Universität Bielefeld

Faculty of Mathematics
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Probability Theory III - Homework Assignment 8

Due date: **Friday, December 12, 12:00 h**

Solutions to the assigned homework problems must be deposited in Christian Wiesel's drop box 55 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise, let $\|\cdot\|$ be the euclidean norm in $\mathbb{R}^d$.

Exercise 8.I [4 pts]
Show that the assertions of Theorem 5.2.9 remain valid if the assumption $\mathbb{E}\|\xi\|^2 < \infty$ is dropped.
Hint: Let k be a positive integer. Show that the proof still works for $\xi_k := \xi\,\mathbb{1}_{\{\|\xi\|\leq k\}}$ and a corresponding $\{\mathcal{F}_t\}_t$-stopping time T_k, which is zero on $\{\|\xi\| > k\}$ and ∞ else. Conclude from this result that there exists a strong solution X with initial condition ξ.

Exercise 8.II [8 pts] (*Kramers-Smoluchowski Approximation, see background information below*)
Let $b(x,t)\colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ be a continuous, bounded function which satisfies a Lipschitz condition, i.e. $\|b(x,t) - b(y,t)\| \leq \kappa\|x-y\|$ for every $0 \leq t < \infty$ and $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, where κ is a positive constant. For every finite $\alpha > 0$ consider the stochastic differential system

$$\begin{aligned} \mathrm{d}X_t^\alpha &= Y_t^\alpha\,\mathrm{d}t, \\ \mathrm{d}Y_t^\alpha &= \alpha\left(b(X_t^\alpha,t) - Y_t^\alpha\right)\mathrm{d}t + \alpha\,\mathrm{d}W_t, \end{aligned} \qquad \text{for } 0 \leq t < \infty, \tag{1}$$

with initial condition $(X_0^\alpha, Y_0^\alpha) = (\xi, \eta)$, where ξ, η are a.s finite random variables, jointly independent of the Brownian motion $(W_t)_{t\geq 0}$. Furthermore X is the unique, strong solution to

$$\mathrm{d}X_t = b(X_t,t)\,\mathrm{d}t + \mathrm{d}W_t, \quad \text{for } 0 \leq t < \infty, \tag{2}$$

with initial condition $X_0 = \xi$.

a) Show that the system (1) admits a unique, strong solution for every value $\alpha \in (0,\infty)$.
Hint: Rewrite the coupled system (1) as special case of $\mathrm{d}Z_t = \tilde{b}(Z_t,t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$.

b) Prove that for every fixed, finite $T > 0$, we have

$$\lim_{\alpha\to\infty} \sup_{t\in[0,T]} \|X_t^\alpha - X_t\| = 0, \quad \text{a.s..} \tag{3}$$

Therefore use the following outline:

i) On an arbitrary time interval $[t_n, t_{n+1}] \subseteq [0,T]$, $n \in \mathbb{N}$, holds

$$X_t^\alpha - X_t = X_{t_n}^\alpha - X_{t_n} + \frac{Y_{t_n}^\alpha - Y_t^\alpha}{\alpha} + \int_{t_n}^t b(X_s^\alpha, s) - b(X_s,s)\,\mathrm{d}s, \quad \text{for all } t \in [t_n, t_{n+1}].$$

ii) Choose a value for t_n such that the integral in i) is bounded by $\sup_{s\in[t_n,t_{n+1}]}\|X_s^\alpha - X_s\|/2$ and conclude

$$\sup_{s\in[t_n,t_{n+1}]}\|X_s^\alpha - X_s\| \le 2\|X_{t_n}^\alpha - X_{t_n}\| + 4\sup_{s\in[t_n,t_{n+1}]}\frac{\|Y_s^\alpha\|}{\alpha}, \quad \text{for all } n\in\mathbb{N}\,.$$

iii) Suppose we can prove that

$$\lim_{\alpha\to\infty}\sup_{s\in[t_n,t_{n+1}]}\frac{\|Y_s^\alpha\|}{\alpha} = 0, \quad \text{a.s. for all } n\in\mathbb{N}, \tag{4}$$

and show the claim (3). *Hint: Use an induction over n.*

c) Consider the time interval from b.ii) and give a proof for the statement (4). For this purpose show:

i) $Y_t^\alpha = e^{-\alpha(t-t_n)}Y_{t_n}^\alpha + \alpha\int_{t_n}^t e^{-\alpha(t-s)}b(X_s^\alpha, s)\,\mathrm{d}s + \alpha\int_{t_n}^t e^{-\alpha(t-s)}\,\mathrm{d}W_s$ a.s.
Hint: Use a similar approach as for the solution of homework assignment 7 exercise I.i.b).

ii) It holds, for all $n\in\mathbb{N}$,

$$\sup_{s\in[t_n,t_{n+1}]}\|b(X_s^\alpha,s)\| \le 2\|b(X_{t_n}^\alpha,t_n)\| + 4\kappa\sup_{s\in[t_n,t_{n+1}]}\frac{\|Y_s^\alpha\|}{\alpha} + 2\kappa\sup_{s\in[t_n,t_{n+1}]}\|W_s - W_{t_n}\|,$$

and thereby follows with c.i)

$$\sup_{s\in[t_n,t_{n+1}]}\frac{\|Y_s^\alpha\|}{\alpha} \le 2\frac{\|Y_{t_n}^\alpha\|}{\alpha} + 4\frac{\|b(X_{t_n}^\alpha,t_n)\|}{\alpha} + \varepsilon_n(\alpha), \quad \text{for all } n\in\mathbb{N},$$

where

$$\varepsilon_n(\alpha) := \frac{4\kappa}{\alpha}\sup_{s\in[t_n,t_{n+1}]}\|W_s - W_{t_n}\| + 2\sup_{s\in[t_n,t_{n+1}]}\left\|\int_{t_n}^s e^{-\alpha(s-u)}\,\mathrm{d}W_u\right\|.$$

iii) Show $\varepsilon_n(\alpha)\to 0$ by $\alpha\to\infty$, for all $n\in\mathbb{N}$, and conclude with this (4).

Exercise 8.III

Prepare a mini-presentation for the tutorial on Wednesday, December 17, on the proof of Proposition 5.2.13 by the use of the auxiliary facts of the function

$$\psi_n(x) := \int_0^{|x|}\int_0^y \varrho_n(u)\,\mathrm{d}u\,\mathrm{d}y, \quad x\in\mathbb{R},$$

which have been presented in the proof of Proposition 5.2.18. Suppose that there are two strong solutions $X^{(1)}$ and $X^{(2)}$ of

$$\mathrm{d}X_t = b(X_t,t)\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}W_t,$$

with $X_0^{(1)} = X_0^{(2)}$ a.s and show the indistinguishably of $X^{(1)}$ and $X^{(2)}$ under the assumption[1]

$$\mathbb{E}\left[\int_0^t |\sigma(X_s^{(i)}, s)|^2 \,\mathrm{d}s\right] < \infty, \quad 0 \le t < \infty,\ i = 1, 2.$$

Some background information for Exercise 8.II:
The motion of a particle of mass $1/\alpha$ in a force field $b(x_t^\alpha, t)$ with the friction proportional to the velocity is defined by the *Newton law* (mass $*$ acceleration $=$ force):

$$\tfrac{1}{\alpha}\ddot{x}_t^\alpha = b(x_t^\alpha, t) - \dot{x}_t, \qquad x_0^\alpha = \xi,\ \dot{x}_0^\alpha = \eta,$$

where x_t is the position of the particle, $\dot{x}_t$ the velocity and $\ddot{x}_t$ the acceleration. This second order ODE can be rewritten as the following differential system

$$\begin{aligned} \dot{x}_t^\alpha &= y_t^\alpha, \\ \dot{y}_t^\alpha &= \alpha\left(b(x_t^\alpha, t) - y_t\right), \end{aligned}$$

with initial value $(x_0^\alpha, y_0^\alpha) = (\xi, \eta)$. If we now add some random perturbation to the force field, i.e. $b(x_t^\alpha, t)\,\mathrm{d}t + \mathrm{d}W_t$, we end up with the stochastic differential system (1), thereby considering a particle which is exposed to some random impulses given by $\mathrm{d}W_t$.
The Property (3) is called *Kramers-Smoluchowski Approximation* (of X_t^α by X_t) which states that the solution of (1) converges to the solution of equation (2) as mass goes to zero. This result is the main justification for using the 'simpler' equation (2) for describe small particle motion.
Some nice animation for an equation of type (2) can be found at `http://www.math.uni-bielefeld.de/~daltemeier/simulator1.html`. Here you can see the effect of so called *stochastic resonance*, where the red ball states the deterministic case and the red ball the stochastic case.

[1]Otherwise use (iii) of Definition 2.1 and go on with a localization argument.