Multi-scale problems for Markov evolutions in the continuum

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NON-LINEAR PDE AND SPDE

as phenomenological macroscopic models of complex systems.

SOME TOPICS FROM MATHEMATICAL ECOLOGY:

Non-linear waves and dissipative structures in biology of populations
Non-linear diffusion equations
Ecology and catastrophes theory
Chaos in models of simple ecological systems
Ecological systems in random environment
Stochastic models in mathematical ecology

FRAMEWORK: infinite dimensional dynamical systems
Problem: derivation of meso- and macroscopic equations from microscopic models via

- scaling limits for dynamics (hydrodynamic, Vlasov, Landau etc.)
- scaling of fluctuations (equilibrium or non-equilibrium)
- closure of (infinite linear) moment systems
- hierarchical chains (BBGKY etc.)
Three levels in physics

In context of the theory of rarefied gases:

*(Mi)* is the level of particle dynamics (Newton’s laws)

*(Me)* is the level of Boltzmann description

*(Ma)* is the level of continuum description.
Interacting Particle Systems

IPS as models in
physics (gases, fluids, condensed matter)
chemical kinetics
population biology, ecology (individual based models=IBM)
sociology, economics (agent based models=ABM)

**Lattice and Continuous frameworks**
Individual Based Models in Ecology

IBM is a stochastic (Markov) process with events comprising
birth,
death,
and movement.

Ecological models:
Bolker/Pacala, 1997, ...
Dieckmann/Law/Metz, 2000, ...
............................
Meleard et al., 2004
Birch/Young, 2006
Kondratiev/Srorokhud, 2006
Finkelshtein/Kondratiev/Kutovyi, 2007-2009
Microscopic Stochastic Systems

In mathematical terms we are interested in the links between the following mathematical structures:

\( (\text{Mi}) \) the micro–scale of stochastically interacting entities (cells, individuals, . . .), in terms of continuous (linear) semigroups of Markov operators – continuous stochastic semigroups

\( (\text{Me}) \) the meso–scale of statistical entities, in terms of continuous nonlinear semigroups related to the solutions of nonlinear Boltzmann–type nonlocal kinetic equations

\( (\text{Ma}) \) the macro–scale of densities of interacting entities (in terms of dynamical systems related to reaction–diffusion type equations).
**Configuration space:**

\[ \Gamma := \{ \gamma \subset \mathbb{R}^d | |\gamma \cap \Lambda| < \infty \text{ for any compact } \Lambda \subset \mathbb{R}^d \} . \]

| · | = cardinality of the set.

**Remark:** \( \Gamma \) is a Polish space.

**n-point configuration space:**

\[ \Gamma^{(n)} := \{ \eta \subset \mathbb{R}^d | |\eta| = n \} , \quad n \in \mathbb{N}_0. \]

**Space of finite configurations:**

\[ \Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}. \]
Dynamics of configurations

Deterministic dynamics:

- Hamiltonian dynamics
- Interacting particle dynamical systems

Vlasov equation
due to Braun/Hepp, 1977 and Dobrushin, 1979:
we study asymptotic for $N \to \infty$ of the solution to

$$\frac{d x_i(t)}{dt} = A(x_i(t)) + N^{-1} \sum_{j=1}^{N} B(x_i(t) - x_j(t)),$$

$x_i \in \mathbb{R}^d$, $i = 1, \ldots, N.$
Empirical measure:

\[ \mu^N_t = \frac{1}{N} \sum_i \delta_{x_i(t)} \]

**VE for the limiting density (sic!):**

\[ \frac{\partial \rho_t(x)}{\partial t} = -Tr(\nabla_x(A(x)\rho_t(x))) - Tr\nabla_x\{\rho_t(x) \int_{\mathbb{R}^d} B(x-y)\rho_t(y)dy\}. \]

**VE for Hamiltonian dynamics from BBGKY hierarchy (heuristic derivation):**

Spohn, 1980.

Rigorous derivation meets problems: the lack of detailed knowledge about BBGKY.

**VE appeared originally in plasma physics and in the stellar dynamical problem.**
Markov evolutions in continuum:

- Diffusions (e.g., gradient diffusion)
- Jumping particles Markov processes (e.g., Kawasaki dynamics)
- Birth-and-death stochastic dynamics (e.g., Glauber, IBM in spatial ecology)
- Other stochastic IPS in $\mathbb{R}^d$

Questions:

What is possible concept of related Vlasov equations?

Is there a notion of a limiting IPS dynamics which creates Vlasov equation?
Framework: kinetic equations from stochastic dynamics (e.g., lattice stochastic dynamics and hydrodynamic scaling).

Markov dynamics of IPS $\Rightarrow$ evolution of states (measures)

Particular scalings $\Rightarrow$ kinetic equations

General discussion: see, e.g., [Dobrushin/Sinai/Suhov], 1985.

Interacting diffusions: McKean-Vlasov limit.
The projection of the Lebesgue product measure \((dx)^n\) to \((\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))\) we denote by \(m^{(n)}\). We set \(m^{(0)} := \delta_{\emptyset}\). The Lebesgue–Poisson measure \(\lambda\) on \(\Gamma_0\) is defined as
\[
\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}.
\]

Poisson measure \(\pi\) on \((\Gamma, \mathcal{B}(\Gamma))\) is given as the projective limit of the family of measures \(\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}\), where \(\pi^\Lambda := e^{-m(\Lambda)} \lambda\) is the probability measure on \((\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\). Here \(m(\Lambda)\) is the Lebesgue measure of \(\Lambda \in \mathcal{B}_b(\mathbb{R}^d)\). For any measurable function \(f : \mathbb{R}^d \to \mathbb{R}\) we define the Lebesgue–Poisson exponent
\[
e^{\lambda}(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0; \quad e^{\lambda}(f, \emptyset) := 1.
\]
The following mapping plays a key role:

\[ KG(\gamma) := \sum_{\eta \in \gamma} G(\eta), \quad \gamma \in \Gamma. \]

For any fixed \( C > 1 \) we consider the following Banach space of \( B(\Gamma_0) \)-measurable functions

\[ \mathcal{L}_C := \left\{ G : \Gamma_0 \to \mathbb{R} \left| \|G\|_C := \int_{\Gamma_0} |G(\eta)|C^{\mid\eta\mid}d\lambda(\eta) < \infty \right. \right\}. \]
Harmonic analysis on configuration space

$L : \frac{dF_t}{dt} = LF_t$

$\hat{L} := K^{-1}LK$

$K\downarrow\uparrow K^{-1}$

$\Gamma, F \xleftarrow{\langle F, \mu \rangle = \int_{\Gamma} F \, d\mu} M_{fm}^1(\Gamma)$

$\Gamma_0, G \xleftarrow{\langle G, k \rangle = \int_{\Gamma_0} G \, k \, d\lambda} \mathcal{K}(\Gamma_0)$

$L^* : \frac{d\mu_t}{dt} = L^* \mu_t$

$\hat{L}^* : \frac{dk_t}{dt} = \hat{L}^* k_t$

$K$-transform:

$KG(\gamma) := \sum_{\xi \in \gamma} G(\xi), \quad \gamma \in \Gamma.$
Markov evolutions

Let $L$ be a Markov pre-generator defined on some set of functions $\mathcal{F}(\Gamma)$ given on the configuration space $\Gamma$.

**Kolmogorov equation:**

\[
\frac{\partial F_t}{\partial t} = LF_t,
\]

\[
F_t|_{t=0} = F_0;
\]

**Duality:** $<F, \mu> := \int_\Gamma F d\mu$

**Dual Kolmogorov equation:**

\[
\frac{\partial \mu_t}{\partial t} = L^* \mu_t,
\]

\[
\mu_t|_{t=0} = \mu_0.
\]
Let
\[ \hat{L} := K^{-1}LK \]

*K*- transform or symbol of the operator \( L \).

We consider
\[ \hat{L} : D(\hat{L}) \subset \mathcal{L}_R \to \mathcal{L}_R \]
in Fock type space
\[ \mathcal{L}_R := L^1(\Gamma_0, R d\lambda) = \bigoplus_{n=0}^{\infty} L^1\left(\Gamma^{(n)}, R^{(n)}\sigma^{(n)}\right). \]
Existence of Semigroup: Suppose that \((\hat{L}, D(\hat{L}))\) is a generator of a semigroup in \(\mathcal{L}_R\):

\[
\hat{L} \rightarrow \hat{T}_t, \quad t \geq 0.
\]

Introducing duality between Banach spaces \(\mathcal{L}_R\) and

\[
\mathcal{K}_R := \{ k : \Gamma_0 \rightarrow \mathbb{R} \mid k \cdot R^{-1} \in L^\infty(\Gamma_0, \lambda_1) \}:
\]

\[
\ll G, k \rr := \int_{\Gamma_0} G \cdot k \, d\lambda_1 = \int_{\Gamma_0} G \cdot \frac{k}{R} \cdot R \, d\lambda_1, \quad G \in \mathcal{L}_R,
\]

we construct semigroup \(\hat{T}_t^*, t \geq 0\) on \(\mathcal{K}_R\):

\[
\ll \hat{T}_t G, k \rr =: \ll G, \hat{T}_t^* k \rr.
\]
Vlasov scaling

Initial distribution: $\mu_0 \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ with correlation function $k_0$. $\mu_t \in \mathcal{M}^1(\Gamma)$ the distribution of at time $t > 0$ and $k_t$ its correlation function.

\[
\begin{align*}
\frac{d\mu_t}{dt} &= L^* \mu_t \\
\mu_t \big|_{t=0} &= \mu_0,
\end{align*}
\]

where $L^*$ is the adjoint to the generator of functions

\[
\begin{align*}
\frac{dF_t}{dt} &= LF_t \\
F_t \big|_{t=0} &= F_0.
\end{align*}
\]

\[
\begin{align*}
\frac{dk_t}{dt} &= L^\Delta k_t \\
k_t \big|_{t=0} &= k_0
\end{align*}
\]

where $L^\Delta := \hat{L}^*$ is the generator of a semigroup $T_t^\Delta := \hat{T}_t^*$. 

Yuri Kondratiev (Bielefeld)
Choose the initial state of the system:

∀ε > 0 correlation functions $k_0^{(\varepsilon)}$ as $\varepsilon \to 0$:

$$k_{0,\text{ren}}^{(\varepsilon)}(\eta) := \varepsilon |\eta| k_0^{(\varepsilon)}(\eta) \to r_0(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0,$$

where correlation function $r_0$ will be chosen properly.

In the case of

$$r_0(\eta) = e^{\lambda(\rho_0, \eta)}, \quad \eta \in \Gamma_0$$

the assumption about the initial conditions means:

$$\rho_0 : \mathbb{R}^d \to (0, +\infty)$$

$$\mu^{(\varepsilon)}_{0,\text{ren}} \to \pi \rho_0,$$

where $\mu_{0,\text{ren}}^{(\varepsilon)}$ has correlation function $\varepsilon |\eta| k_0^{(\varepsilon)}(\eta)$. 
2nd step in VS

Scaling of the generator:

\[ L \mapsto L_\varepsilon. \]

The concrete type of this scaling will depend on the model.

Suppose that there exist solution of the correlation functional evolution

\[
\begin{aligned}
\frac{dk^{(\varepsilon)}_t}{dt} &= L_\varepsilon k^{(\varepsilon)}_t \\
k^{(\varepsilon)}_t \big|_{t=0} &= k^{(\varepsilon)}_0
\end{aligned}
\]

We expect (and this will be shown in concrete models) that order of the singularity in \( \varepsilon \) for this solution will be the same as for initial function \( k^{(\varepsilon)}_0 \).
3rd step in VS

We consider

\[ k_{t, \text{ren}}^{(\varepsilon)}(\eta) := \varepsilon |\eta| k_t^{(\varepsilon)}(\eta), \quad \eta \in \Gamma_0, \]

and want to show that

\[ k_{t, \text{ren}}^{(\varepsilon)}(\eta) \to r_t(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0. \]

In fact, we consider renormalized version of the evolution equation

\[
\begin{cases}
\frac{dk_{t, \text{ren}}^{(\varepsilon)}}{dt} = L_{\varepsilon, \text{ren}} k_{t, \text{ren}}^{(\varepsilon)} \\
k_{t, \text{ren}}^{(\varepsilon)}|_{t=0} = k_0^{(\varepsilon), \text{ren}}
\end{cases}
\]

where

\[ L_{\varepsilon, \text{ren}}^{\triangle} = \varepsilon |\eta| L_{\varepsilon}^{\triangle} \varepsilon^{-1} |\eta|. \]
Consider the semigroup $T^\Delta_\varepsilon(t)$ which corresponds to $L^\Delta_\varepsilon$.

Scaling $L \mapsto L_\varepsilon$ s.t. $T^\Delta_\varepsilon(t)$ preserves singularity:

$$(T^\Delta_\varepsilon(t)k_0^{(\varepsilon)})(\eta) \sim \varepsilon^{-|\eta|}r_t(\eta), \quad \varepsilon \to 0, \ \eta \in \Gamma_0.$$

\forall \ \varepsilon > 0 consider a mapping of functions on $\Gamma_0$

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|}r(\eta),$$

$$R^{-1}_\varepsilon = R_{\varepsilon^{-1}}.$$

Then $k_0^{(\varepsilon)} \sim R_{\varepsilon^{-1}}r_0$, and we need

$$r_t \sim R_\varepsilon T^\Delta_\varepsilon(t)k_0^{(\varepsilon)} \sim R_\varepsilon T^\Delta_\varepsilon(t)R_{\varepsilon^{-1}}r_0.$$
To show: for any $t \geq 0$ the operator family

$$R_\varepsilon T_\varepsilon \triangle(t)R_{\varepsilon^{-1}}, \ \varepsilon > 0,$$

has a limiting operator $U(t)$.

But

$$R_\varepsilon T_\varepsilon \triangle(t)R_{\varepsilon^{-1}} = \exp \{t R_\varepsilon L_\varepsilon \triangle R_{\varepsilon^{-1}}\}.$$

We search for an operator $V \triangle$ such that

$$\exp \{t R_\varepsilon L_\varepsilon \triangle R_{\varepsilon^{-1}}\} \to \exp \{t V \triangle\} =: U(t).$$

Weak limit of

$$L_\varepsilon, \text{ren} := R_\varepsilon L_\varepsilon \triangle R_{\varepsilon^{-1}}$$

will be a proper candidate for $V \triangle$. 
We want to show that the solution $k_{t, \text{ren}}^{(\varepsilon)}$ converges to $r_t$ which satisfied

\[ \begin{align*}
\frac{dr_t}{dt} &= V \triangle r_t \\
[r_t]_{t=0} &= r_0
\end{align*} \]

This equation describes an evolution of a virtual interacting particle system appearing in the Vlasov limit.
Consider the case of an initial Poisson measure:

\[ r_0(\eta) = e^{\lambda(\rho_0, \eta)}. \]

Under some general conditions, the scaling leads to the solution \( r_t \) of the same form:

\[ r_t(\eta) = e^{\lambda(\rho_t, \eta)}, \quad \eta \in \Gamma_0. \]

The Vlasov hierarchical equation in this case implies a non-linear equation for \( \rho_t \):

\[ \frac{\partial}{\partial t} \rho_t(x) = v(\rho_t)(x), \quad x \in \mathbb{R}^d, \]

which we will call \textit{Vlasov-type equation} corresponding to the considered Markov evolution.
Derivation of Vlasov hierarchies

Birth, death and hopping evolutions

Two type continuous models: the **birth-and-death** generator $L_{\text{bad}} = L^- + L^+$ and the **hopping** generator $L_{\text{hop}}$, where

\[
\begin{align*}
(L^- F)(\gamma) &:= \sum_{x \in \gamma} d(x, \gamma \setminus x) \left[ F(\gamma \setminus x) - F(\gamma) \right], \\
(L^+ F)(\gamma) &:= \int_{\mathbb{R}^d} b(x, \gamma) \left[ F(\gamma \cup x) - F(\gamma) \right] dx, \\
(L_{\text{hop}} F)(\gamma) &:= \sum_{x \in \gamma} \int_{\mathbb{R}^d} c(x, y, \gamma) \left[ F(\gamma \setminus x \cup y) - F(\gamma) \right] dy.
\end{align*}
\]
Denote $L^- = L^-(d)$, $L^+ = L^+(b)$, $L_{\text{hop}} = L_{\text{hop}}(c)$.

We will use scaling of rates $b, d, c$, say, $b_\varepsilon, d_\varepsilon, c_\varepsilon$, correspondingly, $\varepsilon > 0$.

Scaling of $L_{\text{bad}}$ and $L_{\text{hop}}$:

\[
L_{\text{bad}, \varepsilon} = L^-(d_\varepsilon) + \varepsilon^{-1}L^+(b_\varepsilon),
\]

\[
L_{\text{hop}, \varepsilon} = L_{\text{hop}}(c_\varepsilon).
\]

General conditions for the weak convergence of $L_{\varepsilon}^\triangle$ to the limiting Vlasov generator $V^\triangle$ considered in

Birth-and-death systems

Example (Contact model = branching with mortality)

\[
(LF)(\gamma) = m \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)]
+ \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) [F(\gamma \cup y) - F(\gamma)] dy;
\]

\[(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + \lambda (\rho_t * a)(x).\]

Scaling:

\[
\lambda \mapsto \varepsilon^{-1} \lambda, \quad a \mapsto \varepsilon a
\]
### Example (Migration model)

\[(LF)(\gamma) = \sum_{x \in \gamma} \sum_{y \in \gamma \setminus x} a(x - y)\left[ F(\gamma \setminus x) - F(\gamma) \right] \]

\[+ \sigma \int_{\mathbb{R}^d} \left[ F(\gamma \cup x) - F(\gamma) \right] dx; \]

\[\text{(VE)} : \quad \frac{\partial}{\partial t} \rho_t(x) = -\rho_t(x)(\rho_t * a)(x) + \sigma. \]

### Scaling:

\[a \mapsto \varepsilon a \]

\[\sigma \mapsto \varepsilon^{-1} \sigma \]
Example (Bolker–Pacala model in spatial ecology)

\[(LF)(\gamma) = \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^-(x - y) \right) \left[ F(\gamma \setminus x) - F(\gamma) \right] \]
\[+ \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x - y) \left[ F(\gamma \cup y) - F(\gamma) \right] dy; \]

\[(VE) : \frac{\partial}{\partial t} \rho_t(x) = -m\rho_t(x) - \rho_t(x)(\rho_t \ast a^-)(x) + (\rho_t \ast a^+)(x). \]

Scaling:
\[a^- \mapsto \varepsilon a^- \]
\[a^+ \mapsto \varepsilon^{-1} \varepsilon a^+ = a^+. \]
Example (Ecological model with establishment)

\[(LF)(\gamma) = m \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)]\]
\[+ \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y)e^{-\sum_{u \in \gamma} \phi(y - u)}[F(\gamma \cup y) - F(\gamma)]dy;\]

\[(VE): \quad \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + \lambda (a \ast \rho_t)(x)e^{-(\phi \ast \rho_t)(x)}.\]

Scaling:

\[a \mapsto \varepsilon a, \quad \phi \mapsto \varepsilon \phi\]
\[\lambda \mapsto \varepsilon^{-1} \lambda\]
**Example (Ecological model with fecundity)**

\[(LF)(\gamma) = m \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)] + \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) e^{-\sum_{u \in \gamma \setminus x} \phi(x - u)} [F(\gamma \cup y) - F(\gamma)] dy;\]

\[(VE) : \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) + \lambda (a \ast (\rho_t e^{-\phi \rho_t}))(x).\]

**Scaling:**

\[a \mapsto \varepsilon a, \quad \phi \mapsto \varepsilon \phi\]

\[\lambda \mapsto \varepsilon^{-1} \lambda\]
Example (Dieckmann--Law model)

\[
(LF)(\gamma) = \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^-(x - y) \right) \left[ F(\gamma \setminus x) - F(\gamma) \right] + \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x - y) \left( \lambda + \sum_{u \in \gamma \setminus x} b(x - u) \right) \left[ F(\gamma \cup y) - F(\gamma) \right] dy;
\]

\[
(VE): \quad \frac{\partial}{\partial t} \rho_t(x) = -m \rho_t(x) - \rho_t(x)(\rho_t * a^-)(x) + \lambda(\rho_t * a^+)(x) + (((b * \rho_t) \rho_t) * a^+)(x).
\]

Scaling:

\[
 a^- \mapsto \varepsilon a^-, \quad a^+ \mapsto \varepsilon^{-1} \varepsilon a^+ = a^+.
\]

\[
b \mapsto \varepsilon b
\]
Example (Glauber $G^+$ dynamics)

$$(LF)(\gamma) = \sum_{x \in \gamma} [F(\gamma \setminus x) - F(\gamma)]$$

$$+ z \int_{\mathbb{R}^d} e^{-\sum_{u \in \gamma} \phi(y-u)} [F(\gamma \cup y) - F(\gamma)] dy;$$

$$(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = -\rho_t(x) + ze^{-(\rho_t * \phi)(x)}.$$

Scaling:

$$\phi \mapsto \varepsilon \phi$$

$$z \mapsto \varepsilon^{-1} z$$
Stationary VE for Glauber dynamics

Stationary hierarchy for stochastic Glauber dynamics = Kirkwood-Salsburg equation (K/Oliveira,'06)

KSE = Gibbs state = Bogoliubov equation for generating functional (all appear from Hamiltonian dynamics).

Stationary solution to VE satisfies

\[ \rho(x) = ze^{-\left(\rho\ast\phi\right)(x)}. \]

The latter in nothing but the well-known Kirkwood-Monroe equation introduced in 1941 (!) in the theory of freezing.

We see a dynamical source of the Kirkwood-Monroe equation.
Example (Glauber $G^-$ dynamics)

$$(LF)(\gamma) = \sum_{x \in \gamma} e^{\sum_{u \in \gamma} \phi(x-u)} [F(\gamma \setminus x) - F(\gamma)]$$

$$+ z \int_{\mathbb{R}^d} [F(\gamma \cup y) - F(\gamma)]dy;$$

$$(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = -\rho_t(x)e^{(\rho_t*\phi)(x)} + z.$$ 

Scaling:

$$\phi \mapsto \varepsilon \phi$$

$$z \mapsto \varepsilon^{-1} z$$
Conservative particle systems

Example (Free Kawasaki)

\[(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) \left[ F(\gamma \setminus x \cup y) - F(\gamma) \right] dy;\]

\[(VE): \quad \frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x) - \rho_t(x) \langle a \rangle.\]
Example (Density dependent Kawasaki)

\[(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) \sum_{u \in \gamma} b(x, y, u) [F(\gamma \setminus x \cup y) - F(\gamma)] \, dy;\]

\[\frac{\partial}{\partial t} \rho_t(x) = \int_{\mathbb{R}^d} \rho_t(y) a(x - y) \int_{\mathbb{R}^d} \rho_t(u) b(y, x, u) \, dudy\]

\[- \rho_t(x) \int_{\mathbb{R}^d} a(x - y) \int_{\mathbb{R}^d} \rho_t(u) b(x, y, u) \, dudy.\]

In particular, if \(b(x, y, u) = b(x - u)\) then

\[(VE) : \quad \frac{\partial}{\partial t} \rho_t(x) = (\rho_t(\rho_t * b)) * a)(x) - \langle a \rangle \rho_t(x) (\rho_t * b)(x).\]

If \(b(x, y, u) = b(y - u)\) then

\[\frac{\partial}{\partial t} \rho_t(x) = (\rho_t * b)(x) (\rho_t * a)(x) - \rho_t(x) (\rho_t * a * b)(x).\]
Example (Gibbs--Kawasaki)

\[(LF')(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x - y) e^{-E\phi(x,\gamma)} \left[ F(\gamma \setminus x \cup y) - F(\gamma) \right] dy;\]

\[(VE): \quad \frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x) \exp \left\{ - (\rho_t * \phi)(x) \right\} \]
\[\quad - \rho_t(x) (a * \exp \{ - \rho_t * \phi \})(x).\]

Scaling:

\[\phi \mapsto \varepsilon \phi\]
Example (Gradient diffusion)

\[ (LF)(\gamma) = \sum_{x \in \gamma} \Delta x F(\gamma) - \sum_{x \in \gamma} \sum_{y \in \gamma \setminus x} \langle \nabla \phi(x - y), \nabla_x F \rangle \]

\[ (VE) \quad \frac{\partial}{\partial t} \rho_t(x) = \Delta \rho_t(x) - \int \phi(x - y) \langle \nabla \rho_t(x), \nabla \rho_t(y) \rangle \, dy \]

\[ -\rho_t(x) \int \langle \nabla \phi(x - y), \nabla \rho_t(y) \rangle \, dy \]

Scaling:

\[ \phi \mapsto \varepsilon \phi \]
Convergence problem

In all models above the weak convergence of the generators

\[ L_{\varepsilon}^{\Delta} \to V^{\Delta}, \quad \varepsilon \to 0 \]

is proven.

A difficult question: convergence of solutions of hierarchical equations to the Vlasov hierarchy solution.

Known results (Finkelshtein/K/Kutoviy) concern

- Contact Model
- Glauber Dynamics
- BDLP Model in spatial ecology
Potts model

Consider a continuous system consisting of two types of particles:

\[ \gamma^+ \in \Gamma^+ = \Gamma(\mathbb{R}^d), \quad \gamma^- \in \Gamma^- = \Gamma(\mathbb{R}^d). \]

Gibbs measure (heuristic)

\[ d\mu(\gamma^+, \gamma^-) = \frac{1}{Z} \exp(-\beta \sum_{x \in \gamma^+, y \in \gamma^-} \phi(x - y)) d\pi_z(\gamma^+) d\pi_z(\gamma^-). \]

Here \( \beta, z > 0, \ \phi \geq 0. \)
Glauber dynamics in Potts model

Generator

$$(LF)(\gamma^+, \gamma^-) = \sum_{x \in \gamma^+} [F(\gamma^+ \setminus x, \gamma^-) - F(\gamma^+, \gamma^-)]$$

$$+ z \int_{\mathbb{R}^d} e^{-\beta \sum_{u \in \gamma^-} \phi(y-u)} [F(\gamma^+ \cup y, \gamma^-) - F(\gamma^+, \gamma^-)] dy$$

$$+ \text{symm.} \ (\leftrightarrow)$$
Scaling of generator:

\[ \phi \mapsto \varepsilon \phi, \quad z \mapsto \varepsilon^{-1} z. \]

Vlasov equation:

\[
\begin{align*}
\frac{\partial}{\partial t} \rho_t^+(x) &= -\rho_t^+(x) + z e^{-\beta (\phi^* \rho_t^-)}(x) \\
\frac{\partial}{\partial t} \rho_t^-(x) &= -\rho_t^-(x) + z e^{-\beta (\phi^* \rho_t^+)}(x)
\end{align*}
\]
Consider space-homogeneous Vlasov equation (SHVE):

\[
\begin{align*}
\frac{\partial}{\partial t} \rho^+_t &= -\rho^+_t + ze^{-\beta<\phi>} \rho^-_t \\
\frac{\partial}{\partial t} \rho^-_t &= -\rho^-_t + ze^{-\beta<\phi>} \rho^+_t
\end{align*}
\]

Denote \( a := z\beta <\phi> \).

**Theorem (F/K/K,’09)**

If \( a \leq e \), then SHVE has a unique stationary point \((x_0, x_0)\).

If \( a > e \) we have exactly three stationary points

\[
(x_1, x_3), \ (x_2, x_2), \ (x_3, x_1).
\]

All stationary points of SHVE are stable focuses.
Therefore, the Vlasov hierarchy exhibits the dynamical phase transition. For low temperature-high density regime corresponding virtual particle system has broken $\mathbb{Z}_2$-symmetry.

The latter may be considered a sign of a phase transition in the initial Potts model. Actually, this phase transition was shown by H.-O.Georgii et.al. using the percolation techniques. Our approach may lead to the dynamical explanation of this phenomena.
(\text{I}\!\!F)(\gamma^+, \gamma^-) := \sum_{x \in \gamma^+} \left( m^+ + \sum_{x' \in \gamma^+ \setminus x} a_1^-(x - x') + \sum_{y \in \gamma^-} b_1^-(x - y) \right) 
\times \left[ F(\gamma^+ \setminus x, \gamma^-) - F(\gamma^+, \gamma^-) \right]

+ \sum_{y \in \gamma^-} \left( m^- + \sum_{y' \in \gamma^- \setminus y} a_2^-(y - y') + \sum_{x \in \gamma^+} b_2^-(x - y) \right) 
\times \left[ F(\gamma^+, \gamma^- \setminus y) - F(\gamma^+, \gamma^-) \right]
\[\begin{align*}
\sum_{x' \in \gamma^+} \int_{\mathbb{R}^d} & \left( a_1^+ (x - x') + \sum_{y \in \gamma^-} b_1^+ (y, x, x') \right) \\
& \left[ F(\gamma^+ \cup x, \gamma^-) - F(\gamma^+, \gamma^-) \right] \, dx \\
\sum_{y' \in \gamma^-} \int_{\mathbb{R}^d} & \left( a_2^+ (y - y') + \sum_{x \in \gamma^+} b_2^+ (x, y, y') \right) \\
& \left[ F(\gamma^+, \gamma^- \cup y) - F(\gamma^+, \gamma^-) \right] \, dy.
\end{align*}\]
Scaling:

\[ a_{1,\varepsilon}^{\pm} = \varepsilon a_{1}^{\pm}, \quad b_{1,2,\varepsilon}^{-} = \varepsilon b_{1,2}^{-} \]

\[ b_{1,2,\varepsilon}^{+} = \varepsilon^2 b_{1,2}^{+} \]

Birth intensity:

\[ 1 \mapsto \varepsilon^{-1} \]
Vlasov equation in ecology

\[
\frac{\partial}{\partial t} \rho_t^+ (x) = -m^+ \rho_t^+ (x) - \rho_t^+ (x) (a_1^- * \rho_t^+) (x) \\
\quad - \rho_t^+ (x) (\rho_t^- * b_1^-) (x) + (\rho_t^+ * a_1^+) (x) \\
\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_t^+ (x') \rho_t^- (y) b_1^+ (y, x, x') \, dx' \, dy
\]

and

\[
\frac{\partial}{\partial t} \rho_t^- (y) = -m^- \rho_t^- (y) - \rho_t^- (y) (a_1^+ * \rho_t^-) (y) \\
\quad - \rho_t^- (y) (\rho_t^+ * b_1^+) (y) + (\rho_t^- * a_1^-) (y) \\
\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_t^- (x) \rho_t^+ (y') b_1^+ (x, y, y') \, dx \, dy'
\]
Relations with jump generators

\[(\rho * a)(x) = \int a(x-y)\rho(y)dy =
\]
\[
\int a(x-y)(\rho(y) - \rho(x))dy + <a> \rho(x) = L_a \rho(x) + <a> \rho(x).
\]
In translation invariant case

\[
\begin{align*}
\frac{d}{dt} \rho^+_t &= (a_1^+ - m^+) \rho^+_t - a_1^- (\rho^+_t)^2 + (b_1^+ - b_1^-) \rho^+_t \rho^-_t \\
\frac{d}{dt} \rho^-_t &= (a_2^+ - m^-) \rho^-_t - a_2^- (\rho^-_t)^2 + (b_2^+ - b_2^-) \rho^+_t \rho^-_t.
\end{align*}
\]