Universität Bielefeld
Faculty of Mathematics
Prof. Dr. Barbara Gentz
Diana Kämpfe, Christian Wiesel WS 2014/2015

Probability Theory III - Homework Assignment 9

Due date: **Friday, January 9, 12:00 h**

Solutions to the assigned homework problems must be deposited in Christian Wiesel's drop box 55 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 9.I [4 pts]

Show that

a) the one-dim. stochastic differential equation

$$\mathrm{d}X_t = -\operatorname{sgn}(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 0,$$

has a weak solution which is unique in the sense of probability law holds.

b) if (X, W), $(\Omega, \mathcal{F}, Q)$, $\{\mathcal{F}_t\}_t$ is a solution considered in a) and $L_t(0)$ is the local time at the origin for the Brownian family $\{(X_t)_t, \{\mathbb{P}^x\}_{x\in\mathbb{R}}\}$ on $(\Omega, \mathcal{F})$, adapted on a filtration $\{\mathcal{F}_t\}_t$, then

$$Q\{X_t \in \Gamma\} = e^{-t/2}\,\mathbb{E}^0\left[\mathbb{1}_{\{X_t\in\Gamma\}}\, e^{-|X_t|+L_t(0)}\right], \quad \text{for } \Gamma \in \mathscr{B}(\mathbb{R}).$$

Exercise 9.II [4 pts]

Consider following d-dim. stochastic differential equation

$$\mathrm{d}X_t = b(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}W_t, \tag{1}$$

with $\sigma(x, t)$ a nonsingular $(d \times d)$-matrix for every $t \geq 0$ and $x \in \mathbb{R}^d$. Assume that $b(x, t)$ is uniformly bounded, the smallest eigenvalue of $\sigma(x, t)\sigma(x, t)^T$ is bounded away from zero, and the equation

$$\mathrm{d}\tilde{X}_t = \sigma(\tilde{X}_t, t)\,\mathrm{d}W_t, \quad \text{for } 0 \leq t \leq T,$$

has a weak solution with initial distribution μ. Show that (1) also has a weak solution for $0 \leq t \leq T$ with initial distribution μ.

Exercise 9.III [4 pts]

Prove the following claim. For fixed $t \geq 0$ and $F \in \mathscr{B}_t(\mathcal{C}[0,\infty)^d)$, the mapping $(x, \omega) \mapsto Q_j(x, \omega; F)$ is $\hat{\mathscr{B}}_t$-measurable, where $\{\hat{\mathscr{B}}_t\}_t$ is the augmentation of the filtration $\{\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}_t(\mathcal{C}[0,\infty)^r)\}_t$ by the null sets of $\mu(\mathrm{d}x)\,\mathbb{P}_\star(\mathrm{d}\omega)$.

Hint: Consider the regular conditional probabilities $Q_j^t(x, \omega; F)\colon\ \mathbb{R}^d \times \mathcal{C}[0,\infty)^r \times \mathscr{B}_t(\mathcal{C}[0,\infty)^d) \to [0,1]$ *for* $\mathscr{B}_t(\mathcal{C}[0,\infty)^d)$*, given* $(x, \varphi_t\omega)$*. These enjoy properties analogous to those of* $Q_j(x, \omega; F)$*; in particular, for every* $F \in \mathscr{B}_t(\mathcal{C}[0,\infty)^d)$*, the mapping* $(x, \omega) \mapsto Q_j^t(x, \omega; F)$ *is* $\mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}_t(\mathcal{C}[0,\infty)^r)$*-measurable, and*

$$\mathbb{P}_j(G \times F) = \int_G Q_j^t(x, \omega; F)\mu(\mathrm{d}x)\,\mathbb{P}_\star(\mathrm{d}\omega) \tag{2}$$

for every $G \in \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}_t(\mathcal{C}[0,\infty)^r)$. If (2) is shown to be valid for every $G \in \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathcal{C}[0,\infty)^r)$, then comparison of (2) with

$$\forall F \in \mathscr{B}_t(\mathcal{C}[0,\infty)^d) : (x,\omega) \mapsto Q_j(x,\omega;F) \text{ is } \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}_t(\mathcal{C}[0,\infty)^r)\text{-measurable}$$

shows that $Q_j(x,\omega;F) = Q_j^t(x,\omega;F)$ for $\mu \times \mathbb{P}_\star$-a.e. (x,ω), and the conclusion follows. Establish (2), first for sets of the form

$$G = G_1 \times (\varphi_t^{-1} G_2 \cap \sigma_t^{-1} G_3), \quad \text{for } G_1 \in \mathscr{B}(\mathbb{R}^d), \quad G_2, G_3 \in \mathscr{B}(\mathcal{C}[0,\infty)^r),$$

where $(\sigma_t\omega)(s) := \omega(t+s) - \omega(t)$, $s \geq 0$, and then in the generality required.

Exercise 9.IV

Prepare a mini-presentation for the tutorial on Wednesday, January 14, on weak solutions to functional stochastic differential equations.

Suppose the following definition is known:

Definition: Let $b_i(y,t)$ and $\sigma_{ij}(y,t)$, $1 \leq i \leq d$, $1 \leq j \leq r$, be progressively measurable functionals from $\mathcal{C}[0,\infty)^d \times [0,\infty)$ into $\mathbb{R}$. A *weak solution to the functional stochastic differential equation*

$$\mathrm{d}X_t = b(X,t)\,\mathrm{d}t + \sigma(X,t)\,\mathrm{d}W_t, \quad \text{for } 0 \leq t < \infty, \tag{3}$$

is a triple (X,W), $(\Omega,\mathcal{F},\mathbb{P})$, $\{\mathcal{F}_t\}_t$ satisfying

i) $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, and $\{\mathcal{F}_t\}_t$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions;

ii) $X = (X_t)_t$ is a continuous, $\{\mathcal{F}_t\}_t$-adapted, $\mathbb{R}^d$-valued process, and $W = (W_t)_t$ is an r-dim., $\{\mathcal{F}_t\}_t$-adapted Brownian motion

iii) $\int_0^t \{|b_i(X,s)| + \sigma_{ij}^2(X,s)\}\,\mathrm{d}s < \infty$, for $t \geq 0$ and $1 \leq i \leq d$, $1 \leq j \leq r$;

iv) $X_t = X_0 + \int_0^t b(X,s)\,\mathrm{d}s + \int_0^t \sigma(X,s)\,\mathrm{d}W_s$, for $0 \leq t < \infty$ a.s.

Assume $b_i(y,t)$ and $\sigma_{ij}(y,t)$ ($1 \leq i \leq d$, $1 \leq j \leq r$) are progressively measurable functionals from $\mathcal{C}[0,\infty)^d \times [0,\infty)$ into $\mathbb{R}$ satisfying

$$\|b(y,t)\|^2 + \|\sigma(y,t)\|^2 \leq K\left(1 + \max_{0\leq s\leq t} \|y(s)\|^2\right) \quad \text{for all } 0 \leq t < \infty, \quad y \in \mathcal{C}[0,\infty)^d,$$

where K is a positive constant. Show that, if (X,W), $(\Omega,\mathcal{F},\mathbb{P})$, $\{\mathcal{F}_t\}_t$ is a weak solution to (3) with $\mathbb{E}\,\|X_0\|^{2m} < \infty$ for some $m \geq 1$, then for any finite $T > 0$, we have

i) $\mathbb{E}\left[\max_{0\leq s\leq t} \|X_s\|^{2m}\right] \leq C\left(1 + \mathbb{E}\,\|X_0\|^{2m}\right)e^{Ct}$, for $0 \leq t \leq T$;

ii) $\mathbb{E}\,\|X_t - X_s\|^{2m} \leq C\left(1 + \mathbb{E}\,\|X_0\|^{2m}\right)(t-s)^m$, for $0 \leq s < t \leq T$;

where C is a positive constant depending only on m, T, K and d.