(\mathbb{W}_t)_{t \geq 0}$ the following inequality holds

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left\| \int_0^t \sigma(u, X_u) \, \mathrm{d}\mathbb{W}_u \right\|^2 \right] \leq 4 \, \mathbb{E}\left[\int_0^T \left\| \sigma(u, X_u) \right\|^2 \, \mathrm{d}u \right].$$

(*This is an oral exercise, to be prepared for a mini-presentation on Wednesday, December 10*)