Construction of Random LaguerreTessellations

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Outline

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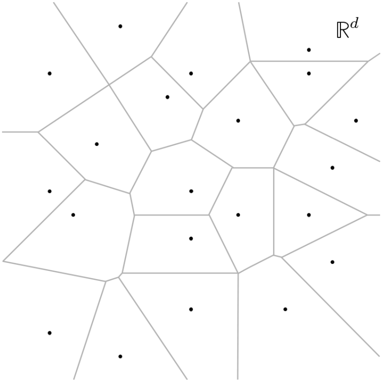
(Possible) Applications



Generalization of Voronoi tilings

- Voronoi tilings are a special case of *Laguerre tilings*
- Might describe certain problems in more detail

Simulation of granular media

- "detect" collision in sets of spheres with different radius
  (s.f. Ferrez, Liebling & Müller in *Lecture Notes in Physics* Vol. 554: "Dynamic Triangulations for Granular Media Simulations")

. . .

Outline

- **Phase spaces:** $(X, \mathscr{A}, \mathcal{B}(X))$, where
  - X is the set where the points lie,
  - $\mathscr{A}$ a σ-Algebra in X, containing at least all sets of the kind $\{x\}$, $x \in X$ and
  - $\mathcal{B}(X)$, the bounded sets in X, a valid set of subsets of X. They define locality. ($\mathcal{B}_0(X)$: measurable bounded sets.)
- **Measure spaces:**
  - $\mathscr{M}(X)$ locally finite measures, prepared with σ-Algebra $\mathscr{F}(X) = \sigma(\zeta_B; B \in \mathcal{B}_0(X))$, $\zeta_B : \mathscr{M}(X) \to \mathbb{R}_0^+$, $\zeta_B(\mu) := \mu(B)$;
  - $\mathscr{M}^{\cdot\cdot}(X)$ counting measures, $\mathscr{F}^{\cdot\cdot}(X) := \mathscr{M}^{\cdot\cdot}(X) \cap \mathscr{F}(X)$;
  - $\mathscr{M}^{\cdot}(X)$ simple counting measures, $\mathscr{F}^{\cdot\cdot}(X) := \mathscr{M}^{\cdot}(X) \cap \mathscr{F}^{\cdot\cdot}(X)$.
- **Probabilities on measure spaces:** random measures, point processes, simple point processe respectively.

(All point process theory used in this talk is based on Kerstan, Matthes, Mecke "Unbegrenzt teilbare Punktprozesse" and Ripley "Locally finite random sets: foundations for point process theory" in *Ann. Probability* 4(6):983-994, 1976.)

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Cluster Space

- Consider the phase space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, where $\mathscr{B}(\mathbb{R}^d)$ is the set of Borel sets in $\mathbb{R}^d$ with respect to the Euclidean topology and $\mathcal{B}(\mathbb{R}^d)$ is the set of metrically bounded subsets of $\mathbb{R}^d$.

Γ, $\mathscr{G}$

We define the cluster space as follows

$$\Gamma := \left\{ x \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \middle| x(\mathbb{R}^d) < +\infty \right\}.$$

And the corresponding σ-Algebra

$$\mathscr{G} := \Gamma \cap \mathscr{F}^{\cdot}\left(\mathbb{R}^d\right).$$

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Locality in Cluster Space

Bounded Sets

A subset F of Γ belongs to the bounded sets $\mathcal{B}(\Gamma)$, iff there exists some $B \in \mathcal{B}_0(\mathbb{R}^d)$ such that

$$F \subseteq \mathcal{F}_B := \{x \in \Gamma \,|\, x(B) > 0\}$$

Remarks: $\mathcal{F}_B \in \mathscr{G}$ for all $B \in \mathcal{B}_0(\mathbb{R}^d)$, even $\mathscr{G} = \sigma\left(\mathcal{F}_B;\ B \in \mathcal{B}_0(\mathbb{R}^d)\right)$.

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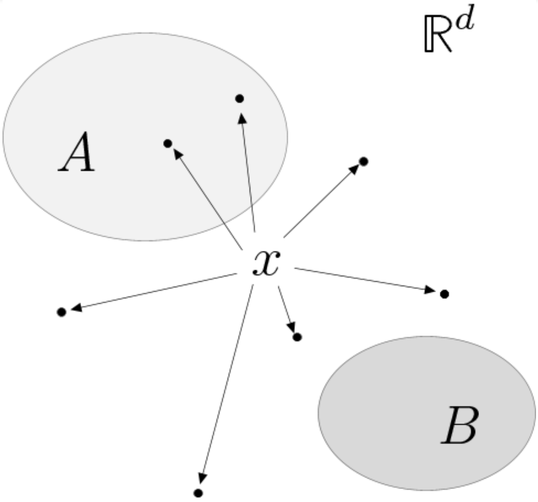

Figure: x is in $\mathcal{F}_A$ but not in $\mathcal{F}_B$

Locally Finite Cluster Configurations

Proposition 1

$(\Gamma, \mathscr{G}, \mathcal{B}(\Gamma))$ is a valid phase space.

- We can now talk about $\mathscr{M}^{\cdot}(\Gamma)$, (locally finite) cluster configurations

"Locally finite" in this context means that only finitely many clusters of a configuration have points in a fixed bounded set of $\mathbb{R}^d$.

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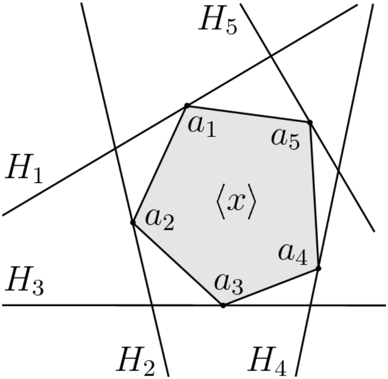
Convex Polytopes

Discrete Convex Polytopes

A cluster $x \in \Gamma$ is called discrete convex polytope (in $\mathbb{R}^d$), if for all points $a \in x$ there exists some $(d-1)$-dimensional hyperplane H with $H \cap \langle x \rangle = \{a\}$. We denote the set of discrete convex polytopes by $\mathscr{K}(\mathbb{R}^d)$.



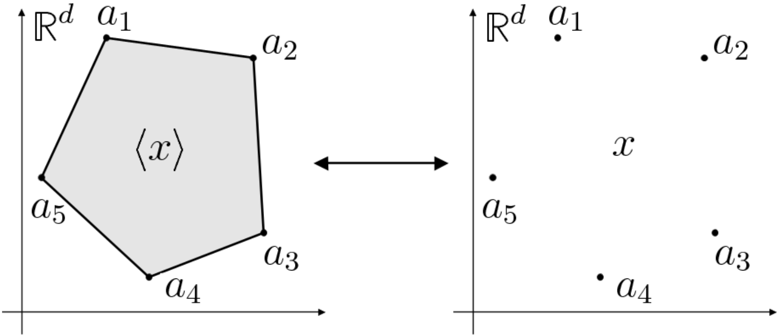
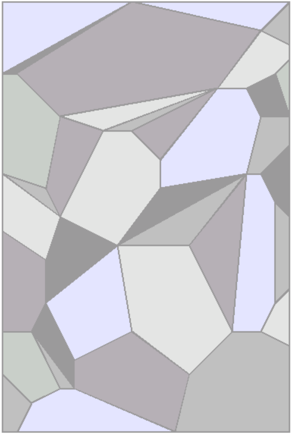


Figure: We identify convex polytopes with their vertices

Special Cluster Configurations: Tilings



Tilings

A configuration $\mu \in \mathscr{M}^{\cdot}(\Gamma)$ is called tiling in $\mathbb{R}^d$, if

1. $x \in \mu \Rightarrow x \in \mathscr{K}(\mathbb{R}^d)$,
2. the convex hulls of the elements of μ are face-to-face and
3. $\bigcup_{x\in\mu}\langle x\rangle = \mathbb{R}^d$.

The set of all tilings in $\mathbb{R}^d$ will be denoted by $\mathbb{M}(\mathbb{R}^d)$.

Random Tilings: Point Processes in Cluster Space

Random Tilings

A probability P on $\mathscr{M}^{\cdot}(\Gamma)$ is called random tiling, if all measurable sets M such that $\mathbb{M}(\mathbb{R}^d) \subseteq M$ have probability 1.

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Preparations

Now we go over to $E = \mathbb{R}^d \times \mathbb{R}$ and prepare it with the standard Borel sets $\mathscr{B}(E)$. We define the projections $q : E \to \mathbb{R}^d$, $e = (q_e, g_e) \mapsto q(e) = q_e$ and $g : E \to \mathbb{R}$, $e = (q_e, g_e) \mapsto g(e) = g_e$.

What subsets of E should be considered bounded to have an apropriate notion of "locally finite"?

- The metrically bounded sets
- but we need more...

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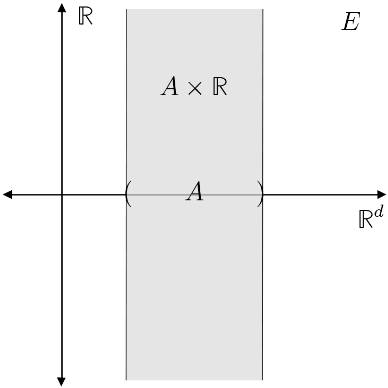
Cylindrical Sets



All subsets of cylindrical sets $A \times \mathbb{R}$, $A \in \mathcal{B}(\mathbb{R}^d)$, should be bounded, because the projection

$$\begin{aligned} q: \quad \mathscr{M}(E) &\longrightarrow \mathscr{M}\left(\mathbb{R}^d\right), \\ \eta &\longmapsto q(\eta), \end{aligned}$$

needs to be well defined. (This projection is the induced mapping of $q: E \to \mathbb{R}^d$.)

A Special Symmetric Form

Consider the following symmetric form:

$$
\begin{aligned}
s: \quad E \times E & \longrightarrow \mathbb{R}\,, \\
(e, f) & \longmapsto (q(e) - q(f))^2 - (g(e) + g(f))\,.
\end{aligned}
$$

Remark: $s\left((q_1, 0), (q_2, 0)\right) = (d(q_1, q_2))^2$.

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Remark: $s\left((q_1,0),(q_2,0)\right) = (d(q_1,q_2))^2$.

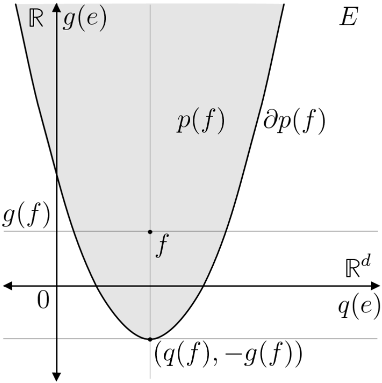
Paraboloid Sets

Paraboloid Sets

We define the paraboloid with focus in $f \in E$ by

$$p(f) := \{e \in E \,|\, s(e, f) \leq 0\} \; .$$

- Finite unions of such paraboloids should also belong to the bounded sets $\mathcal{B}(E)$.



Locally Finite Measures in $(E, \mathscr{B}(E), \mathcal{B}(E))$

Proposition 2

Let $\rho = z \cdot \lambda^d \otimes \tau$, where $z \in \mathbb{R}^+$, λ^d is the Lebesgue measure in $\mathbb{R}^d$ and τ is some finite measure in $\mathbb{R}$. If for all $g \in \mathbb{R}$

$$\int_{-g}^{+\infty} \lambda^d \left(B_{\sqrt{g+t}}(0) \right) \tau(\mathrm{d}t) < +\infty ,$$

then $\rho \in \mathscr{M}(E)$.

Examples:

- $\tau = \delta_{g_1} + \ldots + \delta_{g_n}$, $n \in \mathbb{N}$, $g_1, \ldots, g_n \in \mathbb{R}$,
- τ concentrated on some bounded interval in $\mathbb{R}$,
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Laguerre Configurations

Analogously to the Voronoi tessellations we need an additional property of the underlying point configurations to get proper tilings:

$\mathscr{L}$

We define the Laguerre Configurations $\mathscr{L} \subset \mathscr{M}^{\cdot}(E)$ by

$$\eta \in \mathscr{L} \quad :\Longleftrightarrow \quad q(\eta)\left(H^{+}(u,\alpha)\right) \geq 1 \quad \forall u \in \mathbb{Q}^{d} \setminus \{0\}, \forall \alpha \in \mathbb{Q},$$

where $H^{+}(u,\alpha) := \left\{v \in \mathbb{R}^{d} \,\middle|\, u \cdot v \geq \alpha\right\}$.

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The Cells

Laguerre Cells

Let $\eta \in \mathscr{L}$. The Laguerre cell of a point $e \in \eta$ is defined by

$$L_\eta(e) := \left\{ v \in \mathbb{R}^d \,\middle|\, s\left(e, (v, 0)\right) \leq s\left(f, (v, 0)\right), \forall f \in \eta \right\}$$

A special case:

Voronoi Cells

Let $\eta \in \mathscr{L}$ such that $g(e) = g =$ const. for all $e \in \eta$, then the Laguerre cells "are" the Voronoi cells of the configuration $\mu = q(\eta)$:

$$L_\eta(e) = V_\mu\left(q(e)\right) = \left\{ v \in \mathbb{R}^d \,\middle|\, d\left(v, q(e)\right) \leq d\left(v, u\right) \forall u \in \mu \right\}.$$

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Example Cells

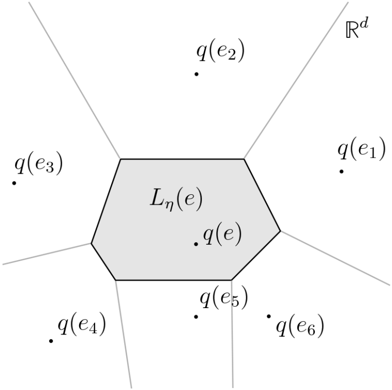

Figure: Same weights: the Voronoi case

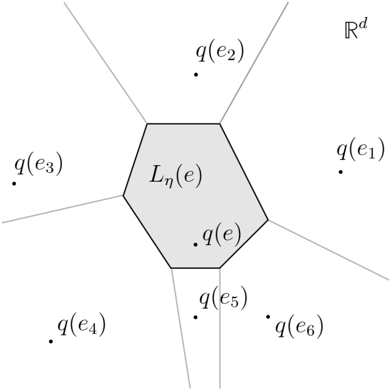

Figure: General case: the faces are shiftet according to the relative weights

Properties of the Cells

Proposition 3

For $\eta \in \mathscr{L}$ the Laguerre cells $L_\eta(e)$, $e \in \eta$ are convex polytopes.

- Cells are compact: due to the points in "enough" half spaces,
- Cells are polygones: because of $\eta\left(p(f_1) \cup \cdots \cup p(f_n)\right) < +\infty$ for arbitrary $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \eta$.

Proposition 4

For $\eta \in \mathscr{L}$ the collection of the Laguerre cells $L_\eta(e)$, $e \in \eta$ is a face-to-face collection.

Follows almost immediately from definition of the cells, because the sets $\{v \in \mathbb{R}^d \mid s(e,(v,0)) = s(f,(v,0))\}$, $e, f \in \eta$ are hyperplanes. Proofs of these results are modifications of the ones in M. Schlottmann, "Periodic and Quasi-Periodic Laguerre Tilings" in *International Journal of Modern Physics B*, Vol. 7 (1993).

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Laguerre Cluster Property

$D_{\mathscr{L}}$

We define the Laguerre cluster property $D_{\mathscr{L}} \subset \Gamma \times \mathscr{M}^{\cdot}(E)$ by $(x, \eta) \in D_{\mathscr{L}}$, iff

(L1) $\eta \in \mathscr{L}$ and

(L2) there exists some $e \in \eta$ such that $L_\eta(e) \neq \emptyset$ and $x = \sum_{q \in \text{vert } L_\eta(e)} \delta_q$

Remark: $D_{\mathscr{L}} \in \mathscr{G} \otimes \mathscr{F}^{\cdot}(E)$.

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The Laguerre Cluster Function

$\varphi_{\mathscr{L}}$

We define the Laguerre cluster function by

$$\begin{aligned} \varphi_{\mathscr{L}} : \quad \mathscr{L} &\longrightarrow \mathscr{M}^{\cdot}(\Gamma)\,, \\ \eta &\longmapsto \sum_{(x,\eta)\in D_{\mathscr{L}}} \delta_x\,. \end{aligned}$$

Proposition 5

The Laguerre cluster function is well defined, that is $\varphi_{\mathscr{L}}(\eta)$ is locally finite for all $\eta \in \mathscr{L}$, and measurable.

Main Lemma

If $\eta \in \mathscr{L}$, then $\varphi_{\mathscr{L}}(\eta)$ is a tiling.

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Random Laguerre Tessellations

Just by applying the transformation theorem we get the following result:

Proposition 6

Let P be a probability on $\mathscr{M}^{\cdot}(E)$ such that $P(\mathscr{L}) = 1$. Then $Q := \varphi_{\mathscr{L}}(P)$ is a random tessellation.

We call such a cluster process a random Laguerre tessellation (or random Laguerre tiling).

A Prominent Example

Proposition 7

Let $\rho = z \cdot \lambda^d \otimes \tau$, such that τ complies the condition of proposition 2, that is, for all $g \in \mathbb{R}$

$$\int_{-g}^{+\infty} \lambda^d \left(B_{\sqrt{g+t}}(0) \right) \tau(\mathrm{d}t) < +\infty \,.$$

Then $P_\rho(\mathscr{L}) = 1$.

Poisson Laguerre Process

Thus we get the main result for this talk:

Theorem

The image of P_ρ, ρ as in proposition 7, under the laguerre cluster function $\varphi_{\mathscr{L}}$ is a random tiling in $\mathbb{R}^d$.

We call this process the Poisson Laguerre process.

Corollary

Let $\rho = z \cdot \lambda^d \otimes \delta_r$ with $z \in \mathbb{R}^+$ and $r \in \mathbb{R}$. Then P_ρ-almost surely all $\varphi_{\mathscr{L}}(\eta)$ are tilings in $\mathbb{R}^d$, consisting of the Voronoi cells of $q(\eta)$.

Such a process is called Poisson Voronoi process.

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We call this process the Poisson Laguerre process.

Corollary

Let $\rho = z \cdot \lambda^d \otimes \delta_r$ with $z \in \mathbb{R}^+$ and $r \in \mathbb{R}$. Then P_ρ-almost surely all $\varphi_{\mathscr{L}}(\eta)$ are tilings in $\mathbb{R}^d$, consisting of the Voronoi cells of $q(\eta)$.

Such a process is called Poisson Voronoi process.

Outline

Existing Expansion of the Theory

The Dual Tiling

Analogously to the Voronoi and Delone Tilings there exists some dual Laguerre Tiling. But the construction differs slightly:

- You take the vertices of the Laguerre tiling as point configurations,
- give them "apropriate" weights and then
- consider the Laguerre Cells on the new configurations.

Outlook

Possible, not yet Fully Developed Expansions

- Go over from the symmetric form s to some general symmetric form, having certain properties.
- Replace s by some other well known symmetric forms, for instance the Minkowski quadratic form $m : E \times E \to \mathbb{R}$, $m(e, f) := (q(e) - q(f))^2 - (g(e) - g(f))^2$, which might have applications in special relativity.
- Go over to tilings in E and not in the projected space $\mathbb{R}^d$.

Thank you for your audience!

Questions or remarks?

Thank you again!