Construction of Random Laguerre Tessellations

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3 Clusters

4 Laguerre Tilings

5 Expansions of the Theory & Outlook
Outline

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2. Context and Notations
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Motivation

Outline

1. Introduction
   - Motivation

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Motivation

(Possible) Applications

Generalization of Voronoi tilings

- Voronoi tilings are a special case of *Laguerre tilings*
- Might describe certain problems in more detail

Simulation of granular media

- "detect" collision in sets of spheres with different radius

...
Outline

1. Introduction

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4. Laguerre Tilings

5. Expansions of the Theory & Outlook
• **Phase spaces:** \((X, \mathcal{A}, \mathcal{B}(X))\), where
  
  - \(X\) is the set where the points lie,
  - \(\mathcal{A}\) a \(\sigma\)-Algebra in \(X\), containing at least all sets of the kind \(\{x\}, x \in X\) and
  - \(\mathcal{B}(X)\), the **bounded sets** in \(X\), a valid set of subsets of \(X\).

  They define **locality**. \((B_0(X): \text{measurable bounded sets})\)

• **Measure spaces:**
  
  - \(\mathcal{M}(X)\) locally finite measures, prepared with \(\sigma\)-Algebra \(\mathcal{F}(X) = \sigma(\zeta_B; B \in B_0(X))\), \(\zeta_B : \mathcal{M}(X) \to \mathbb{R}_0^+\), \(\zeta_B(\mu) := \mu(B)\);
  - \(\mathcal{M}^\cdot( X )\) counting measures, \(\mathcal{F}^\cdot( X ) := \mathcal{M}^\cdot( X ) \cap \mathcal{F}( X )\);
  - \(\mathcal{M}^\cdot( X )\) simple counting measures, \(\mathcal{F}^\cdot( X ) := \mathcal{M}^\cdot( X ) \cap \mathcal{F}^\cdot( X )\).

• **Probabilities on measure spaces:** random measures, point processes, simple point processes respectively.

(All point process theory used in this talk is based on Kerstan, Matthes, Mecke "Unbegrenzt teilbare Punktprozesse" and Ripley "Locally finite random sets: foundations for point process theory" in *Ann. Probability* 4(6):983-994, 1976.)

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4. Laguerre Tilings
5. Expansions of the Theory & Outlook
The Phase Space of Clusters

Outline

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3. Clusters
   - The Phase Space of Clusters
   - Special Clusters or Geometry

4. Laguerre Tilings

5. Expansions of the Theory & Outlook
Consider the phase space \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))\), where \(\mathcal{B}(\mathbb{R}^d)\) is the set of Borel sets in \(\mathbb{R}^d\) with respect to the Euclidean topology and \(\mathcal{B}(\mathbb{R}^d)\) is the set of metrically bounded subsets of \(\mathbb{R}^d\).

\[ \Gamma := \left\{ x \in \mathcal{M} \cdot (\mathbb{R}^d) \mid x(\mathbb{R}^d) < +\infty \right\} . \]

And the corresponding \(\sigma\)-Algebra

\[ \mathcal{G} := \Gamma \cap \mathcal{F} \cdot (\mathbb{R}^d) . \]
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We define the cluster space as follows

\[
\Gamma := \left\{ x \in \mathcal{M} \cdot (\mathbb{R}^d) \mid x(\mathbb{R}^d) < +\infty \right\}.
\]

And the corresponding \(\sigma\)-Algebra

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\mathcal{G} := \Gamma \cap \mathcal{F} \cdot (\mathbb{R}^d).
\]
Bounded Sets

A subset $F$ of $\Gamma$ belongs to the bounded sets $\mathcal{B}(\Gamma)$, iff there exists some $B \in \mathcal{B}_0(\mathbb{R}^d)$ such that

$$F \subseteq \mathcal{F}_B := \{ x \in \Gamma \mid x(B) > 0 \}$$

Remarks: $\mathcal{F}_B \in \mathcal{G}$ for all $B \in \mathcal{B}_0(\mathbb{R}^d)$, even $\mathcal{G} = \sigma (\mathcal{F}_B; B \in \mathcal{B}_0(\mathbb{R}^d))$. 
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The Phase Space of Clusters

Construction of Random Laguerre Tessellations

Figure: $x$ is in $\mathcal{F}_A$ but not in $\mathcal{F}_B$
Proposition 1

$(\Gamma, \mathcal{G}, \mathcal{B}(\Gamma))$ is a valid phase space.

We can now talk about $\mathcal{M}(\Gamma)$, (locally finite) cluster configurations.

"Locally finite" in this context means that only finitely many clusters of a configuration have points in a fixed bounded set of $\mathbb{R}^d$. 

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The Phase Space of Clusters

Locally Finite Cluster Configurations

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   - The Phase Space of Clusters
   - Special Clusters or Geometry

4 Laguerre Tilings

5 Expansions of the Theory & Outlook
A cluster \( x \in \Gamma \) is called a **discrete convex polytope** (in \( \mathbb{R}^d \)), if for all points \( a \in x \) there exists some \((d - 1)\)-dimensional hyperplane \( H \) with \( H \cap \langle x \rangle = \{a\} \). We denote the set of discrete convex polytopes by \( \mathcal{K}(\mathbb{R}^d) \).
Figure: We identify convex polytopes with their vertices
Special Clusters or Geometry

Special Cluster Configurations: Tilings

A configuration $\mu \in \mathcal{M}(\Gamma)$ is called **tiling** in $\mathbb{R}^d$, if

1. $x \in \mu \Rightarrow x \in \mathcal{H}(\mathbb{R}^d)$,
2. the convex hulls of the elements of $\mu$ are face-to-face and
3. $\bigcup_{x \in \mu} \langle x \rangle = \mathbb{R}^d$.

The set of all tilings in $\mathbb{R}^d$ will be denoted by $\mathcal{M}(\mathbb{R}^d)$. 

Tilings

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Construction of Random Laguerre Tessellations
Random Tilings: Point Processes in Cluster Space

Random Tilings

A probability $P$ on $\mathcal{M}(\Gamma)$ is called random tiling, if all measurable sets $M$ such that $M(\mathbb{R}^d) \subseteq M$ have probability 1.
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Locality and Bounded Sets for Laguerre Configurations

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   - Construction of the Cluster Process
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Preparations

Now we go over to $E = \mathbb{R}^d \times \mathbb{R}$ and prepare it with the standard Borel sets $\mathcal{B}(E)$. We define the projections $q : E \to \mathbb{R}^d$, $e = (q_e, g_e) \mapsto q(e) = q_e$ and $g : E \to \mathbb{R}$, $e = (q_e, g_e) \mapsto g(e) = g_e$.

What subsets of $E$ should be considered \textit{bounded} to have an appropriate notion of "locally finite"?

- The metrically bounded sets
- but we need more...
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Cylindrical Sets

All subsets of cylindrical sets $A \times \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R}^d)$, should be bounded, because the projection

$$q : \mathcal{M}(E) \longrightarrow \mathcal{M}(\mathbb{R}^d),$$

$$\eta \longrightarrow q(\eta),$$

needs to be well defined. (This projection is the induced mapping of $q : E \rightarrow \mathbb{R}^d$.)
Consider the following symmetric form:

\[ s : \ E \times E \to \mathbb{R}, \]
\[ (e, f) \mapsto (q(e) - q(f))^2 - (g(e) + g(f)). \]

Remark: \[ s ((q_1, 0), (q_2, 0)) = (d(q_1, q_2))^2. \]
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**Remark:** \( s((q_1, 0), (q_2, 0)) = (d(q_1, q_2))^2. \)
We define the paraboloid with focus in \( f \in E \) by
\[
p(f) := \{ e \in E \mid s(e, f) \leq 0 \}.
\]
Finite unions of such paraboloids should also belong to the bounded sets \( \mathcal{B}(E) \).
Proposition 2

Let \( \rho = z \cdot \lambda^d \otimes \tau \), where \( z \in \mathbb{R}^+ \), \( \lambda^d \) is the Lebesgue measure in \( \mathbb{R}^d \) and \( \tau \) is some finite measure in \( \mathbb{R} \). If for all \( g \in \mathbb{R} \)

\[
\int_{-g}^{+\infty} \lambda^d \left( B_{\sqrt{g+t}(0)} \right) \tau(dt) < +\infty,
\]

then \( \rho \in \mathcal{M}(E) \).

Examples:

- \( \tau = \delta_{g_1} + \ldots + \delta_{g_n} \), \( n \in \mathbb{N} \), \( g_1, \ldots, g_n \in \mathbb{R} \),
- \( \tau \) concentrated on some bounded interval in \( \mathbb{R} \),
- \( \tau \) has some density \( f \) with respect to \( \lambda^1 \) such that for all \( g \in \mathbb{R} \)

\[
\int_{-g}^{+\infty} (g + t)^{d/2} f(t) \lambda^1(dt) < +\infty.
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Proposition 2

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Laguerre Tilings

Laguerre Configurations

Analogously to the Voronoi tessellations we need an additional property of the underlying point configurations to get proper tilings:

We define the Laguerre Configurations $\mathcal{L} \subset \mathcal{M} \cdot (E)$ by

$$\eta \in \mathcal{L} \iff q(\eta) \left( H^+(u, \alpha) \right) \geq 1 \quad \forall u \in \mathbb{Q}^d \setminus \{0\}, \forall \alpha \in \mathbb{Q},$$

where $H^+(u, \alpha) := \{ v \in \mathbb{R}^d \mid u \cdot v \geq \alpha \}$.

Remark: $\mathcal{L} \in \mathcal{F} \cdot (E)$
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Laguerre Tilings

The Cells

Laguerre Cells
Let $\eta \in \mathcal{L}$. The Laguerre cell of a point $e \in \eta$ is defined by

$$L_{\eta}(e) := \left\{ v \in \mathbb{R}^d \mid s(e, (v, 0)) \leq s(f, (v, 0)), \forall f \in \eta \right\}$$

A special case:

Voronoi Cells
Let $\eta \in \mathcal{L}$ such that $g(e) = g = \text{const.}$ for all $e \in \eta$, then the Laguerre cells "are" the Voronoi cells of the configuration $\mu = q(\eta)$:

$$L_{\eta}(e) = V_\mu (q(e)) = \left\{ v \in \mathbb{R}^d \mid d(v, q(e)) \leq d(v, u), \forall u \in \mu \right\}.$$
Laguerre Tilings

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**Laguerre Tilings**

**Example Cells**

**Figure:** Same weights: the Voronoi case

**Figure:** General case: the faces are shifted according to the relative weights
Laguerre Tilings

Properties of the Cells

Proposition 3
For $\eta \in \mathcal{L}$ the Laguerre cells $L_\eta(e), e \in \eta$ are convex polytopes.

- Cells are compact: due to the points in "enough" half spaces,
- Cells are polygons: because of $\eta(p(f_1) \cup \cdots \cup p(f_n)) < +\infty$ for arbitrary $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \eta$.

Proposition 4
For $\eta \in \mathcal{L}$ the collection of the Laguerre cells $L_\eta(e), e \in \eta$ is a face-to-face collection.

Follows almost immediately from definition of the cells, because the sets $\{v \in \mathbb{R}^d | s(e, (v, 0)) = s(f, (v, 0))\}$, $e, f \in \eta$ are hyperplanes. Proofs of these results are modifications of the ones in M. SCHLOTTMANN, "Periodic and Quasi-Periodic Laguerre Tilings" in International Journal of Modern Physics B, Vol. 7 (1993).
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Construction of the Cluster Process

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5. Expansions of the Theory & Outlook
We define the Laguerre cluster property $D_L \subset \Gamma \times \mathcal{M} \cdot (E)$ by

$$(x, \eta) \in D_L, \text{ iff }$$

(L1) $\eta \in \mathcal{L}$ and

(L2) there exists some $e \in \eta$ such that $L_\eta(e) \neq \emptyset$ and

$$x = \sum_{q \in \text{vert } L_\eta(e)} \delta_q$$

Remark: $D_L \in \mathcal{G} \otimes \mathcal{F} \cdot (E)$.

$(x, \eta) \in D_L$ means that $x$ "is" a Laguerre cell of the point configuration $\eta$. 
Construction of the Cluster Process

Laguerre Cluster Property

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We define the Laguerre cluster property \( D_\mathcal{L} \subset \Gamma \times \mathcal{M}^\cdot (E) \) by \((x, \eta) \in D_\mathcal{L}\), iff

\((L1)\) \(\eta \in \mathcal{L}\) and

\((L2)\) there exists some \(e \in \eta\) such that \(L_\eta(e) \neq \emptyset\) and

\[ x = \sum_{q \in \text{vert} L_\eta(e)} \delta_q \]

**Remark:** \( D_\mathcal{L} \in \mathcal{G} \otimes \mathcal{F}^\cdot (E) \).

\((x, \eta) \in D_\mathcal{L}\) means that \(x\) "is" a Laguerre cell of the point configuration \(\eta\).
The Laguerre Cluster Function

We define the Laguerre cluster function by

\[ \varphi_L : \mathcal{L} \rightarrow \mathcal{M} \cdot (\Gamma), \]
\[ \eta \mapsto \sum_{(x, \eta) \in D_L} \delta_x. \]

**Proposition 5**

The Laguerre cluster function is well defined, that is \( \varphi_L(\eta) \) is locally finite for all \( \eta \in \mathcal{L} \), and measurable.

**Main Lemma**

If \( \eta \in \mathcal{L} \), then \( \varphi_L(\eta) \) is a tiling.
We define the Laguerre cluster function by

\[ \varphi_{\mathcal{L}} : \mathcal{L} \rightarrow M(\Gamma), \]
\[ \eta \mapsto \sum_{(x, \eta) \in D_{\mathcal{L}}} \delta_x. \]

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5. Expansions of the Theory & Outlook
Just by applying the transformation theorem we get the following result:

**Proposition 6**

Let $P$ be a probability on $\mathcal{M}(E)$ such that $P(\mathcal{L}) = 1$. Then $Q := \varphi(\mathcal{L}(P))$ is a random tessellation.

We call such a cluster process a **random Laguerre tessellation** (or **random Laguerre tiling**).
Proposition 7

Let $\rho = z \cdot \lambda^d \otimes \tau$, such that $\tau$ complies the condition of proposition 2, that is, for all $g \in \mathbb{R}$

$$\int_{-g}^{+\infty} \lambda^d \left(B_{\sqrt{g+t}(0)}\right) \tau(dt) < +\infty.$$ 

Then $P_\rho(\mathcal{L}) = 1$. 

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Thus we get the main result for this talk:

**Theorem**
The image of $P_\rho$, $\rho$ as in proposition 7, under the laguerre cluster function $\varphi_L$ is a random tiling in $\mathbb{R}^d$.

We call this process the **Poisson Laguerre process**.

**Corollary**
Let $\rho = z \cdot \lambda^d \otimes \delta_r$ with $z \in \mathbb{R}^+$ and $r \in \mathbb{R}$. Then $P_\rho$-almost surely all $\varphi_L(\eta)$ are tilings in $\mathbb{R}^d$, consisting of the Voronoi cells of $q(\eta)$.

Such a process is called **Poisson Voronoi process**.
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Introduction

Context and Notations

Clusters

Laguerre Tilings

Expansions of the Theory & Outlook
The Dual Tiling

Analogously to the Voronoi and Delone Tilings there exists some dual Laguerre Tiling. But the construction differs slightly:

- You take the vertices of the Laguerre tiling as point configurations,
- give them ”appropriate” weights and then
- consider the Laguerre Cells on the new configurations.
Outlook

Possible, not yet Fully Developed Expansions

- Go over from the symmetric form $s$ to some general symmetric form, having certain properties.
- Replace $s$ by some other well known symmetric forms, for instance the Minkowski quadratic form $m : E \times E \to \mathbb{R}$,
  
  \[ m(e, f) := (q(e) - q(f))^2 - (g(e) - g(f))^2, \]

  which might have applications in special relativity.
- Go over to tilings in $E$ and not in the projected space $\mathbb{R}^d$. 
Thank you for your audience!
Questions or remarks?
Thank you again!