Slightly Randomised Silver-Mean Chains

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Outline

Why (random) aperiodic tilings?

- *For Mathematicians:* Because they are nice.
- *For the rest:* Similar structures can be found in nature:
  - Aperiodic tilings describe quasicrystals, e.g. decagonal $Al_{72}Ni_{20}Co_8$
  - even almost crystalline structures often have random shifts of the molecules.

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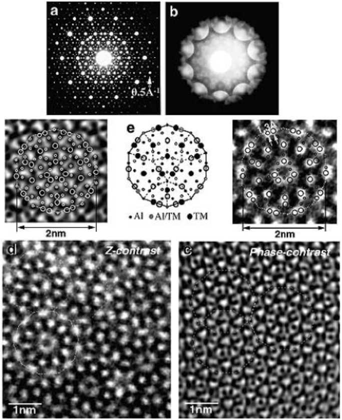


Figure: $Al_{72}Ni_{20}Co_8$

Strategy for Modelling

- To describe quasicrystals and aperiodic tilings we apply *model sets*, point sets based on the *cut and project scheme*.
- To get randomness into the model, we use *point* respectively *cluster processes*.
- The *tiles* are represented by their *vertex sets*.

Outline

Cut and Project Schemes

$$
\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times G & \xrightarrow{\;\pi_{\text{int}}\;} & G \\
\cup & & \cup & & \cup \\
L = \pi\big(\tilde{L}\big) & \xleftarrow{\;\pi|_{\tilde{L}}\;} & \tilde{L} & \xrightarrow{\;\pi_{\text{int}}|_{\tilde{L}}\;} & \pi_{\text{int}}\big(\tilde{L}\big)
\end{array}
$$

Commutative diagram called *cut and project scheme* if

- G locally compact group ('*internal space*'),
- π, π_{int} canonical projections,
- $\tilde{L}$ lattice,
- $\pi_{\text{int}}\big(\tilde{L}\big)$ dense in G and
- $\pi|_{\tilde{L}} : \tilde{L} \to L$ is injective.

($\mathbb{R}^d$ in this context called '*physical space*')

Model Sets

Let a cut and project scheme with $G, \tilde{L}, L, \pi, \ldots$ as above be given. Further let $w \subset G$ such that w nonempty and $w = \overline{\operatorname{int} w}$ is compact. Then the set

$$\Lambda = \Lambda(w) := \left\{ \pi(e) \,\middle|\, e \in \tilde{L}, \pi_{\text{int}}(e) \in w \right\}$$

is called *model (or cut and project) set with window w*.

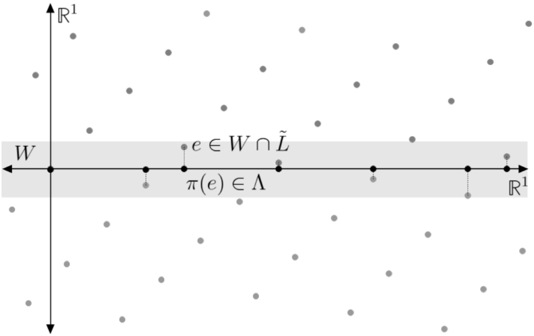
- If we set $W := \mathbb{R}^d \times w$ (called '*stripe*'), we have

$$\Lambda = \pi\left(W \cap \tilde{L}\right).$$

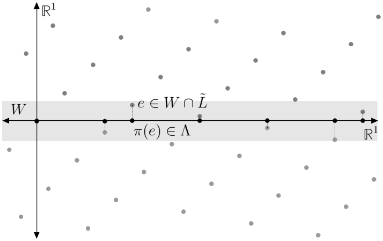
Our Example Model Set – The Silver-Mean Chain

Let $d = 1$, $G = \mathbb{R}^1$,

- $\tilde{L} := \left\{ \left(u + v\sqrt{2}\,,\, u - \sqrt{2}\right) \mid u, v \text{ integers} \right\} \subset \mathbb{R}^1 \times \mathbb{R}^1$
- $w = \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$



Some Remarks on the Silver-Mean Chain



- Two neighboured points in Λ have either the distance 1 or $1 + \sqrt{2}$.
- The tiling, where the tiles are convex hulls of two neighboured points, is aperiodic.
- The ratio between long and short tiles is the silver-mean (like 'golden ratio'), again $1 + \sqrt{2}$, positive root of the equation $\lambda^2 = 2\lambda + 1$. (Golden ratio: root of $\lambda^2 = \lambda + 1$.)

Other (Aperiodic) Tilings Obtained from Model Sets

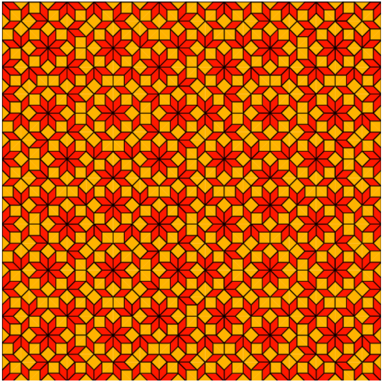


Figure: Ammann-Beenker-Tiling

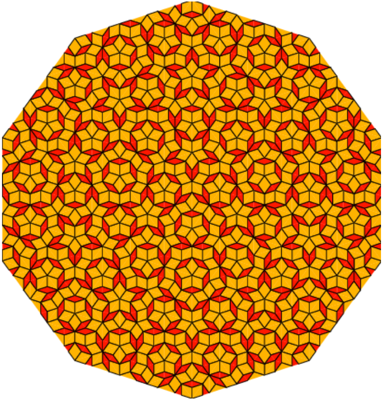


Figure: Penrose-Tiling

Outline

Connection to Model Sets and Tilings

- The generating lattices $\tilde{L}$ and model sets Λ are *locally finite* sets of points.
  Point processes are probability measures on locally finite point sets of certain spaces. They might be imagined as *random point configurations*.
- The corresponding tilings consist of *locally finite* sets of convex polytopes, each of which can be described uniquely by their finite vertex set.
  Cluster processes are probability measures on configurations of finite point sets.

Locally Finite Point Sets – a Convenient Identification

(Although the theory is much more general we stick to the easy example of locally finite point configurations in $\mathbb{R}^d$.)

- A locally finite subset η of $\mathbb{R}^d$ is always at most countable.
- We can therefore uniquely identify such an $\eta = \{e_1, e_2, \dots\}$ with the *simple counting measure*

$$\tilde{\eta} := \sum_{i \in \mathbb{N}} \delta_{e_i} \,.$$

- $\tilde{\eta}(B)$, B Borel set, then just gives the cardinality of the set $\eta \cap B$ – it counts the points in B.

More on Counting Measures

- Set of simple counting measures in $\mathbb{R}^d$ denoted by $\mathscr{M}^{\bullet}(\mathbb{R}^d)$.
- If one allows 'multiple points'

$$\tilde{\eta} := \sum_{i \in \mathbb{N}} n_i \delta_{e_i} \qquad n_i \in \mathbb{N}$$

  we speak of *counting measures*, $\tilde{\eta} \in \mathscr{M}^{\bullet\bullet}(\mathbb{R}^d)$.
- In the Following we omit the $\tilde{}$ and use η for both, point set and corresponding measure ($e \in \eta$, $\eta(B)$, ...).

'Interesting events' and Probabilities

- Since we want to talk about probability measures on the spaces $\mathscr{M}^{\bullet}(\mathbb{R}^d), \mathscr{M}^{\bullet\bullet}(\mathbb{R}^d)$, we need to talk about 'interesting events' which should be given probabilities.
- Somehow obvious choice for generating class of events:

$$\left\{\eta \in \mathscr{M}^{\bullet}(\mathbb{R}^d) \,\middle|\, \eta \text{ has } k \text{ points in } B\right\}, \; k \in \mathbb{N}_0, \; B \text{ Borel set.}$$

(Analogously for $\mathscr{M}^{\bullet\bullet}(\mathbb{R}^d)$.)

- Probability measures P on the σ-algebras generated by these events are called *(simple) point processes*
- Since B can be chosen arbitrarily big or small we get a lot of information about the distribution of the 'random points'.

An Example

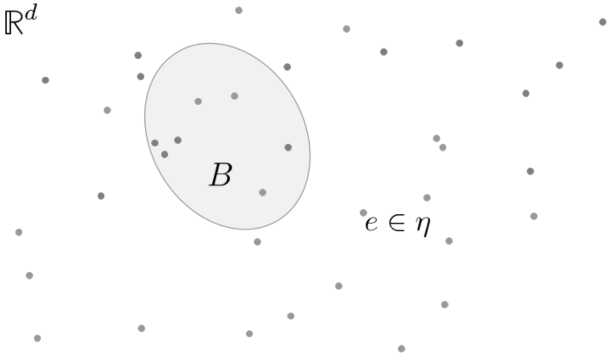


Figure: An example realisation of the set $\{\eta \text{ has } 7 \text{ points in } B\}$

The Poisson Point Process

Theorem

There is exactly one simple point process P_λ, $\lambda \in \mathbb{R}^+$, with the following two properties:

1. The two events
$$\{\eta \text{ has } k \text{ points in } B\}\ ,\ \{\eta \text{ has } l \text{ points in } C\}\ ,$$
$k, l \in \mathbb{N}_0$, are independent as long as B and C are disjoint.
2. The typical events are given the following probabilities:
$$P_\lambda\left(\{\eta \text{ has } k \text{ points in } B\}\right) = \frac{(\lambda \cdot \mathrm{vol}(B))^k}{k!}\, \mathrm{e}^{-\lambda \cdot \mathrm{vol}(B)}\ .$$

Poisson Point Process – Continued

- P_λ expects λ points in a Borel set of volume 1.
- P_λ describes an *ideal gas* at a fixed moment in time.
- The probabilities are invariant under a shift of B.
- If one omits 'simple' in the theorem, $\lambda \cdot \mathrm{vol}$ in the probabilities may be exchanged by an arbitrary locally finite measure ρ. We then speak of P_ρ, the Poisson point process with intensity ρ.
- P_ρ is simple, as long as ρ is diffuse (has not atoms).

Clusters and Convex Polytopes

- From our point of view, a cluster X in $\mathbb{R}^d$ is a finite subset of $\mathbb{R}^d$.
- We will denote the set of clusters in $\mathbb{R}^d$ by $\mathfrak{X}_d$.
- $X \in \mathfrak{X}_d$ is called *discrete polytope* if for any $a \in X$ there exists some $(d-1)$-dimensional hyperplane that intersects with the convex hull $\langle X \rangle$ only in a.
- Obviously there is a one to one correspondence between convex polytopes and discrete ones.

Illustration of Discrete Polytopes

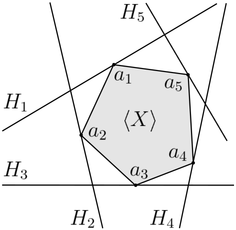


Figure: A discrete polytope

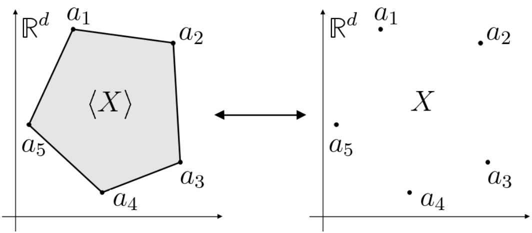


Figure: Correspondence between convex and discrete polytopes

Cluster Properties

- Now that we know about random points and discrete replacements for 'tiles', we need a way to combine them.
- Formally a *cluster property* is a (measurable) subset of $\mathfrak{X}_d \times \mathscr{M}^\bullet(\mathbb{R}^n)$.
- It should be imagined as a 'connection rule': *How do I get the tiles out of the (random) points?*
- Given a cluster property $\mathscr{D} \subset \mathfrak{X}_d \times \mathscr{M}^\bullet(\mathbb{R}^n)$ and $\eta \in \mathscr{M}^\bullet(\mathbb{R}^n), X \in \mathfrak{X}_d$ we call X a cluster of type $\mathscr{D}$ for η, if $(X, \eta) \in \mathscr{D}$.
  ('X is a proper tile in the tiling belonging to η')
- If you think of $\tilde{L}$ and Λ in a model set, it is obvious that there are nice situations with $n \neq d$.

Cluster Processes

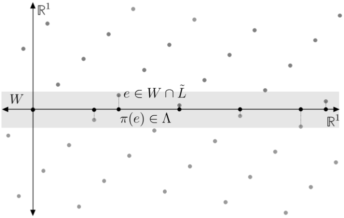
- For our construction, we need a 'nice combination' of a cluster property $\mathscr{D}$ and a point process P: For P-almost all $\eta \in \mathscr{M}^\bullet(\mathbb{R}^n)$ there are at most countably many clusters of type $\mathscr{D}$ in the fixed η.
- In the above situation we can regard the collection $\varphi_{\mathscr{D}}(\eta)$ of clusters of type $\mathscr{D}$ for a P-almost all η:

$$\varphi_{\mathscr{D}}(\eta) := \sum_{X \text{ is a cluster of type } \mathscr{D} \text{ for } \eta} \delta_X$$

- Imagine the point configuration η to be randomly chosen through the 'mechanism' P, then $\varphi_{\mathscr{D}}(\eta)$ is a 'random collection of clusters' (maybe even a tiling). Technically it is given by the image $P \circ \varphi_{\mathscr{D}}^{-1}$ of P under the mapping $\varphi_{\mathscr{D}}$.
(You need a proper σ-algebra on the sets of clusters to do that correctly.)

Outline

An Appropriate $\mathscr{D}$ for the Silver-Mean Chain



- Recall:

$$\begin{aligned} \tilde{L} &= \left\{ \left(u + v\sqrt{2}\,,\, u - \sqrt{2}\right) \,\middle|\, u, v \text{ integers} \right\} \subset \mathbb{R}^1 \times \mathbb{R}^1\,, \\ W &= \mathbb{R}^1 \times \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right], \\ \Lambda &= \pi\left(W \cap \tilde{L}\right) \subset \mathbb{R}^1\,. \end{aligned}$$

- Therefore we need $\mathscr{D} \subset \mathfrak{X}_1 \times \mathscr{M}^{\bullet}\left(\mathbb{R}^2\right)$.

An Appropriate $\mathscr{D}$ for the Silver-Mean Chain (2)

We define (X, η) to be in $\mathscr{D}$, iff

- $\operatorname{card} X = 2$,
- $X = \{a_1 = \pi(e_1)\,,\, a_2 = \pi(e_2)\}$ with $e_1, e_2 \in W \cap \eta$ and
- the intersection of the open interval (a_1, a_2) and $\pi(W \cap \eta)$ is empty.

If we take the point process P such that $P(\{\eta = \tilde{L}\}) = 1$, then $\varphi_{\mathscr{D}}(\eta)$ is our old silver-mean tiling P-almost surely. (Boring!)

Random Holes in the Lattice

- Consider the locally finite measure

$$\rho = \sum_{e \in \tilde{L}} c \cdot \delta_e \,,$$

where c is some positive constant.

- The Poisson point process with intensity ρ then has typical realisations

$$\eta = \sum_{e \in \tilde{L}} k_e(\eta) \cdot \delta_e \,,$$

where $k_e(\eta)$ is some positive integer or 0. P_ρ is not simple.

- The support of such an η is given by

$$\operatorname{supp} \eta = \sum_{e \in \tilde{L} \,,\, k_e(\eta) \neq 0} \delta_e \,,$$

Random Holes in the Lattice (2)

Let P_ρ^* be the image of P_ρ under the mapping supp.

Proposition 1

P_ρ^* is again a simple point process, where the realisations are random subsets of the lattice $\tilde{L}$. The probability for a certain point $e \in \tilde{L}$ to be in such a random subset is $1 - e^{-c}$.

This has the following consequences for the corresponding cluster process:

Proposition 2

$P_\rho^* \circ \varphi_{\mathscr{D}}^{-1}$ is a locally finite random tiling, where the tiles of a realisation have any length of the form $n + m\sqrt{2}$.

Random Holes in the Lattice (3)

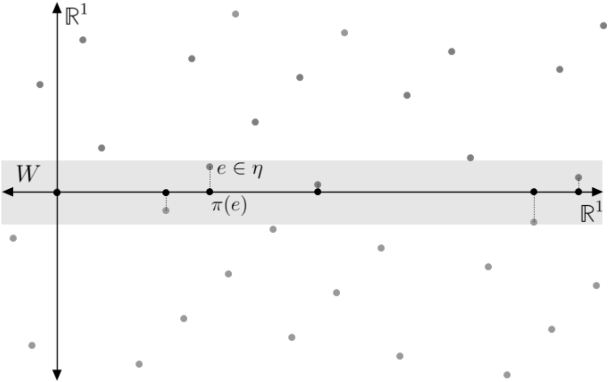


Figure: A typical realisation η of P_ρ^* and the corresponding cluster process

Randomly Shifted Points

- Let $\varepsilon > 0$, small enough that the closed balls $B_\varepsilon(e)$, $e \in \tilde{L}$ do not intersect. Now consider the barycentre mappings $\boldsymbol{b}_e : \mathscr{M}^\bullet(\mathbb{R}^2) \to \mathscr{M}^\bullet(\mathbb{R}^2)$,

$$\boldsymbol{b}_e(\eta) := \begin{cases} \frac{1}{\operatorname{card}(\eta \cap B_\varepsilon(e))} \sum_{f \in \eta \cap B_\varepsilon(e)} f\,, & \text{if } \eta \cap B_\varepsilon(e) \neq \emptyset\,, \\ e\,, & \text{if } \eta \cap B_\varepsilon(e) = \emptyset\,, \end{cases}$$

  which give the centres of weight of the points of a configuration η in $B_\varepsilon(e)$ or e if there are no points in the ball.

- Let $\boldsymbol{b}(\eta)$ be the collection of those barycentres:

$$\boldsymbol{b}(\eta) := \sum_{e \in \varLambda} \boldsymbol{b}_e(\eta)\,.$$

Randomly Shifted Points (2)

- By construction for any $\eta \in \mathscr{M}^{\bullet}(\mathbb{R}^2)$ the configuration $\boldsymbol{b}(\eta)$ has exactly one point in every ball $B_{\varepsilon}(e)$.

Thus:

Proposition 3

If you take some arbitrary simple point process P, e.g. P_{λ}, the image $P_{\boldsymbol{b}}$ of P under the mapping $\boldsymbol{b}$ randomly produces point configurations with exactly one point in every ε-ball centred in the lattice points.

Proposition 4

The corresponding cluster process $P_{\boldsymbol{b}} \circ \varphi_{\mathscr{D}}^{-1}$ again is a random 1-dimensional tiling.

Randomly Shifted Points (3)

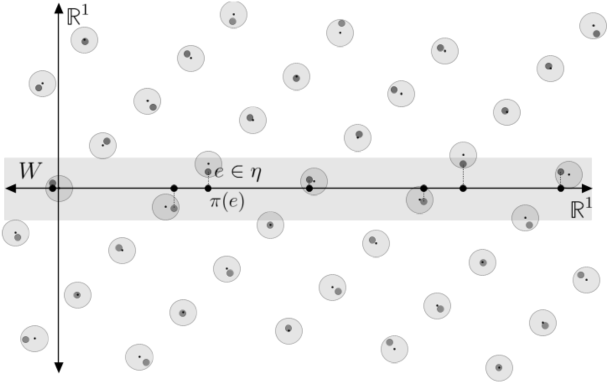


Figure: A typical realisation η of $P_{\boldsymbol{b}}$ and the corresponding cluster process

Randomly Shifted Points (4)

Some remarks:

- The corresponding tiles to the clusters for a random η typically are close to the tiles of the original silver-mean tiling, differing in length up to $4 \cdot \varepsilon$.
- Since certain ε-balls around points in $\tilde{L} \cap W^{\complement}$ intersect with W and others around points in $\tilde{L} \cap W$ intersect with $W^{\complement}$ there might be 'completely new tiles'.
- The probabilities to shift a point inside or vice versa outside of the stripe, therefore the density of the vertices stays the same as the silver-mean chain.

Thank you for your audience!

Questions or remarks?

Thank you again!