Cluster Algebras

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Chapter 1

Informal introduction

1.1 Sequences of Laurent polynomials

In the lecture we wish to give an introduction to Sergey Fomin and Andrei Zelevinsky's theory of *cluster algebras*. Fomin and Zelevinsky have introduced and studied cluster algebras in a series of four influential articles [FZ, FZ2, BFZ3, FZ4] (one of which is coauthored with Arkady Berenstein). Although their initial motivation comes from Lie theory, the definition of a cluster algebra is very elementary.

We will give the precise definition in Chapter 2, but to give the reader a first idea let a and b be two undeterminates and let us consider the map

$$F\colon (a,b) \mapsto \left(b, \frac{b+1}{a}\right).$$

Surprisingly, the non-trivial algebraic identity $F^5 = \mathrm{id}$ holds. This equation has a long and colourful history. The equation in this form dates back to Arthur Cayley [Ca] who remarks that this equation was already essentially known to Carl-Friedrich Gauß [Ga] in the context of spherical geometry and Napier's rules. Latin speaking Gauß refers to the phenomenon as *pentagramma mirificum*, which the author would translate as marvelous pentagram.

But not only the periodicity $F^5 = \mathrm{id}$ after five steps is remarkable. Another interesting feature is the *Laurent phenomenon*: all terms occurring on the way are actually Laurent polynomials in the initial values a and b. Figure 1.1 illustrates the five Laurent polynomials.

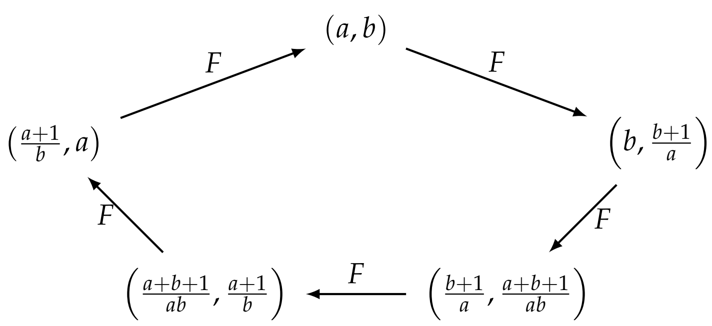


Figure 1.1: A pentagon of Laurent polynomials

Fomin-Reading [FR, Section 1.1] provide a variation of the theme. For a natural number $i \in \mathbb{N}$ let F_i be the map that we obtain from F by replacing the polynomial $b+1$ in the numerator of

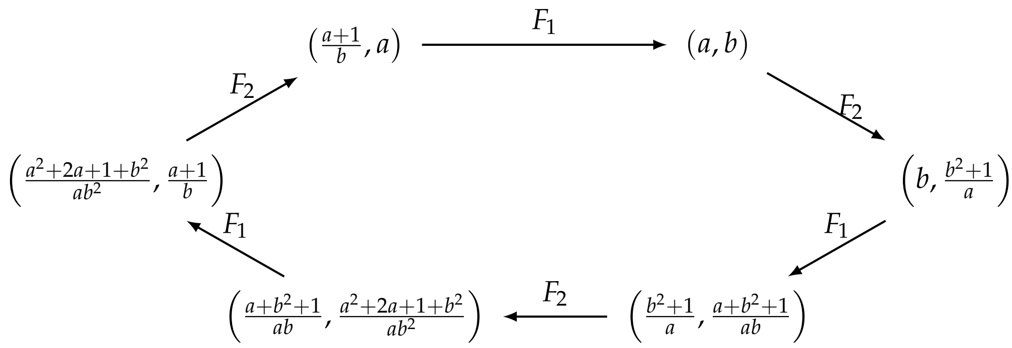


Figure 1.2: A hexagon of Laurent polynomials

the second entry with the polynomial $b^i + 1$. Surprisingly, we have the relations $(F_1F_2)^3 = \mathrm{id}$ and $(F_1F_3)^4 = \mathrm{id}$. Furthermore, all occurring terms are again Laurent polynomials as Figure 1.2 indicates in the case of F_1 and F_2.

Various mathematicians have developed different approaches to these marvels. Whereas dynamical system theorists try to explain the Laurent phenomenon by the notion of *algebraic entropy*, cf. Hone [Ho, Section 2], Fomin and Zelevinsky wish to explain it by Lie theory. The authors observe that the exponents in the denominators $a^r b^s$ (with $r, s \geq 0$) of the non-initial variables yield vectors $(r, s) \in \mathbb{R}^2$ that have an interpretation in terms of *root systems*. In Figure 1.1 we get the vectors $(1, 0)$, $(1, 1)$ and $(0, 1)$ and in Figure 1.2 we get the vectors $(1, 0)$, $(1, 1)$, $(1, 2)$ and $(0, 1)$. These are the coordinates of the positive roots in the basis of simple roots for root systemss of type A_2 and B_2. For the maps F_1 and F_3 we will get a corresponding interpretation in terms of the root system G_2. The term root system (and the notion of positive and simple roots) will be made precise during the lecture. Roughly speaking this means that every reflection across a line orthogonal to one of the vectors preserves the configuration of vectors. Figure 1.3 displays the three root systems and the positive roots are coloured gray.

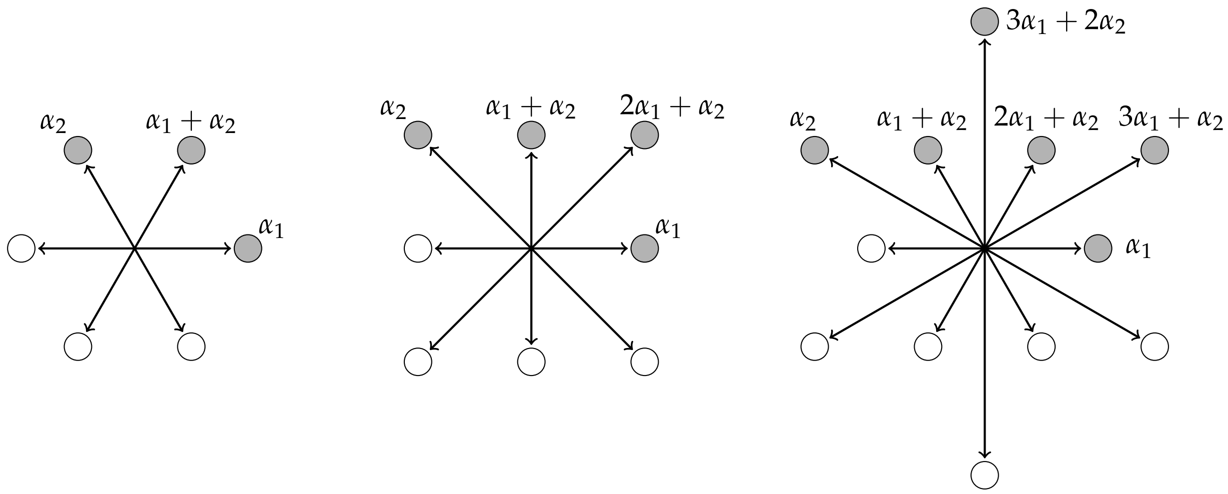


Figure 1.3: The root systems of type A_2, B_2 and G_2

Readers familiar with the work of H. S. M. Coxeter [Co1, Co2] might also notice that the relations $(F_1F_2)^3 = \mathrm{id}$ and $(F_1F_3)^4 = \mathrm{id}$ resemble relations in certain Coxeter groups.

At the beginning of the lecture we will provide a precise definition of a *cluster algebra*. It is a commutative algebra which is generated by so-called *cluster variables*. We obtain the relations among the cluster variables by generalising the recursions in the above examples and by applying the recursions to more general input data. We can encode the relations among the cluster variables by *quivers* and *mutations*.

As the definition of a cluster algebra is very abstract, we want to discuss examples. Most importantly, we will discuss cluster algebras of rank 2, which serve as a good prototype for the whole theory. In this case, the cluster variables form a sequence and we can describe the relations by two parameters. The parameters are positive integers and usually called b and c. If $|bc| \leq 3$, then there are only finitely many cluster variables (and the three cases $bc = 1, 2, 3$ yield the examples above). We also study the case $b = c = 2$. Zelevinksy [Ze, Equation 13] observes that the non-linear exchange relations for cluster variables degenerates to a linear recurrence relation and Caldero-Zelevinsky [CZ, Theorem 4.1] give interesting formulae for the coefficients. A second class of examples that we will consider are cluster algebras of type A, whose structure admits an interpretation by triangulations of polygons. Here, the relations among cluster variables become *Ptolemy relations*.

After the examples, we will state and prove Fomin-Zelevinsky's two main theorems, namely the classification of cluster algebras of finite type and the Laurent phenomenon. As indicated above we can classify cluster algebras of finite type by finite type root systems and Dynkin diagrams.

The core of the lecture will be connections to other fields of mathematics. First of all, the Caldero-Chapoton map [CC] connects cluster algebras with representations of quivers, where the quivers of finite representation type are also classified by Dynkin diagrams. Furthermore, we will study the connection to Lie theory. Especially, we will study canonical bases and totally positive matrices. Both topics had been a main motivation for the defining relations of a cluster algebra.

As the reader might have noticed, explicit calculation can not be avoided and we will see how the computer algebra software SAGE might help. If time permits, we will discuss cluster varieties.

1.2 Exercises

Exercise 1.1. Verify the equation $(F_1F_3)^4 = \mathrm{id}$ and draw the corresponding octogon. What exponents occur in the denominators?

Exercise 1.2. By considering the orbit of $(1, 1)$ prove that F_2 is of infinite order, i.e. there does not exist a natural number k such that $F_2^k = \mathrm{id}$.

Chapter 2

What are cluster algebras?

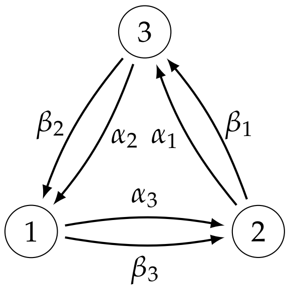
2.1 Quivers and adjacency matrices

2.1.1 Quivers

In this section we wish to introduce quivers. Quivers turn out o be a crucial tool to construct cluster algebras. We start with the following definition.

Definition 2.1.1 (Quiver)**.** A *quiver* is a tuple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are finite sets and $s, t\colon Q_1 \to Q_0$ are arbitrary maps. If Q is a quiver, then elements in the set Q_0 will be called *vertices* and elements in the set Q_1 will be called *arrows*. For an arrow $\alpha \in Q_1$, we refer to the vertex $s(\alpha) \in Q_0$ as the *starting point* and to the vertex $t(\alpha) \in Q_1$ as the *terminal point* of α.

It is very convenient to visualize a quiver by a picture. For every vertex we draw a point in the plane and we connect the points by corresponding arrows. Figure 2.1 shows two examples. We call such a picture a *drawing* of the quiver in the plane. Often we use natural numbers for vertices and small greek letters for arrows.



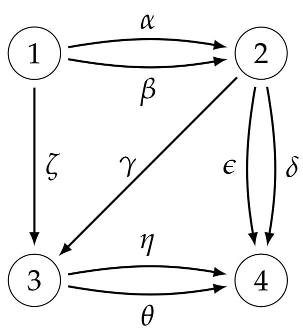


Figure 2.1: Two quivers

The readers familiar with discrete mathematics might know the concept under the name directed graph. Pierre Gabriel has introduced the terminology quiver in the context of quiver representation (which we will study in later). The idea for introducing a new word for well-known concept is emphasize a new and different way to look at the concept. Cluster theorists (who often have a background in representation theory) have adopted the terminology.

In the rest of the section we wish to introduce some notions which will become important in later chapters. Most notions are classical and have originated from graph theory.

Definition 2.1.2 (Isomorphism)**.** Let $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q_0', Q_1', s', t')$ be two quivers. An *isomorphism* between Q and Q' is a pair (f_0, f_1) of bijective maps $f_0 \colon Q_0 \to Q_0'$ and $f_1 \colon Q_1 \to Q_1'$ such that the diagrams

$$\begin{array}{ccc} Q_1 & \xrightarrow{s} & Q_0 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_1'} \\ Q_1' & \xrightarrow{s'} & Q_0' \end{array} \qquad\qquad \begin{array}{ccc} Q_1 & \xrightarrow{t} & Q_0 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_1'} \\ Q_1' & \xrightarrow{t'} & Q_0' \end{array}$$

commute, i.e. for all arrows $\alpha \in Q_1$ we have $f_0(s(\alpha)) = s'(f_1(\alpha))$ and $f_0(t(\alpha)) = t'(f_1(\alpha))$. If there is an isomorphism between Q and Q', then the quiver Q and Q' are said to be *isomorphic*. In this case we will write $Q \cong Q'$.

The name isomorphism is of Greek origin and means the *having the same structure*. Informally speaking, two quivers are isomorphic and only if the quivers have the same structure in the sense that we can obtain one quiver from the other by renaming the vertices (via the map f_0) and edges (via the map f_1). The notion clearly induces an equivalence relation.

Definition 2.1.3 (Sinks and sources)**.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A vertex $i \in Q_0$ is called a *source* if there is no arrow $\alpha \in Q_1$ with $t(\alpha) = i$. A vertex $i \in Q_0$ is called a *sink* if there is no arrow $\alpha \in Q_1$ with $s(\alpha) = i$.

For example, in the second quiver of Figure 2.1 the vertex 1 is a source and the vertex 4 is a sink. The first quiver contains neither sources nor sinks.

Definition 2.1.4 (Path)**.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver. If $m \geq 1$ is a positive integer, then a sequence $p = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in Q_1^m$ of arrows such that $t(\alpha_k) = s(\alpha_{k+1})$ for all $k \in \{1, 2, \ldots, m-1\}$ will be called a *path of length m* in Q. In this case we will call the vertex $s(\alpha_1)$ the *starting point* of p and vertex $t(\alpha_m)$ the *terminal point* of p and we will write $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_m)$. For every vertex $i \in Q_0$ we introduced a *lazy path e_i* of length 0 and we set $s(e_i) = t(e_i) = i$. A path p is called *closed* if $s(p) = t(p)$. A closed path of length 1 is called a *loop*.

For example, the sequence $(\alpha_1, \alpha_2, \alpha_3)$ of arrows in the first quiver of Figure 2.1 is a closed path with starting and terminal point 2, the sequence (α_1, α_2) is a path with starting point 2 and ending point 1 and the sequence (α_2, α_1) is not a path.

Definition 2.1.5 (Cycle)**.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver. Two closed paths $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $(\alpha_1', \alpha_2', \ldots, \alpha_m')$ of the same length $m \geq 1$ in Q are *equivalent* if there is an integer $k \in \{1, 2, \ldots, m\}$ such that $(\alpha_1', \alpha_2', \ldots, \alpha_m') = (\alpha_k, \alpha_{k+1}, \ldots, \alpha_m, \alpha_1, \alpha_2, \ldots, \alpha_{k-1})$. Let $m \in \mathbb{N}$ be a positive integer. An *oriented cycle of length m* or an m-cycle is an equivalence class of a path of length m. An oriented cycle of length 3 is called a *triangle*.

For example, the three sequences $(\alpha_1, \alpha_2, \alpha_3)$, $(\alpha_2, \alpha_3, \alpha_1)$ and $(\alpha_3, \alpha_1, \alpha_1)$ in the first quiver of Figure 2.1 are equivalent paths. Their equivalence class is a triangle. Altogether, there are $2^3 = 8$ triangles in the quiver Q.

Definition 2.1.6 (Subquiver)**.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A quiver $Q' = (Q_0', Q_1', s', t')$ with $Q_0' \subseteq Q_0$ and $Q_1' = Q_1$ is called a subquiver of Q if for every arrow $\alpha \in Q_1'$ the starting and terminal points $s(\alpha), t(\alpha) \in Q_0'$ and satisfy equations $s'(\alpha) = s(\alpha)$ and $t'(\alpha) = t(\alpha)$. A subquiver Q' of Q is called *full* if every arrow $\alpha \in Q_1$ with $s(\alpha), t(\alpha) \in Q_0'$ lies in Q_1'.

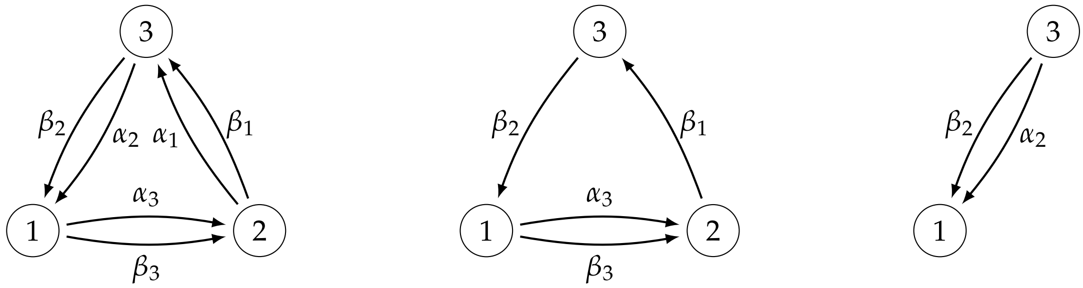


Figure 2.2: Subquivers

For example, Figure 2.2 displays a (well-known) quiver Q, a subquiver Q' with $Q'_0 = \{1,2,3\}$ and $Q'_1 = \{\alpha_3, \beta_1, \beta_2, \beta_3\}$ (which is not full) and a full subquiver Q'' with $Q''_0 = \{1,3\}$. Clearly, a full subquiver is uniquely determined by its set of vertices.

Definition 2.1.7 (Acyclic)**.** A quiver $Q = (Q_0, Q_1, s, t)$ is called *acyclic* if it contains no oriented cycle (or equivalently, if it contains no closed path).

If a quiver contains a loop, then it cannot be acyclic. For another example, the first quiver in Figure 2.1 is not acyclic as it admits various oriented cycles. The second quiver in the figure is acyclic. It is easy to see that a quiver is acyclic if and only if it contains only finitely many paths.

Proposition 2.1.8. Let $Q = (Q_0, Q_1, s, t)$ be an acyclic quiver with $n = |Q_0|$ vertices. Then Q is isomorphic to a quiver $Q' = (Q'_0, Q'_1, s', t')$ with $Q'_0 = \{1, 2, \ldots, n\}$ such that for every arrow $\alpha' \in Q'_1$ we have $s'(\alpha') < t'(\alpha')$.

We refer to such a numbering of the vertices of an acyclic quiver as a *topological ordering*. The vertices of the second quiver in Figure 2.1 are in topological order.

Proof. We prove the statement by mathematical induction on the number n of vertices. If $n = 1$, then Q contains no arrows as it contains no loops. In this case Q is isomorphic to the quiver with one vertex 1 and no arrows. Now let $n \geq 2$. Among all of the finitely many paths in Q, we consider a path p of maximal length. The terminal point $t(p)$ must be a sink in Q, because otherwise we could form a path of longer length. By induction hypothesis we know that the full subquiver Q' with the set $Q'_0 = Q_0 \backslash \{t(p)\}$ as vertices admits a topological ordering. Rename the vertices of Q'_0 by $\{1, 2, \ldots, n-1\}$ (according to the topological order on Q') and rename $t(p)$ by n. The result is a topological ordering on Q. □

Definition 2.1.9 (Connected)**.** We say that a quiver $Q = (Q_0, Q_1, s, t)$ is *disconnected* if there exists a partition $Q_0 = Q'_0 \sqcup Q''_0$ of the set of vertices into two disjoint and non-empty sets such that the starting and terminal points $s(\alpha), t(\alpha)$ of every arrow $\alpha \in Q_1$ belong to same part of the partition, i.e. we either have $s(\alpha), t(\alpha) \in Q'_0$ or we have $s(\alpha), t(\alpha) \in Q''_0$. We say that Q is *connected* if it is not disconnected.

Of course, a quiver is connected if and only if every drawing of Q in the plane is connected in the topological sense. We finish the section by two examples. The names come from Lie theory and will be explained later.

Example 2.1.10. Let $n \geq 1$ be an integer. Often we consider the following two quivers, which are also shown in Figures 2.3, 2.4.

(a) The quiver $Q = (Q_0, Q_1, s, t)$ with $Q_0 = \{1, 2, \ldots, n\}$ and $Q_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ such that $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$ for all indices $i \in \{1, 2, \ldots, n-1\}$ is called the *linearly oriented quiver of type A_n*.

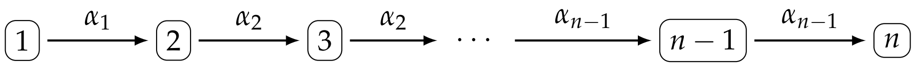


Figure 2.3: The linearly oriented quiver of type A

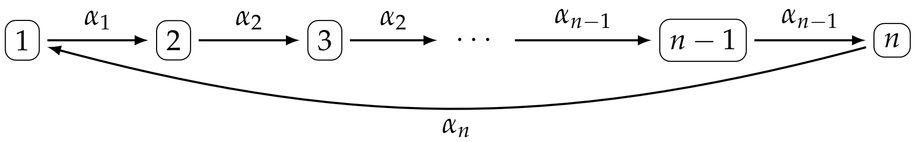


Figure 2.4: A circle

(b) The quiver $Q = (Q_0, Q_1, s, t)$ with $Q_0 = \{1, 2, \ldots, n\}$ and $Q_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $s(\alpha_i) = i$ for all $i \in \{1, 2, \ldots, n\}$, $t(\alpha_i) = i + 1$ for all $i \in \{1, 2, \ldots, n-1\}$ and $t(\alpha_n) = 1$ is called a *circle* of length n.

2.1.2 Signed and non-signed adjacency matrices

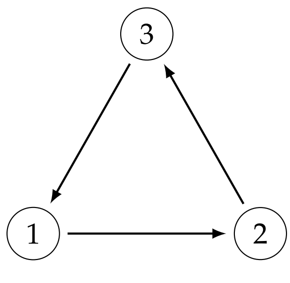
Sometimes – especially in computer science – it is convenient to encode a quiver by a matrix. This can be done in two different ways, namely by incidence matrices, which encode incidence relations between vertices and arrows, and by adjacency matrices, which encode adjacency relations between the vertices. For our purposes adjacency matrices will become the most important way to encode quivers.

Definition 2.1.11. (a) Let $Q = (Q_0, Q_1, s, t)$ be an arbitrary quiver with $n = |Q_0|$ vertices. The *adjacency matrix* of Q is the $n \times n$ integer matrix $A = A(Q) = (a_{ij})_{i,j \in Q_0}$ where a_{ij} is the number of arrows $i \to j$ with starting point $i \in Q_0$ and terminal point $j \in Q_0$.

(b) Let $Q = (Q_0, Q_1, s, t)$ be an arbitrary quiver with $n = |Q_0|$ vertices. Assume that Q contains neither loops nor 2-cycles. The *signed adjacency matrix* of Q is the $n \times n$ integer matrix $B = B(Q) = (b_{ij})_{i,j \in Q_0}$ where $b_{ij} = a_{ij} - a_{ji}$.

By construction the signed adjacency matrix $B = B(Q) = (b_{ij})_{i,j \in Q_0}$ of a quiver Q is a skew-symmetric matrix, i.e. it satisfies the equation $B = -B^T$. Especially, all diagonal entries b_{ii} (with $i \in Q_0$) are zero. Figure 2.5 shows an example of the signed and non-signed adjacency matrix of a quiver.

Proposition 2.1.12. Let A be the adjacency matrix of a quiver $Q = (Q_0, Q_1, s, t)$ and let k be a positive integer. We denote the k-th power of A by $A^k = (a_{ij}^{(k)})_{i,j \in Q_0}$. Then for all vertices $i, j \in Q_0$



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Figure 2.5: An adjacency and a signed adjacency matrix

the entry $a_{ij}^{(k)}$ is equal to the number of paths p with starting point $s(p) = i$ and terminal point $t(p) = j$ of length k

Let $I_n \in \mathrm{Mat}(n \times n, \mathbb{Q})$ be the $n \times n$ identity matrix. Note that the notation suits the convention $A^0 = I_n$, because for every vertex $i \in Q_0$ there is a lazy path of length 0 starting and ending in i, but there is no path from vertex i to vertex j of length 0 if $i \neq j$.

Proof. We prove the statement by mathematical induction on k. The claim is true for $k = 1$ by definition. Let us now assume that $k \geq 2$ and that the statement is true for $k - 1$. Consider the equation $A^k = AA^{k-1}$. By the definition of matrix multiplication we have

$$a_{i,j}^{(k)} = \sum_{r \in Q_0} a_{ir} a_{rj}^{(k-1)}$$

for all vertices $i, j \in Q_0$. The term $a_{ir} a_{rj}^{(k-1)}$ is equal to the number of paths from i to j of length k such that the first step is $i \to r$. Then the right hand side is the number of paths from i to j of length k, parametrised by the first step. □

Especially, in the above situation the sum of all the entries is A^k is equal to the number of all paths of length k in Q. Let us denote this number by $P_Q(k)$. Now let $S \in \mathrm{Gl}_n(\mathbb{C})$ be an invertible matrix such that the matrix $J = S^{-1}AS$ is a Jordan canonical form. Then $A = SJS^{-1}$ and thus $A^k = SJ^kS^{-1}$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of largest absolute value. The number $r = r(A) = |\lambda| \in \mathbb{R}^+$ is also known as the *spectral radius* of A. We deduce that the function $\mathbb{N} \to \mathbb{N}, k \mapsto P_Q(k)$ lies in the class $O(r^k)$ for $k \to \infty$. Later we will see characterization of Dynkin diagrams by spectral properties. Let us illustrate this circle of ideas by some examples.

Example 2.1.13. (a) The matrix A in Figure 2.5 is a permutation matrix and so the sequence $(A^k)_{k \in \mathbb{N}}$ of matrices is periodic:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A^3 = I_3.$$

Especially, there is a path from 1 to 1 of length k if and only if k is divisible by 3. The roots of the characteriztic polynomial $\chi_A(X) = X^3 - 1 \in \mathbb{C}[X]$ are the third roots of unity $1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}} \in \mathbb{C}$. The absolute value of all these numbers is 1 which implies that the function $k \mapsto P_Q(k)$ is bounded by a constant. In fact, it is the constant function 3.

(b) For another example, let A' be the adjacency of the first quiver in Figure 2.1. Then $A' = 2A$ so that $(A')^k = 2^k A^k$ for all $k \geq 0$. In other words, the number of paths of length k is equal to $3 \cdot 2^k$ and increases exponentially in k. This is reflected by the fact that the eigenvalues, i.e. the roots of the characteriztic polynomial $x^3 - 8$, all have absolute value 2.

(c) For another example, let A be the adjacency of the second quiver in Figure 2.1. Here, the quiver Q is acyclic and so that the matrix A is nilpotent:

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^4 = 0.$$

The characteriztic polynomial is $\chi_A(X) = X^3 \in \mathbb{C}[X]$ and $\lambda = 0$ is the only eigenvalue so that $P_Q(k)$ is eventually 0. More generally, a quiver Q is acyclic if and only if the adjacency matrix $A(Q)$ is nilpotent.

It is noteworthy to remark that we do not loose information about the quiver when we pass from a quiver Q to its adjacency matrix $A(Q)$. More precisely, let Q and Q' be two quivers with adjacency matrices $A(Q) = (a_{ij})_{i,j \in Q_0}$ and $A(Q') = (a_{ij})_{i,j \in Q_0}$. Then Q and Q' are isomorphic if and only if there exists a bijection $\sigma\colon Q_0 \to Q_0'$ such that $a_{i,j} = a'_{\sigma(i),\sigma(j)}$ for all $i, j \in Q_0$. Moreover, for every matrix $A \in \mathrm{Mat}(n \times n, \mathbb{Z})$ there exists a quiver Q with n vertices such that $A(Q) = A$. Similar statements for the assignment $Q \mapsto B(Q)$ are not true, but they become true if we restrict ourselves to the class of quivers that contain neither loops nor 2-cycles.

2.2 Quiver mutation

2.2.1 The definition of quiver mutation

In this section we define quiver mutation. We assume that $Q = (Q_0, Q_1, s, t)$ is a quiver without loops and 2-cycles. For the definition it is important to partition both the set of vertices and the set of arrows into four parts. Let $k \in Q_0$ be a vertex.

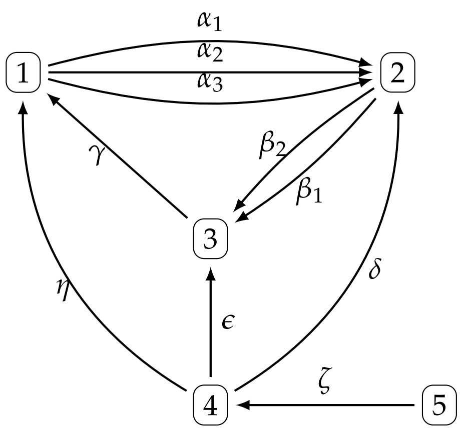


Figure 2.6: A quiver on five vertices

We call a vertex $i \in Q_0$ a *direct predecessor* of k if there exists an arrow $i \to k$ in Q_1 and we call a vertex $j \in Q_0$ a *direct successor* of k if there exists an arrow $k \to j$ in Q_1. We denote the sets of direct predecessors and successors by $\mathrm{DP}(k)$ and $\mathrm{DS}(k)$, respectively. The sets $\mathrm{DP}(k)$ and $\mathrm{DS}(k)$ are disjoint, because there are no 2-cycles in Q. Moreover, the vertex k is neither a direct predecessor nor a direct successor of itself, because there are no loops in Q. We also consider the set $\mathrm{U}(k) = Q_0 \backslash (\{k\} \cup \mathrm{DP}(k) \cup \mathrm{DS}(k)$ of vertices that are unrelated to (or not adjacent to) k, so that we get a partition

$$Q_0 = \{k\} \sqcup \mathrm{DP}(k) \sqcup \mathrm{DS}(k) \sqcup \mathrm{U}(k).$$

Furthermore, we call an arrow $\alpha \in Q_0$ *outgoing* if $s(\alpha) = k$ and we call it *incoming* if $t(\alpha) = k$. We denote the sets of outgoing and incoming arrows by $\mathrm{S}(k)$ and $\mathrm{T}(k)$, respectively. The sets $\mathrm{S}(k)$ and $\mathrm{T}(k)$ are disjoint, because there are no loops in Q. Moreover, let

$$\mathrm{A}(k) = \{\alpha \in Q_1 \colon s(\alpha) \in \mathrm{DP}(k), t(\alpha) \in \mathrm{DS}(k)\} \cup \{\alpha \in Q_1 \colon s(\alpha) \in \mathrm{DS}(k), t(\alpha) \in \mathrm{DP}(k)\}$$

be the set of arrows that connect a direct predecessor of k with a direct successor of k or vice versa. Furthermore, we denote by $\mathrm{R}(k) = Q_1 \backslash \{\mathrm{A}(k) \cup \mathrm{S}(k) \cup \mathrm{T}(k)\}$ the set of remaining arrows. We get a partition

$$Q_1 = \mathrm{S}(k) \sqcup \mathrm{T}(k) \sqcup \mathrm{A}(k) \sqcup \mathrm{R}(k).$$

For example, the vertex $k = 2$ of the quiver Q that is shown in Figure 2.6 has outgoing arrows $\mathrm{S}(k) = \{\beta_1, \beta_2\}$, outgoing arrows $\mathrm{T}(k) = \{\alpha_1, \alpha_2, \alpha_3, \delta\}$, direct predecessors $\mathrm{DP}(k) = \{1, 4\}$ and direct successors $\mathrm{DS}(k) = \{3\}$. We have $\mathrm{U}(k) = \{5\}$, $\mathrm{A}(k) = \{\epsilon, \gamma\}$ and $\mathrm{R}(k) = \{\zeta, \eta\}$.

Definition 2.2.1 (Quiver mutation)**.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver without loops and 2-cycles and let $k \in Q_0$ be a vertex. The *mutation* of Q at k is a new quiver $\mu_k(Q) = Q' = (Q'_0, Q'_1, s', t')$ constructed as follows:

(a) The set of vertices does not change under mutation, i.e. we have $Q'_0 = Q_0$.

(b) The set arrows does change under mutation and it is equal to the union $Q'_1 = \mathrm{S}^*(k) \cup \mathrm{T}^*(k) \cup \mathrm{A}^*(k) \cup \mathrm{R}(k)$ where the four sets are given as follows:

(M1) We reverse all arrows that terminate in k: if $\alpha \in \mathrm{T}(k)$ is an arrow $i \to k$ in Q_1 for some direct predecessor $i \in \mathrm{DP}(k)$, then let $\alpha^* \in Q'_1$ be an arrow $k \to i$, i.e. we set $s'(\alpha^*) = k$ and $t'(\alpha^*) = i$. Put $\mathrm{S}^*(k) = \{\alpha^* \colon \alpha \in \mathrm{S}(k)\}$.

(M2) We reverse all arrows that start in k: if $\beta \in \mathrm{S}(k)$ is an arrow $k \to j$ in Q_1 for some direct successor $j \in \mathrm{DS}(k)$, then let $\beta^* \in Q'_1$ be an arrow $j \to k$, i.e. we set $s'(\beta^*) = j$ and $t'(\beta^*) = k$. Put $\mathrm{T}^*(k) = \{\alpha^* \colon \beta \in \mathrm{T}(k)\}$.

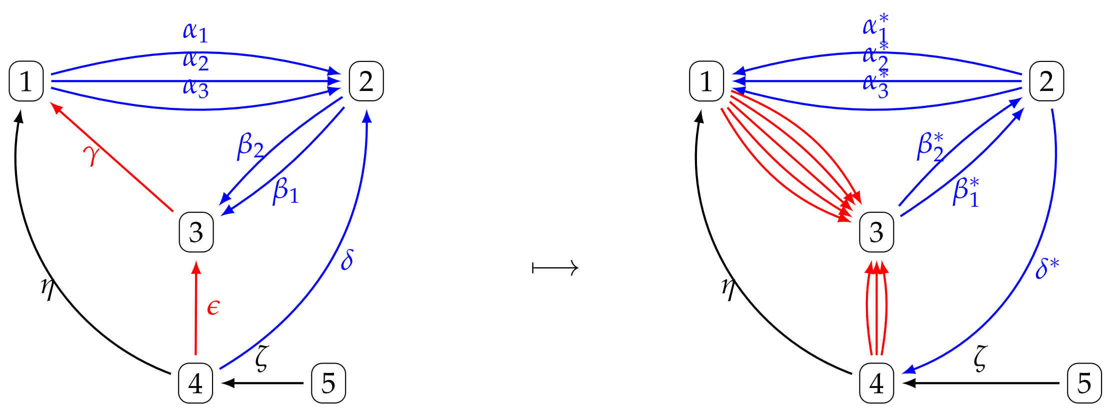
(M3) Let $i \in \mathrm{DP}(k)$ be a direct predecessor and $j \in \mathrm{DS}(k)$ a direct successor of k. Let r_{ij} be the number of paths $(\alpha, \beta) \in Q_1 \times Q_1$ of the form $i \to k \to j$ in Q. Let $A_{ij} = r_{ij} + b_{ij}$. If $A_{ij} \geq 0$, then define $\mathrm{A}^*(k)$ to be the set containing A_{ij} arrows $\alpha_{ij}(r) \colon i \to j$ (for $1 \leq r \leq A_{ij}$). Otherwise define $\mathrm{A}^*(k)$ to be the set containing $-A_{ij}$ arrows $\alpha_{ji}(r) \colon j \to i$ (for $1 \leq r \leq -A_{ij}$).

(M4) The arrows in the set $\mathrm{R}(k)$ do not change under mutation.

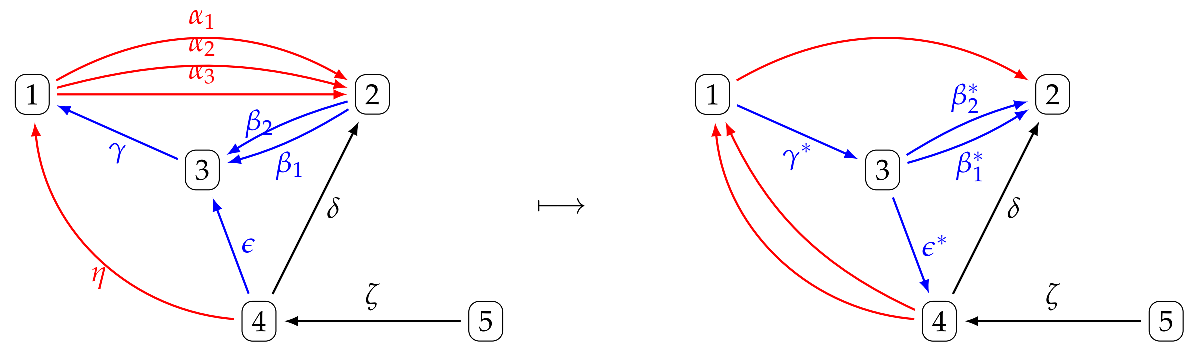
A more intuitive way to describe mutation rule M3 is as follows: for every such path $i \to k \to j$ we add an arrow $i \to j$. Then we remove one possibly created 2-cycle after the other until the quiver does not contain 2-cycles anymore. By definition, the quiver $\mu_k(Q)$ again contains neither loops nor 2-cycles. Let us illustrate the definition by some examples.

Example 2.2.2. (a) First of all, let us consider the quiver $Q = (Q_0, Q_1, s, t)$ from Figure 2.6. We compute the mutation $\mu_2(Q)$. By rules M1 and M2 the mutation reverses the incoming arrows $\alpha \in \mathrm{T}(2) = \{\alpha_1, \alpha_2, \alpha_3, \delta\}$ as well as the outgoing arrows $\beta \in \mathrm{S}(2) = \{\beta_1, \beta_2\}$ (which we both have coloured blue). By M4 the arrows $\eta, \zeta \in \mathrm{R}(2)$ remain unchanged.

We have one (red) arrow $\gamma \colon 3 \to 1$. The six paths (α_r, β_s) for $r \in \{1, 2, 3\}$ and $s \in \{1, 2\}$ yield six new arrows $1 \to 3$. According to mutation rule M3 the quiver $\mu_2(Q)$ has five arrows $\alpha_{13}(1), \dots, \alpha_{13}(5) \colon 1 \to 3$. We have one (red) arrow $\gamma \colon 4 \to 3$. The two paths (δ, β_s) for $s \in \{1, 2\}$ yield two new arrows $4 \to 3$. According to mutation rule M3 the quiver $\mu_2(Q)$ altogether has three arrows $\alpha_{43}(1), \alpha_{43}(2), \alpha_{43}(3) \colon 4 \to 3$.

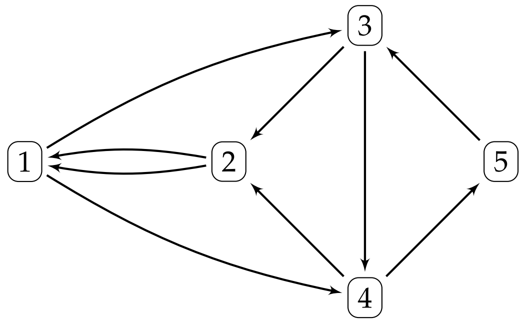


(b) As a second example let us start with the very same quiver Q and calculate the mutation of Q at vertex 3. The (blue) arrows β_1, β_2, γ and ϵ are incident to the vertex 3 and change direction. The mutation does not affect the arrows $\delta, \zeta \in \mathrm{R}(2)$. The number of arrows $\alpha\colon 1 \to 2$ is equal to 3 and therefore by 1 larger than the number of paths $2 \to 3 \to 1$ giving one arrow $1 \to 2$ in $\mu_3(Q)$. The arrow $\eta\colon 4 \to 1$ together with the path (ϵ, γ) yields two arrows $4 \to 1$ in $\mu_3(Q)$.



(c) In the previous examples we have $\mu_2(\mu_2(Q)) \cong Q$ and $\mu_3(\mu_3(Q)) \cong Q$ (and both isomorphisms are the identity on the set of vertices).

(d) For a completely different example let us consider the following quiver Q:



Mutation at a vertex $k \in \{1,2,3,4\}$ produces an isomorphic quiver: $Q \cong \mu_1(Q) \cong \mu_2(Q) \cong \mu_3(Q) \cong \mu_4(Q)$. More precisely, the following table shows which vertices correspond to each other under the isomorphisms.

Q_0	$\mu_1(Q)_0$	$\mu_2(Q)_0$	$\mu_3(Q)_0$	$\mu_4(Q)_0$
1	2	2	4	2
2	1	1	1	3
3	3	3	2	4
4	4	4	3	1
5	5	5	5	5

Bernhard Keller's Java applet is a useful software to perform mutations. It turns out that isomorphisms $\mu_k(Q) = Q$ are seldom and hence Example 2.2.2 d is exceptional in this respect. On the other hand, we can dramatically generalise the isomorphisms from Example 2.2.2 c as the following proposition shows.

Proposition 2.2.3. The assignment $Q \mapsto \mu_k(Q)$ is involutory, i.e. for all quivers $Q = (Q_0, Q_1, s, t)$ without loops and 2-cycles and all vertices $k \in Q_0$ we have $Q \cong \mu_k(\mu_k(Q))$.

Proof. Let $Q = (Q_0, Q_1, s, t)$ be a quiver without loops and 2-cycles and let $i, j, k \in Q_0$ be vertices with $i \neq j$. As a shorthand notation we put $\mu_k(Q) = Q' = (Q'_0, Q'_1, s', t')$ and $\mu_k(\mu_k(Q)) = Q'' = (Q''_0, Q''_1, s'', t'')$. We prove that number of arrows $i \to j$ in Q_1 is equal to the number of arrows $i \to j$ in Q''_1.

First note that the set of direct predecessors of k in Q' is $\mathrm{DS}(k)$ and that the set of direct successors of k in Q' is $\mathrm{DT}(k)$. Therefore, $\mathrm{S}^*(k)$ is the set of arrows in Q' that terminate in k and $\mathrm{T}^*(k)$ is the set of arrows in Q' that start in k. It follows that the claim is true in the case that $i = k$ or $j = k$, because we reverse the arrows incident to k twice. Furthermore, we see that all the arrows between i and j remain unchanged under both mutations except when $i \in \mathrm{DP}(k)$ is a direct predecessor of k in Q and $j \in \mathrm{DS}(k)$ is a direct successor of k in Q or vice versa.

Suppose that $i \in \mathrm{DP}(k)$ is a direct predecessor of k in Q and that $j \in \mathrm{DS}(k)$ is a direct successor of k in Q. If there are no arrows from j to i in Q and $a_{ij} \geq 0$ arrows from i to j in Q, then the first mutation yields $a_{ik}a_{kj} + a_{ij}$ arrows from i to j in Q'. This number is greater than or equal to the number $a_{kj}a_{ik}$ of paths $j \to k \to i$ in Q', so after the cancellation of 2-cycles in second mutation a_{ij} arrows from i to j remain in Q''. Now let us assume that there are $a_{ji} > 0$ arrows from j to i in Q (and hence no arrows from i to j). We distinguish two cases. If $a_{ji} \geq a_{ik}a_{kj}$, then the first mutation yields $a_{ji} - a_{ik}a_{kj}$ arrows from j to i in Q'. The second mutation adds $a_{kj}a_{ik}$ arrows from j to i, so that altogether and without cancellation we get a_{ji} arrows from j to i in Q''. If $a_{ji} < a_{ik}a_{kj}$, then the first mutation yields $a_{ik}a_{kj} - a_{ji}$ arrows from i to j in Q'. This number is smaller than the number $a_{kj}a_{ik}$ of paths $j \to k \to i$ in Q', so after the cancellation of 2-cycles in second mutation $a_{kj}a_{ik} - (a_{ik}a_{kj} - a_{ij}) = a_{ij}$ arrows from i to j remain in Q''. □

2.2.2 Mutation classes

As the mutation of a quiver $Q = (Q_0, Q_1, s, t)$ without loops and 2-cycles at a vertex $k \in Q_0$ again contains no loops and 2-cycles, we may form iterated mutations. If $k, k' \in Q_0$ are two vertices of Q, then we will denote the quiver $\mu_k(\mu_{k'}(Q))$ also by $(\mu_k \circ \mu_{k'})(Q)$.

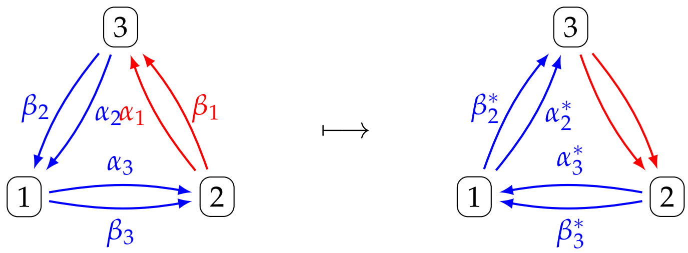
Definition 2.2.4 (Mutation equivalence)**.** We say that two quivers Q and Q' are *mutation equivalent* if there exists a sequence $(k_1, k_2, \ldots, k_r) \in Q_0^r$ of vertices of Q of length $r \geq 0$ such that the quiver $(\mu_{k_1} \circ \mu_{k_2} \circ \cdots \circ \mu_{k_r})(Q)$ is isomorphic to Q'.

Mutation equivalence defines an equivalence relation on the class of all quivers without loops and 2-cycles: it clearly is transitive and reflexive and it is symmetric by Proposition 2.2.3. If the quivers Q and Q' are mutation equivalent, then we will also write $Q \sim Q'$.

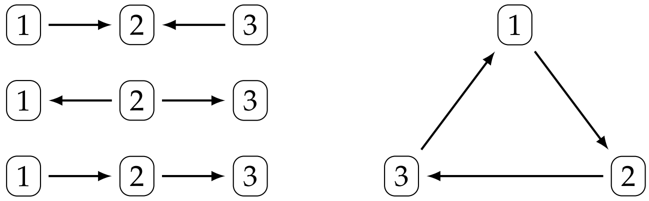
Definition 2.2.5 (Mutation class)**.** Let Q be a quiver without loops and 2-cycles. The *mutation class* of Q is the set of all isomorphism classes that contain a representative Q' with $Q \sim Q'$.

Example 2.2.6. (a) Every acyclic quiver with two vertices is isomorphic to a quiver with vertex set $Q_0 = \{1, 2\}$ and b arrows $1 \to 2$ for some natural number $b \in \mathbb{N}$. We refer to this quiver as $Q(b)$. For every $b \in \mathbb{N}$ we have $Q(b) \cong \mu_1(Q(b)) \cong \mu_2(Q(b))$. Hence, the mutation class of an acyclic quiver with two vertices is a singleton.

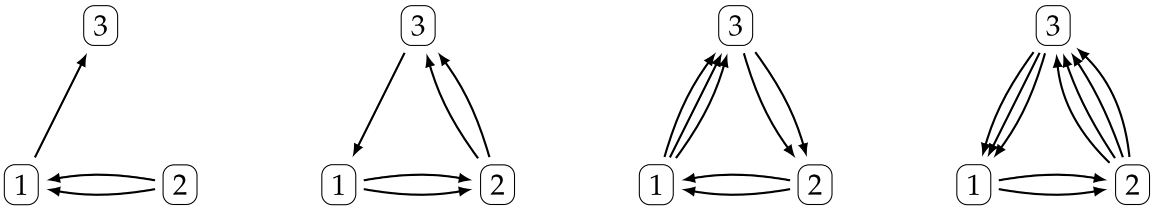
(b) Let Q be the first quiver from Example 2.1. By chance we have $Q \cong \mu_1(Q) \cong \mu_2(Q) \cong \mu_3(Q)$. Hence, the mutation class of Q is a singleton.



(c) The isomorphism classes of the following four quivers form a mutation class of size 4.



(d) The following example shows that mutation classes can be infinite. For every natural number $n \in \mathbb{N}$ let $T(n+1, n, 2)$ be the quiver with $Q_0 = \{i, j, k\}$ such that there are n arrows from i to j, 2 arrows from j to k and $n+1$ arrows from k to i. Then $\mu_k(T(n+1, n, 2)) \cong T(n+2, n+1, 2)$, so that $(T(n+1, n, 2))_{n \in \mathbb{N}}$ is an infinite family of mutation equivalent, pairwise non-isomorphic quivers.



Definition 2.2.7. We say that a quiver Q is *mutation finite* if its mutation class is finite. Otherwise it is called *mutation infinite*.

By the above discussion, the quivers in Example 2.2.6 a-c are mutation finite, whereas the quivers in Example 2.2.6 d are not. In general, it is difficult to decide whether a given quiver is mutation finite.

To formulate a generalization of Example 2.2.6 c we wish to introduce a further piece of notation. If $Q = (Q_0, Q_1, s, t)$ is a quiver, then we will call the graph on vertices Q_0 such that there is an edge from i to j for every arrow $\alpha\colon i \to j$ in Q_1 the *underlying diagram* $\Gamma = \Gamma(Q)$ of Q. In this case we also say that Q is an *orientation* of Γ. Note that a mutation at a sink or a source does not change the underlying diagram. A graph is called a *tree* if it is connected and does not contain closed paths.

Proposition 2.2.8. Any two orientations of the same tree are mutation equivalent.

Proof. Let T be a tree and let $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q_0, Q_1', s', t')$ be orientations of T. We claim that there exists a sequence $(k_1, k_2, \ldots, k_r) \in Q_0^r$ of length $r \geq 0$ such that $(\mu_{k_1} \circ \mu_{k_2} \circ \cdots \circ \mu_{k_r})(Q) \cong Q'$ and every mutation is a mutation at a sink or a source.

We prove the claim by mathematical induction on the number $n = |Q_0|$ of vertices. The case $n = 1$ is trivial. Let $n \geq 2$. By Euler's formula the tree has $n-1$ edges. Hence there must exist a vertex i of T that is incident to only one edge. Let us denote the unique vertex that is adjacent to i by j. By induction hypothesis there exists a sequence $(j_1, j_2, \ldots, j_s)$ of vertices in $Q_0 \backslash \{i\}$ such that the mutations $\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_s}$ transform the full subquiver of Q with vertices $Q_0 \backslash \{i\}$ into the full subquiver of Q' with vertices $Q_0 \backslash \{i\}$. To transform Q into Q' we use the same sequence of vertices except that we possibly include μ_i before we perform μ_j to ensure that j is a sink or a source at that step. The other mutations do not affect the vertex i. In this way we get a sequence of mutations at sinks or sources that transform Q into Q'. □

We say that a quiver is of type A_n if it has the same underlying diagram as the linearly oriented quiver of type A_n. Proposition 2.2.8 implies that for a given natural number $n \in \mathbb{N}$ all quivers of type A_n are mutation equivalent to each other.

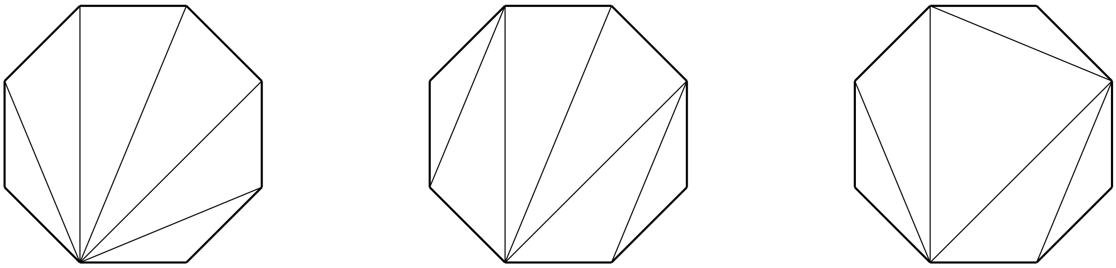


Figure 2.7: Three triangulations of a regular octogon

2.2.3 The mutation class of quivers of type A

In this section we wish to introduce a combinatorial model for the mutation class of quivers of type A in terms of triangulations of a regular polygon. For every natural number $n \geq 3$ let P_n be a regular polygon with n vertices, embedded in the Euclidean plane. A line that joins two different and non-consecutive vertices of P_n is called a *diagonal*. We say two diagonals d_1, d_2 are *crossing* if d_1 and d_2 intersect in a point that lies in the interior of P_n. Let us start with a basic definition.

Definition 2.2.9 (Triangulation)**.** Let $n \geq 1$ be a natural number. A *triangulation* of the regular $(n+3)$-gon P_{n+3} is a collection of non-crossing diagonals that dissect the polygon in triangles.

It is easy to see that every triangulation of P_{n+3} contains exactly n diagonals and that it dissects the polygon in $n+1$ triangles. Furthermore, note that every diagonal in a triangulation borders exactly two triangles. In other words, we could define a triangulation as a maximal collection of pairwise non-crossing diagonals of P_{n+3}. Figure 2.7 shows three different triangulations of a regular octogon.

Now let us fix a natural number $n \geq 1$ and a regular polygon P_{n+3}. To every triangulation T of P_{n+3} we attach a quiver $Q(T) = Q = (Q_0, Q_1, s, t)$ as follows. First of all, we put $Q_0 = T$, i.e. the vertices of Q correspond to the diagonals in the triangulation. We introduce an arrow from the diagonal d_1 to the diagonal d_2 in Q_1 whenever d_1 and d_2 are two sides of a triangle in the triangulation such that d_1 *directly* precedes d_2 when traversing the boundary of the triangle in counterclockwise orientation. For an example, Figure 2.8 displays the quivers associated with two triangulations of the regular octogon.

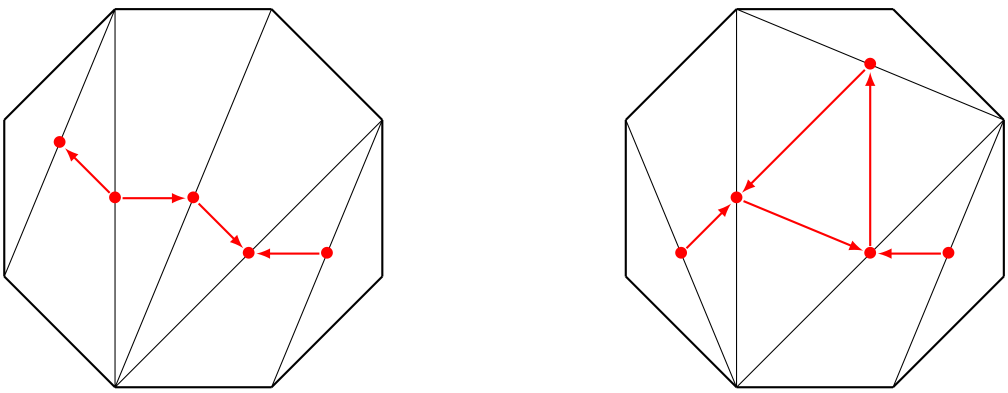


Figure 2.8: The quiver attached to a triangulation

For every vertex v of P_{n+3} there is a special triangulation T_v given by all the n diagonals that are incident to v. An example is the first triangulation in Figure 2.7. Note that the quiver $Q(T_v)$ is a linearly oriented quiver of type A_n.

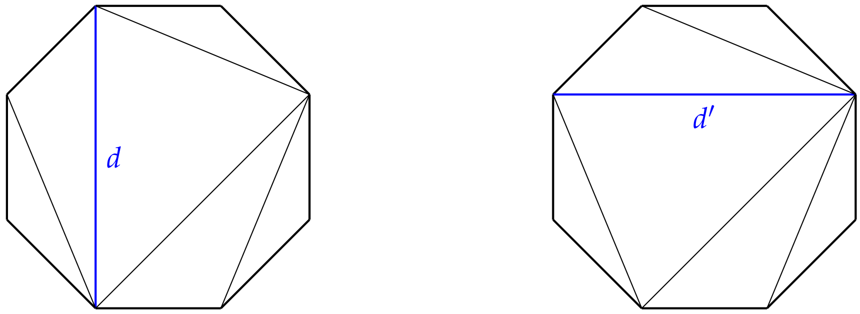


Figure 2.9: The flip $T \rightsquigarrow F_d(T)$ of a triangulation at a diagonal

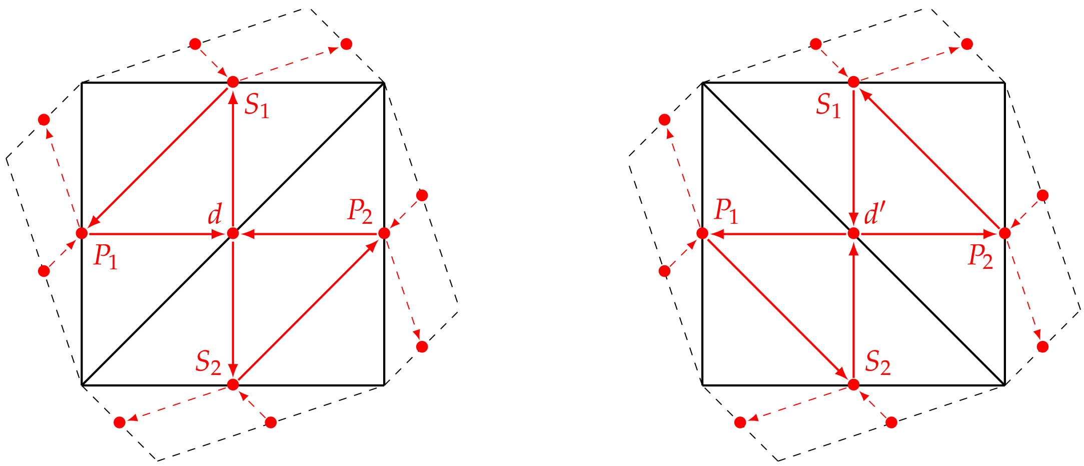
Natural questions arise: Does a mutation of a quiver $Q(T)$ for a triangulation T at a vertex come again from triangulation? Can we even formulate the mutation rule geometrically? It turns out that quiver mutation has a simple graphical description.

Definition 2.2.10 (Flips)**.** Let $d \in T$ be diagonal of a triangulation T of the regular polygon P_{n+3}. If we remove the diagonal d from the triangulation, then the two triangles with side d merge into a quadrilateral. Let d' be the other diagonal of this quadrilateral. The *flip* of the triangulation T at the diagonal d is the triangulation $F_d(T) = (T \cup \{d'\}) \setminus \{d\}$.

By construction, the flip of a triangulation is again a triangulation. Just as the mutation the flip is involutory, i.e. for all d and T we have $F_d(F_d(T)) = T$. Figure 2.9 shows an example of a flip. Moreover, Figure 2.10 displays all 14 triangulations of a hexagon and their flips. The next proposition relates flips of triangulations with mutations of quivers.

Proposition 2.2.11. Let $d \in T$ be diagonal of a triangulation T of the regular polygon P_{n+3}. We denote the quiver of the triangulation T by Q and the quiver of the flip $F_d(T)$ by Q'. Then we have $Q' \cong \mu_d(Q)$.

Proof. The diagonal $d \in T$ is the side of two triangles of the triangulation. Thus the vertex $d \in Q_0$ has at most two direct predecessors and at most two direct successors (depending on whether the segments are sides or diagonals of P_{n+3}). Denote the direct predecessors and successors of d in Q in the two triangles (if existent) by P_1, P_2, S_1 and S_2.



Let P' be the quadrilateral with diagonals d and d'. The flip does not change the triangulation outside of P'. So the arrows in Q' that come from triangles outside of P' remain the same as in Q. But these arrows are neither incident to k nor do they connect a direct predecessor with a direct

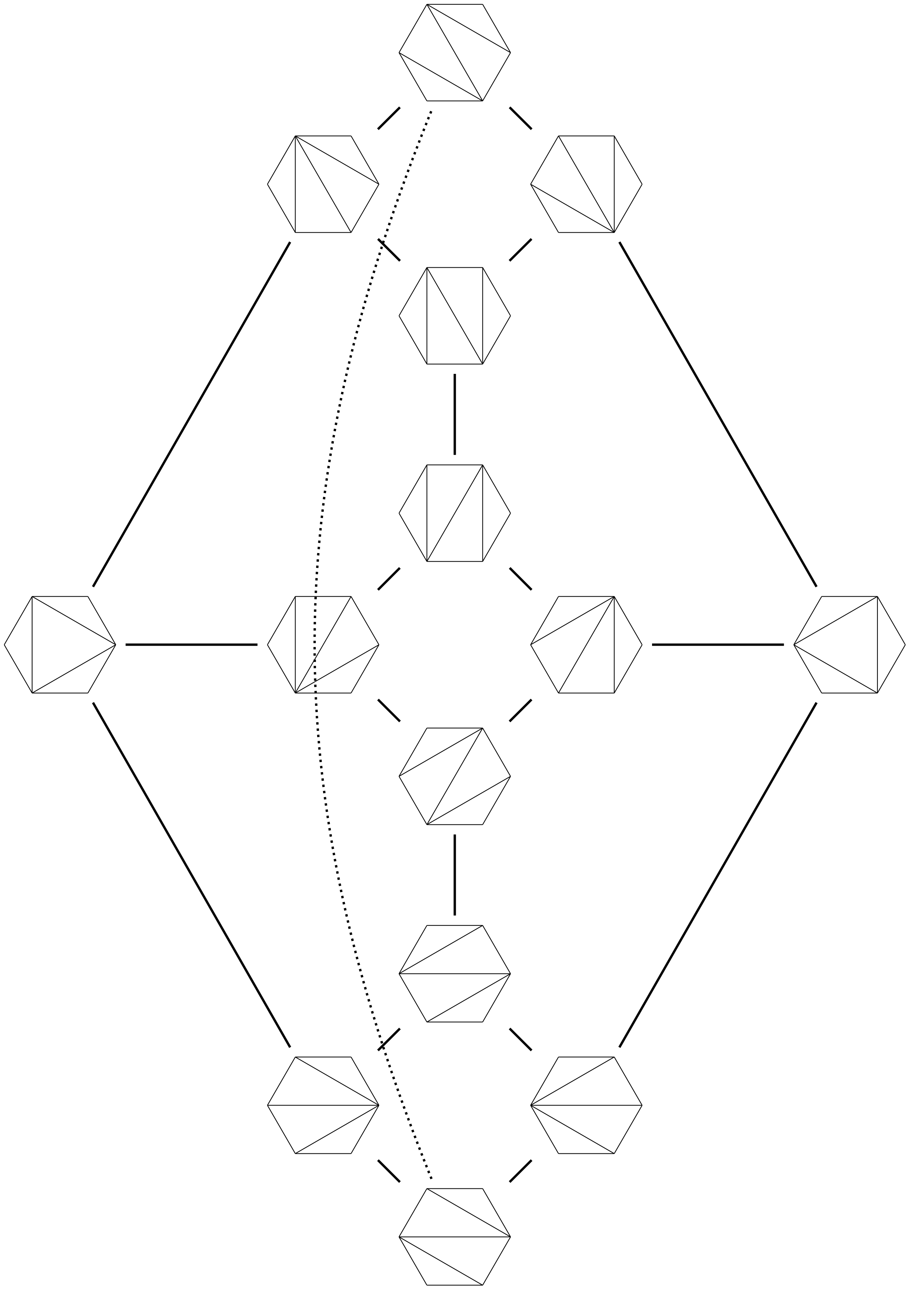


Figure 2.10: The triangulations of a hexagon and their flips

n	0	1	2	3	4	5	6
C_n	1	1	2	5	14	42	132

Figure 2.11: The first Catalan numbers

successor, so they also remain the same under mutation according to rule M4. It is also evident that inside P' the arrows incident to d change direction in agreement with mutation rules M1 and M2. Furthermore, arrows $S_1 \to P_1$ and $S_2 \to P_2$ vanish and arrows $P_1 \to S_2$ and $P_2 \to S_1$ appear in agreement with mutation rule M3. □

Especially, for every quiver Q of type A_n there is a triangulation T of P_{n+3} such that $Q = Q(T)$. Hence, the triangulations form a combinatorial model for the mutation class of quivers of type A_n. The simple description of the mutation by flips is one justification for the definition of quiver mutation, which may seem unintuitive at first sight.

2.2.4 Catalan numbers

Definition 2.2.12 (Catalan numbers)**.** The initial values $C_0 = C_1 = 1$ together with the recursion

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} = C_0 C_n + C_1 C_{n-1} + C_2 C_{n-2} + \ldots + C_n C_0$$

for all $n \geq 1$ define an increasing sequence $(C_n)_{n \in \mathbb{N}}$ of positive integers. Elements of the sequence are called *Catalan numbers*.

The table in Figure 2.11 displays the first Catalan numbers. The numbers have a long and colourful history. They are named after Eugène Charles Catalan, although Leonhard Euler had already considered the sequence. In a famous exercise, Stanley [Sta, Exercise 6.19] enumerates many combinatorial objects by Catalan numbers.

Proposition 2.2.13. The number of triangulations of the regular polygon P_{n+3} is equal to the Catalan number C_{n+1}.

Before we give a proof, let us note that the formula remains true for $n \in \{0, -1\}$ if we adopt the convention that the regular 2- and 3-gons admit exactly one triangulation (consisting of zero diagonals).

Proof. We prove the claim by strong mathematical induction on n. It is obviously true for $n = 1$, because a quadrilateral admits $C_2 = 2$ triangulations.

Now assume that $n \geq 2$. We label the vertices of P_{n+3} consecutively and in counterclockwise order with the numbers $1, 2, \ldots, n+3$. Let T be a triangulation of P_{n+3}. Then we consider the smallest number $k \in \{3, 4, \ldots, n+3\}$ such that vertex 1 is connected with vertex k, either by a diagonal or a side of P_{n+3}. The line $1k$ dissects the whole polygon in an k-gon P' with vertices $1, 2, \ldots, k$ and an $(n+5-k)$-gon P'' with vertices $k, k+1, k+2, \ldots, n+3, 1$. Inside P' there must be a triangle with side $1k$. The third vertex of this triangle must be the vertex 2, because by construction 1 is not connected to vertices $3, 4, \ldots, k-1$. The line $2k$ dissects the polygon P'' in the triangle $12k$ and a polygon P'''. So very triangulation T of P_{n+3} induces a triangulation T'' of the $(n+5-k)$-gon P'' and a triangulation T''' of the $(k-1)$-gon P'''.

Conversely, every pair (T'', T''') of triangulations an $(n+5-k)$-gon and an $(k-1)$-gon defines a triangulation T of P_{n+3} such that k is smallest number connected to vertex 1.

By induction hypothesis the number of triangulations of polygon P'' is equal to C_{n+3-k} and the number of triangulations of polygon P'' is equal to C_{k-3}. We see that

$$\sum_{k=3}^{n+3} C_{n+3-k}C_{k-3} = \sum_{k=0}^{n} C_{n-k}C_k = C_{n+1}$$

is the number of triangulations of the regular polygon P_{n+3}. □

2.2.5 Matrix mutation

In this section we wish to write the mutation rule in terms of matrices. Let us introduce the following notations. For a real number $x \in \mathbb{R}$ we put $[x]_+ = \max(0, x)$. Furthermore, we denote the sign of x by $\operatorname{sgn}(x) \in \{-1, 0, 1\}$.

Definition 2.2.14. Let $n \in \mathbb{N}$ be a natural number and $k \in \{1, 2, \dots, n\}$ be an index. Suppose that $B = (b_{ij})_{1 \le i,j \le n} \in \operatorname{Mat}(n \times n, \mathbb{Z})$ a skew-symmetric matrix. The *mutation* of B at k is the matrix $\mu_k(B) = B' = (b'_{ij})_{1 \le i,j \le n} \in \operatorname{Mat}(n \times n, \mathbb{Z})$ with entries

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases}$$

Some authors formulate the mutation in a different way. For example, the reader sometimes finds the formula $\frac{1}{2}(b_{ik}|b_{kj}| + |b_{ik|}b_{kj})$ instead of $\operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+$. The meaning to both formulae is the same. It is zero unless the numbers b_{ik}, b_{kj} are either both positive or both negative. In this case the term is equal to $\pm|b_{ik}b_{kj}|$ and the sign depends on the sign of b_{ik} and b_{kj}.

Proposition 2.2.15. Let $Q = (Q_0, Q_1, s, t)$ be a quiver without loops and 2-cycles. Let $B = B(Q)$ be the signed adjacency matrix and let $k \in Q_0$ be any vertex of Q. Then the signed adjacency matrix of the mutation $\mu_k(Q)$ of Q at k is equal to the mutation $\mu_k(B)$ of B at k.

Proof. Denote the entries of the signed adjacency matrices of Q and $\mu_k(Q)$ by $B = (b_{ij})_{i,j \in Q_0}$ and $B' = (b'_{ij})_{i,j \in Q_0}$. Let $i, j \in Q_0$ be vertices. According to the rules M1 and M2 the mutation reverses all arrows incident to k, i.e. $b'_{ij} = -b_{ij}$ if $i = k$ or $j = k$. The above discussion shows that $\operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ = 0$ unless i is a direct predecessor of k and j is a direct successor of k (in which case $b_{ik} > 0$ and $b_{kj} > 0$), or vice versa (in which case $b_{ik} < 0$ and $b_{kj} < 0$). In both cases, by the choice of the sign the addition of the term $|b_{ik}b_{kj}|$ corresponds to the augmentation of $|b_{ik}b_{kj}|$ arrows from a direct predecessor to a direct successor. This product is equal to the number of paths of length 2 from the direct predecessor to the direct successor via k, in agreement with mutation rule $M3$. The deletion of 2-cycles does not affect the B-matrix. □

2.2.6 Invariants of mutation

Often *invariants* are a useful tool to study sequences and dynamical system. Quiver mutation has a trivial invariant, namely the number of vertices does not change under mutation. In contrast, the number of arrows may change under mutation. This is one reason why we work with incidence matrices instead adjacency matrices in this context. In fact, most graph theoretic properties are not mutation invariant. For instance, acyclicity is not invariant under mutation as Example 2.2.6 (c) shows. See Exercise 2.4 for two numbers that actually remain invariant under matrix mutation, namely the rank of the matrix and the greatest common divisor of the entries in a column.

2.3 Cluster algebras attached to quivers

2.3.1 Generalities on algebras

Let us briefly recall some relevant notions from algebra. The material is classical, compare for example Artin [Art], Assem-Simson-Skowronski [ASS], Bosch [Bos] and Scheja-Storch [SS].

Definition 2.3.1 (Ring)**.** A *ring* is a triple $(A, +, \cdot)$ that consists of a set A together with two binary operations $+\colon A \times A \to A, (x, y) \mapsto x + y$ and $\cdot\colon A \times A \to A, (x, y) \mapsto x \cdot y$ such that the pair $(A, +)$ is an abelian group, the operation $\cdot$ is associative and distrubutive laws $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$ hold for all elements $x, y, z \in A$.

If $(A, +, \cdot)$ is a ring, then we will denote the zero element in the abelian group $(A, +)$ by the symbol 0. Furthermore, we will say that the ring is *unital* if there exists an element $1 \in A$ such that $1 \cdot x = x = x \cdot 1$ for all elements $x \in A$. In this case we will call the element 1 a *unit*. It is easy to see that there can only be one unit.

Definition 2.3.2 (Algebra)**.** Let k be a field. A *k-algebra* is a unital ring $(A, +, \cdot)$ together with a binary operation $\cdot\colon k \times A \to A, (\lambda, x) \mapsto \lambda \cdot x$, called *scalar multiplication*, such that the abelian group $(A, +)$ together with the scalar multiplication forms k-vector space and the scalar multiplications is compatible with the ring multiplication, i.e. the equations $\lambda \cdot (x \cdot y) = (\lambda \cdot x) \cdot y = x \cdot (\lambda \cdot y)$ hold for all scalars $\lambda \in k$ and all elements $x, y \in A$

Examples of algebras include the polynomial ring $k[X]$ in one variable, the polynomial ring $k[X_1, X_2, \ldots, X_n]$ in several variables and the matrix algebra $\mathrm{Mat}(n \times n, k)$ for every natural number $n \in \mathbb{N}$. The first two examples are infinite-dimensional algebras whereas the third example is a finite-dimensional algebra.

Sometimes we write ab instead of $a \cdot b$ for brevity. We say that the ring A is *commutative* if $xy = yx$ for all elements $x, y \in A$. For example, the polynomial algebra is commutative, but the matrix algebra is not commutative.

Definition 2.3.3 (Subalgebra)**.** Let $(A, +, \cdot)$ be an algebra over a field k. A *subalgebra* is a k-vector subspace $B \subseteq A$ such that the identity $1 \in A$ lies in B and it it closed under multiplication, i.e. we have $b_1 b_2 \in B$ for all elements $b_1, b_2 \in B$.

For example, $k[x^2] \subseteq k[x]$ is the subalgebra of the polynomial algebra of k-linear combinations of powers of x with even degree.

Definition 2.3.4 (Generated subalgebra)**.** Let $(A, +, \cdot)$ be an algebra over a field k and $(x_i)_{i \in I}$ a family of elements $x_i \in A$. The *subalgebra generated by* $(x_i)_{i \in I}$ is the smallest subalgebra which contains all elements of the family. We denote this algebra by $k[x_i\colon i \in I]$ and refer to the family $(x_i)_{i \in I}$ as a *generating set*. We say that A is *finitely generated* if there are finitely many elements $x_1, x_2, \ldots, x_n \in A$ such that $A = k[x_1, x_2, \ldots, x_n]$.

For example, the polynomial algebra $k[X_1, X_2, X_3]$ is generated by the three elements X_1, X_2, X_3. For the rest of the section let us assume that $(A, +, \cdot)$ is a commutative algebra over a field k. In this case, the subalgebra generated by a family $(x_i)_{i \in I}$ of elements is the set of all k-linear combinations of monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdot \ldots \cdot x_{i_r}^{a_r}$ for all sequences $(i_1, i_2, \ldots, i_r) \in I^r$ and $(a_1, a_2, \ldots, a_r) \in \mathbb{N}^r$ and $r \geq 0$.

Definition 2.3.5 (Zero divisors)**.** A non-zero element $x \in A$ is called a *zero divisor* if there exists an element $y \in A$ such that $xy = 0$. An algebra is called *integral domain* if it does not contain any zero divisors.

We say that a subset $S \subseteq A$ is a *multiplicative system* if $1 \in S$, $0 \notin S$ and $st \in S$ for all elements $s, t \in S$. Assume that $S \subseteq A$ is s multiplicative system. We say that two pairs $(x, s), (y, t)$ in $A \times S$ are *equivalent* if $xt = sy$. In this case we also write $(x, s) \sim (y, t)$. We see that the relation $\sim$ defines an equivalence relation on the set $A \times S$: the relation clearly is reflexive and symmetric. For transitivity, assume that $(x, s) \sim (y, t)$ and $(y, t) \sim (z, u)$ are equivalent pairs in $A \times S$. The equations $xt = sy$ and $yu = zt$ imply $xut = suy = szt$ and thus $(xu - sz)t = 0$. Since $t \neq 0$ and A is an integral domain, we conclude that $xu = sz$ so that $(x, s) \sim (z, u)$.

Because of the similarity with identification of fractions, we also use the symbol $\frac{x}{s}$ to denote the equivalence class of $(x, s) \in A \times S$ in $S^{-1}A$. Lastly, let us introduce the notation $S^{-1}A$ for the set of equivalence classes.

Definition 2.3.6 (Localisation)**.** Let A be an integral domain and $S \subseteq A$ a multiplicative system. The *localisation* of A at S is the ring $(S^{-1}A, +, \cdot)$ where we define addition and multiplication by the formulae

$$\frac{x}{s} + \frac{y}{t} = \frac{xt + ys}{st} \in S^{-1}A, \qquad \frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st} \in S^{-1}A$$

for all elements $\frac{x}{s}, \frac{y}{t} \in S^{-1}A$. It is easy to see that both the addition and the multiplication are well-defined, i.e. they are independent of the choice of the representatives of (x, s) and (y, t) in $A \times S$. Furthermore, $S^{-1}A$ inherits a k-vector space structure from A so that it is again an algebra.

Let us discuss two examples of localisations which will become important in the following sections. First of all, note for every integral domain the set $S = \{x \in A \colon x \neq 0\}$ is a multiplicative system. The localisation of A at S is called the *quotient field*. For instance, for every natural number $n \in \mathbb{N}$ the quotient field of the polynomial algebra $k[X_1, X_2, \ldots, X_n]$ is the field $k(X_1, X_2, \ldots, X_n)$ of *rational functions* in n variables with coefficients in k.

For a second example note that the set of monomials $\{X_1^{a_1} X_2^{a_2} \cdot \ldots \cdot X_n^{a_n} \colon (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n\}$ is another multiplicative system in $k[X_1, X_2, \ldots, X_n]$. The localisation is canonically isomorphic to the algebra $k[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}]$ of *Laurent polynomials*.

Definition 2.3.7 (Algebraic independence)**.** Let $k \subseteq \mathcal{F}$ be a field extension. We say that the elements $u_1, u_2, \ldots, u_n$ in $\mathcal{F}$ are *algebraically dependent* over the field k if there exists a polynomial $f \in k[X_1, X_2, \ldots, X_n]$ with coefficients in k such that $f(u_1, u_2, \ldots, u_n) = 0$. The elements are called *algebraically independent* if they are not algebraically dependent.

2.3.2 Cluster algebras associated with quivers

Now we are ready to present Fomin-Zelevinsky's definition of cluster algebras. Moreover, we fix a natural number $n \geq 1$. Furthermore, although there is a more general setup, we stick to the case when the base field $k = \mathbb{Q}$ is the field of rational numbers. First of all, let us $\mathcal{F}$ be a field extension of $\mathbb{Q}$. Typically we have $\mathcal{F} = \mathbb{Q}(u_1, u_2, \ldots, u_n)$ for some algebraically independent variables $u_1, u_2, \ldots, u_n$. The field $\mathcal{F}$ is called the *ambient field*.

We present the definition of a cluster algebra in several steps.

Definition 2.3.8 (Cluster)**.** A *cluster* is a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ of algebraically independent elements of length n. We refer to the elements in a cluster $\mathbf{x} \in \mathcal{F}^n$ as *cluster variables*.

If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ is a cluster, then the field $\mathcal{F}$ must contain the field $\mathbb{Q}(x_1, x_2, \ldots, x_n)$. Thus, if we have a distinguished cluster $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$, then the smallest possible field, namely $\mathbb{Q}(x_1, x_2, \ldots, x_n)$, is a natural choice of an ambient field.

Definition 2.3.9 (Seed)**.** A *seed* is a pair $(\mathbf{x}, Q)$ where $\mathbf{x} \in \mathcal{F}^n$ is a cluster and Q is a quiver with vertices $Q_0 = \{1, 2, \ldots, n\}$ without loops and 2-cycles.

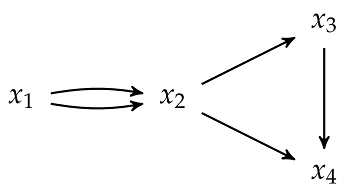


Figure 2.12: An example of a seed

Assume that $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ are two seeds given by clusters $\mathbf{x}, \mathbf{x}' \in \mathcal{F}^n$ and quivers $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q_0', Q_1', s', t')$. We say that the seeds are *isomorphic*, if there exists a quiver isomorphism given by two bijections $\sigma\colon Q_0 \to Q_0'$ and $\tau\colon Q_1 \to Q_1'$ such that $x_i = x'_{\sigma(i)}$ for all indices $i \in \{1, 2, \ldots, n\}$. In other word, two seeds are isomorphic if they are obtained from each other by a simultaneous reordering of cluster variables and quiver vertices. In this case we write $(\mathbf{x}, Q) \cong (\mathbf{x}', Q')$. Often we identify isomorphic seeds. We visualize a seed by drawing the quiver in the plane with cluster variables instead of vertices, see Figure 2.12.

Definition 2.3.10 (Mutation of seeds)**.** Let $(\mathbf{x}, Q)$ be a seed and $k \in \{1, 2, \ldots, n\}$ an index. The mutation of $(\mathbf{x}, Q)$ at k is a seed $\mu_k(\mathbf{x}, Q) = (\mu_k(\mathbf{x}), \mu_k(Q))$ where $\mu_k(Q)$ is the mutation of the quiver Q at vertex k and $\mu_k(x) = (x_1', x_2', \ldots, x_k') \in \mathcal{F}^n$ is the cluster with $x_l' = x_l$ if $l \neq k$ and

$$x_k' = \frac{1}{x_k}\left(\prod_{\alpha\colon i\to k} x_i + \prod_{\beta\colon k\to j} x_j\right) \in \mathcal{F}.$$

Here the product is taken over all arrows in $\alpha \in Q_1$ that start or terminate in vertex k, respectively, counted possibly with multiplicity. Of course, the product is understood to be 1 if there are no such arrows.

Remark 2.3.11. Let $B = B(Q)$ is the signed adjacency matrix of the quiver Q in a seed $(\mathbf{x}, Q)$, then we can rewrite the above equation as

$$x_k x_k' = \prod_{\alpha\colon i\to k} x_i + \prod_{\beta\colon k\to j} x_j = \prod_{i\in\{1,2,\ldots,n\}\colon b_{ik}>0} x_i^{b_{ik}} + \prod_{i\in\{1,2,\ldots,n\}\colon b_{ik}<0} x_i^{-b_{ik}}$$

The equation is also called *exchange relation*.

Remark 2.3.12. It is easy to see that the mutation is well-defined, i.e. the mutation of a seed at an index is again a seed. Moreover, mutation is involutory, i.e. for all seeds $(\mathbf{x}, Q)$ and all indices $k \in \{1, 2, \ldots, n\}$ we have $(\mu_k \circ \mu_k)(\mathbf{x}, Q) \cong (\mathbf{x}, Q)$: the equation $(\mu_k \circ \mu_k)(\mathbf{x}) = \mathbf{x}$ is true, because quiver mutation rules M1, M2 switch the roles of direct predecessors and direct successors, and Proposition 2.2.3 implies $(\mu_k \circ \mu_k)(Q) \cong Q$.

Mutation equivalence defines an equivalence relation on the class of all quivers without loops and 2-cycles: it clearly is transitive and reflexive and it is symmetric by Proposition 2.2.3. If the quivers Q and Q' are mutation equivalent, then we will also write $Q \sim Q'$.

Definition 2.3.13 (Mutation equivalence)**.** We say that two seeds $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ are *mutation equivalent* if there exists a sequence $(k_1, k_2, \ldots, k_r) \in Q_0^r$ of indices of length $r \geq 0$ such that the seed $(\mu_{k_1} \circ \mu_{k_2} \circ \cdots \circ \mu_{k_r})(\mathbf{x}, Q)$ is isomorphic to $(\mathbf{x}', Q')$. In this case, we also write $(\mathbf{x}, Q) \sim (\mathbf{x}', Q')$.

Definition 2.3.14 (Cluster algebra)**.** Let $(\mathbf{x}, Q)$ be a seed. The *cluster algebra* $\mathcal{A}(\mathbf{x}, Q)$ attached to the seed is the subalgebra of the ambient field $\mathcal{F}$ generated by the set

$$\chi(\mathbf{x}, Q) = \bigcup_{(\mathbf{x}', Q')\sim(\mathbf{x}, Q)} \{x_1' x_2', \ldots, x_n'\}.$$

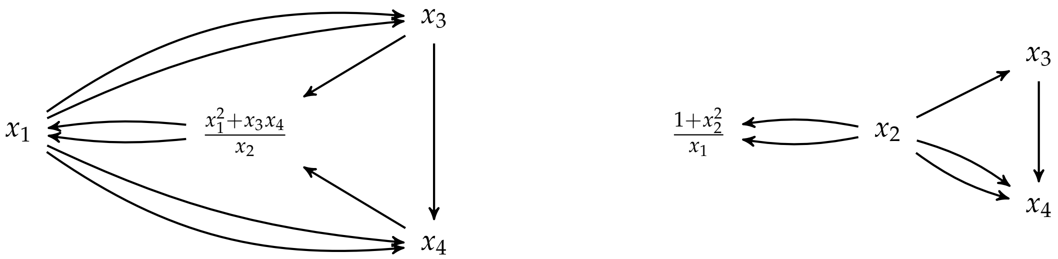


Figure 2.13: The mutations of the previous seed at vertices 2 and 1

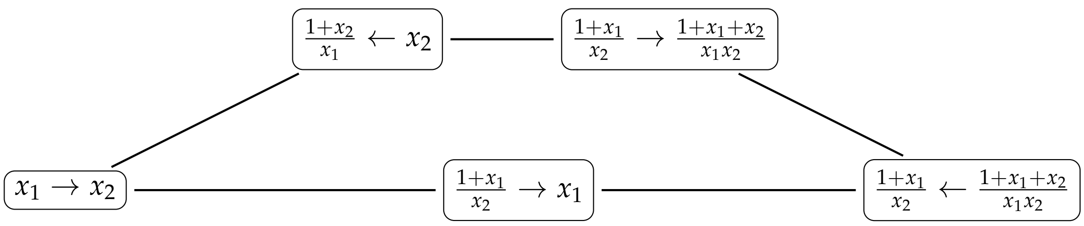
In other words it is generated by all cluster variables in all seeds that are mutation equivalent to the given seed. We also call the seeds $(\mathbf{x}', Q') \sim (\mathbf{x}, Q)$ the *seeds* of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$, the clusters $\mathbf{x}'$ the *clusters* of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ and the elements in $\chi(\mathbf{x}, Q)$ the *cluster variables* of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$. We denote the variable x_k' also by $\mu_k(x_k)$.

Let us make some remarks. First of all, since all cluster variables of a cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ lie in the subfield $\mathbb{Q}(x_1, x_2, \ldots, x_n) \subseteq \mathcal{F}$, the definition is independent of the choice of the ambient field. Of course, if the seeds $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ are mutation equivalent, then the cluster algebras $\mathcal{A}(\mathbf{x}, Q) = \mathcal{A}(\mathbf{x}', Q')$ are the same. Moreover, if $y \in \mathcal{G}^n$ is another cluster of the same length (in a another ambient field $\mathcal{G}$), then the cluster algebras $\mathcal{A}(\mathbf{x}, Q) \cong \mathcal{A}(\mathbf{y}, Q)$ are isomorphic algebras. Therefore, we sometimes write $\mathcal{A}(Q)$ instead if $\mathcal{A}(\mathbf{x}, Q)$.

If we think of the cluster algebra as being associated with a distinguished seed $(\mathbf{x}, Q)$, then we will refer to this seed as the *initial seed*. Moreover, we will refer to the natural number $n \in \mathbb{N}$ as the *rank* of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$.

Example 2.3.15. (a) Let Q be a quiver of type A_1, i.e. the quiver has a single vertex and no arrows. A cluster $\mathbf{x} = (x_1)$ contains only one element and is mutation equivalent to only one other cluster $(\frac{2}{x_1})$. Thus, up to isomorphism the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ admits two seeds, two clusters and two cluster variables and it is isomorphic to the algebra $\mathbb{Q}[x_1^{\pm 1}]$ of Laurent polynomials in one variable.

(b) Let Q be the quiver $1 \to 2$ of type A_2. We choose an initial seed $x_1 \to x_2$. Seed mutation yields the following mutation equivalence classes of seeds.



We get five (isomorphism classes) of seeds, five clusters and five cluster variables. Note the occurring terms are the same as in the introductory example from Section 1.1. Surprisingly, there are only finitely many cluster variables and all of them are elements in the Laurent polynomial ring $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}] \subseteq \mathbb{Q}(x_1, x_2)$.

The cluster algebra is the subalgebra of $\mathbb{Q}(x_1, x_2)$ generated by the set

$$\chi = \left\{ x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2} \right\}$$

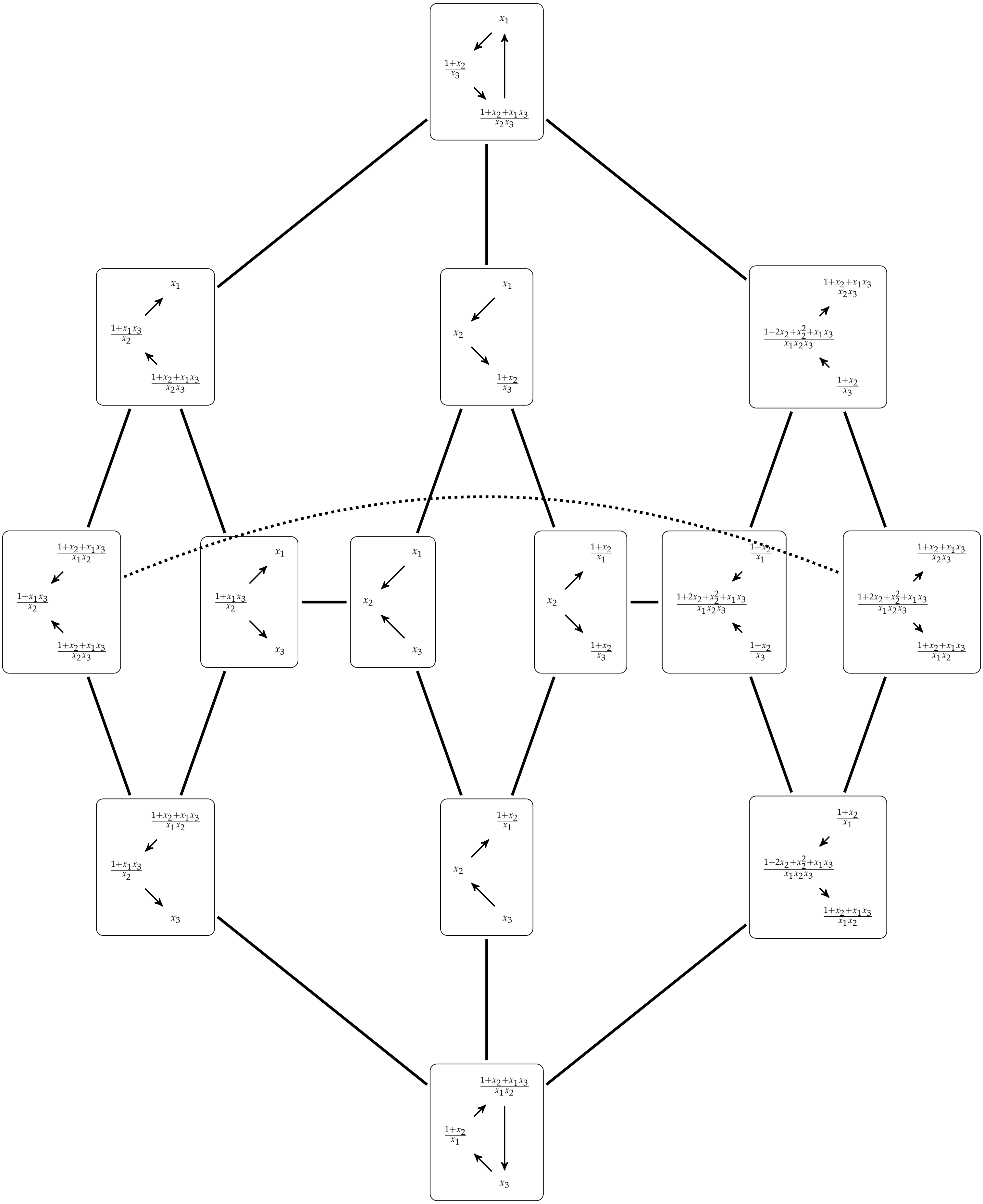


Figure 2.14: The clusters associated with quivers of type A_3

of cluster variables. The generating set is redundant. Note that

$$\frac{1+x_1+x_2}{x_1x_2} = \frac{1+x_2}{x_1}\cdot\frac{1+x_1}{x_2} - 1.$$

So that this generating set is not minimal with respect to inclusion, because a removal of this cluster variable yields the same subalgebra. Note that also $x_2 = \frac{1+x_2}{x_1}\cdot x_1 - 1$. We conclude

$$\mathcal{A}(\mathbf{x},Q) \cong \mathbb{Q}\left[x_1, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}\right] \cong \mathbb{Q}[U,V,W]/(UVW-U-V-1)$$

is actually the coordinate ring of a 2-dimensional hypersurface in 3-dimensional space.

(c) Let Q be a quiver of type $A_1\times A_1$, i.e. the quiver has two vertices and no arrows. A cluster $\mathbf{x}=(x_1)$ contains only two algebraically independent variables and its seed is mutation equivalent to three other seeds with clusters $(x_1, \frac{2}{x_2})$, $(\frac{2}{x_1}, x_2)$ and $(\frac{2}{x_1}, \frac{2}{x_2})$. Thus, the cluster algebra $\mathcal{A}(\mathbf{x},Q)$ admits four seeds, four clusters and four cluster variables and it is isomorphic to the algebra $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm}]$ of Laurent polynomials in two variables.

(d) Let Q be the quiver $1\to 2\to 3$ of type A_3. We choose an initial seed $x_1\to x_2\leftarrow x_3$. The calculations in Figure 2.14 shows that the cluster algebra has 9 cluster variables grouped into 14 seeds. As in the previous example, there are only finitely many cluster variables and all of them are elements in the Laurent polynomial ring $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}] \subseteq \mathbb{Q}(x_1,x_2,x_3)$.

Note that Figure 2.14 has the same shape as Figure 2.10. The shape is known as *associahedron* or *Stasheff polytope*. Generalising the example, we define the *exchange graph* of a cluster algebra $\mathcal{A}(\mathbf{x},Q)$ as follows: the vertices are the isomorphism classes of seeds that are mutation equivalent to the seed $(\mathbf{x},B)$ and we connect two vertices by an arrow if they are related by a single mutation. Note that the exchange graph is always *n-regular*, i.e. every vertex is adjacent to exactly n vertices.

2.4 Skew-symmetrizable matrices, ice quivers and cluster algebras

We wish to generalize the notion of cluster algebras in two ways. Firstly, we forbid mutations at certain indices which we call frozen indices. Secondly, we replace the class of quivers without loops and 2-cycles, which are given by skew-symmetric matrices due to the discussion in Section 2.2.1, with the larger class of skew-symmetrizable matrices. For the rest of the section, let us fix integers m,n with $m\geq n\geq 1$.

Let $\widetilde{B}=(b_{ij})$ be an $m\times n$ matrix with integer entries. Let us write the matrix

$$\widetilde{B} = \begin{bmatrix} B \\ C \end{bmatrix}$$

in block form, where B is an $n\times n$-matrix B and C an $(m-n)\times n$-matrix. We refer to the matrix B as the *principal part* of $\widetilde{B}$. We call the principal part B *skew-symmetrizable* if there exists a diagonal $n\times n$ matrix $D=\operatorname{diag}(d_1,d_2,\ldots,d_n)$ with positive integer diagonal entries such that the matrix DB is skew-symmetric, i.e. the equation $d_ib_{ij}=-d_jb_{ji}$ holds for all $1\leq i,j\leq n$. In this case, the matrix D is called a *skew-symmetrizer* for $\widetilde{B}$. Notice that an entry b_{ij} of a skew-symmetrizable matrix is different from zero if and only if the entry b_{ji} is different from zero. In this case, the entries have different sign. We call the matrix $\widetilde{B}$ an *exchange matrix* if the principal part is skew-symmetrizable.

We say that two exchange matrices $\widetilde{B}=(b_{ij})$ and $\widetilde{B}'=(b'_{ij})$ are *isomorphic* if there exists a permutation $\sigma\in S_m$ such that $\sigma(j)\in\{1,2,\ldots,n\}$ for all $j\in\{1,2,\ldots,n\}$ and $b_{ij}=b_{\sigma(i),\sigma(j)}$ for all $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$.

Let $\widetilde{B}$ be an exchange matrix. We call the indices $k \in \{1, 2, \dots, n\}$ *mutable* and the indices $k \in \{n+1, n+2, \dots, m\}$ *frozen*. A *mutation* of $\widetilde{B}$ at a mutable index k is the matrix $m \times n$ matrix $\mu_k(\widetilde{B}) = \widetilde{B}' = (b'_{ij})$ with entries

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}; \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases}$$

The principal part of the mutation $\widetilde{B}'$ is skew-symmetrizable with the same skew-symmetrizer D: the equations $d_i \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ = -d_j \operatorname{sgn}(b_{jk})[b_{jk}b_{ki}]_+$ for all i, j become obvious after multiplication with d_k.

Let k be a field of characteristic 0 and $\mathcal{F}$ a field extension. An *extended cluster* is a sequence $\mathbf{x} = (x_1, x_2, \dots, x_m)$ of algebraically independent elements in the ambient field $\mathcal{F}$. An *extended seed* is a pair $(\mathbf{x}, \widetilde{B})$, where $\mathbf{x}$ is an extended cluster and $\widetilde{B}$ is an exchange matrix. The *mutation* of a seed $(\mathbf{x}, \widetilde{B})$ at a mutable index k is defined by the same formulae as above: we replace the variable x_k in the extended cluster by the element

$$x'_k = \frac{1}{x_k} \left(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{jk}<0} x_j^{-b_{jk}} \right) \in \mathcal{F},$$

where the sum is taken over all mutable and frozen indices $i, j \in \{1, 2, \dots., m\}$, and replace the exchange matrix $\widetilde{B}$ with its mutation $\mu_k(\widetilde{B})$. As before, the mutation is well-defined and involutory. It defines an equivalence relation on the class of all extended seeds that we will denote by $\sim$.

Let $(\mathbf{x}, \widetilde{B})$ be an extended seed. Authors consider two versions of cluster algebras in this context. The *cluster algebra without invertible coefficients* $\mathcal{A}(\mathbf{x}, \widetilde{B})$ attached to the seed is the subalgebra of the ambient field $\mathcal{F}$ generated by the set

$$\chi(\mathbf{x}, \widetilde{B}) = \bigcup_{(\mathbf{x}', \widetilde{B}') \sim (\mathbf{x}, \widetilde{B})} \{x'_1, x'_2, \dots, x'_m\}.$$

The *cluster algebra with invertible coefficients* $\mathcal{A}(\mathbf{x}, \widetilde{B})^{inv}$ attached to the seed is the subalgebra of the ambient field $\mathcal{F}$ generated by the set

$$\chi(\mathbf{x}, \widetilde{B})^{inv} = \left(\bigcup_{(\mathbf{x}', \widetilde{B}') \sim (\mathbf{x}, \widetilde{B})} \{x'_1 x'_2, \dots, x'_n\} \right) \cup \{x_{n+1}^{-1}, x_{n+2}^{-1}, \dots, x_m^{-1}\}.$$

We will refer to the elements $x'_1, x'_2, \dots, x'_n$ in the above union as the *cluster variables* of the cluster algebra and to the elements $x_{n+1}, x_{n+2}, \dots, x_m$ as the *frozen variables* of the cluster algebra. We will refer to the number n of cluster variables in a single cluster as the *rank* of the cluster algebra.

For example, a cluster algebra $\mathcal{A}(\mathbf{x}, \widetilde{B})$ of rank 1 admits two extended clusters $(x_1, x_2, \dots, x_m)$ and $(x'_1, x_2, \dots, x_m)$. Thus, it is isomorphic to the coordinate ring $k[X'_1, X_1, X_2, \dots, X_m]/(X_1X'_1 - P)$ of an m-dimensional hypersurface for some polynomial $P \in k[X_2, \dots, X_m]$.

Let us remark that the cluster algebras $\mathcal{A}(\mathbf{x}, \widetilde{B})$ and $\mathcal{A}(\mathbf{x}, -\widetilde{B})$ are naturally isomorphic for all extended clusters $\mathbf{x}$ and all exchange matrices $\widetilde{B}$, because they have the same exchange relations.

An *ice quiver* is a quiver $Q = (Q_0, Q_1, s, t)$ without loops and 2-cycles together with a partition of the set of vertices $Q_0 = M \sqcup F$ into two sets, called *mutable* and *frozen* vertices, such that the starting and terminating point of an arrow $\alpha \in Q_1$ cannot both be frozen vertices. An *isomorphism* between ice quivers $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ is an isomorphism (f_0, f_1) of quivers such that $f_0 \colon Q_0 \to Q'_0$ maps mutable vertices to mutable vertices and frozen vertices to frozen vertices. The *mutable part* of an ice quiver Q is the full subquiver on the set of mutable vertices.

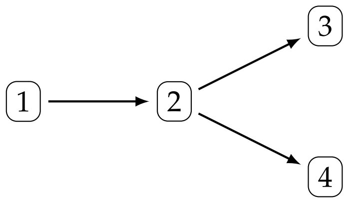
Let $Q = (Q_0, Q_1, s, t)$ be an ice quiver with mutable vertices $M = \{1, 2, \ldots, n\}$ and frozen vertices $F = \{n+1, n+2, \ldots, m\}$. For $i \in Q_0$ and $j \in M$ let $b_{ij} = a_{ij} - a_{ji}$. Then the $m \times n$ matrix $B(Q) = (b_{ij})$ is an exchange matrix with a skew-symmetric principal part. The denote the corresponding cluster algebra by $\mathcal{A}(\mathbf{x}, Q)$. Conversely, an exchange matrix with skew-symmetric principal part defines an ice quiver.

For example, let Q be the ice quiver with one mutable vertex 1, one frozen vertex 2 and one arrow $1 \to 2$. Let (x_1, x_2) be an initial extended cluster. Then the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ without invertible coefficients admits two seeds, (x_1, x_2) and (y_1, x_2) where the two cluster variables x_1 and y_1 satisfy $x_1 y_1 = 1 + x_2$. As x_1 and y_1 are algebraically independent, $\mathcal{A}(\mathbf{x}, Q) \cong k[x_1, y_1]$ is isomorphic to a polynomial ring. We obtain the cluster algebra $\mathcal{A}(\mathbf{x}, Q)^{inv}$ with invertible coefficients by localizing at the multiplicatively closed set generated by the element $x_1 y_1 - 1$.

Let $\widetilde{B}$ be an exchange matrix. For further use let us define an ice quiver $Q(\widetilde{B})$ with vertex sets $M = \{1, \ldots, n\}$ and $F = \{n+1, \ldots, m\}$ by introducing an arrow $i \to j$ between two vertices $i \in Q_0$ and $j \in M$ if and only if $b_{ij} > 0$ and by drawing an arrow $j \to i$ between two vertices $i \in F$ and $j \in M$ if and only if $b_{ji} < 0$. We say that the seed $(\mathbf{x}, \tilde{B})$ is *acyclic* if the mutable part of the quiver $Q(\widetilde{B})$ does not contain an oriented cycle. In this case, we also call the matrix $\widetilde{B}$ *acyclic*. Finally, we call the cluster algebra $\mathcal{A}(\mathbf{x}, \tilde{B})$ *acyclic* if it admits an acyclic seed.

2.5 Exercises

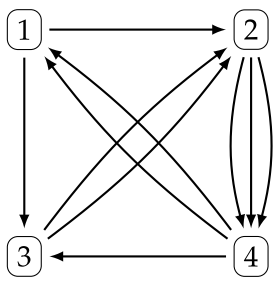
Exercise 2.1. Let Q be the following acyclic quiver with four vertices and three arrows:



(a) Compute $\mu_2(Q)$.

(b) Describe the mutation class of Q.

Exercise 2.2. Let Q be the following quiver:



(a) Compute the mutations $\mu_1(Q)$, $\mu_2(Q)$ and $\mu_2(\mu_2(Q))$.

(b) Among the four quivers Q, $\mu_1(Q)$, $\mu_2(Q)$ and $\mu_2(\mu_2(Q))$, decide which pairs are isomorphic.

Exercise 2.3. Construct all triangulations of a regular pentagon. What are their flips?

Exercise 2.4. Let $n \in \mathbb{N}$ be a natural number and B a skew-symmetric matrix with integer entries. Assume that $k \in \{1, 2, \ldots, n\}$ and $B' = \mu_k(B)$ is the mutation of B at k.

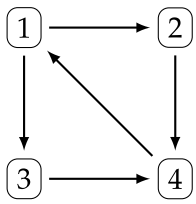
(a) Prove that the rank is mutation invariant, i.e. prove that $\operatorname{rank}(B) = \operatorname{rank}(B')$.

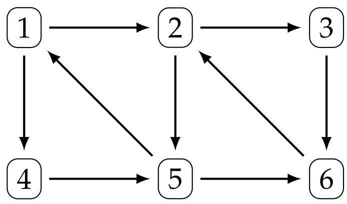
(b) Prove that the greatest common divisor of the entries in a column is mutation invariant, i.e. prove that the equation $\gcd(b_{ij}\colon i \in \{1,2,\dots,n\}) = \gcd(b'_{ij}\colon i \in \{1,2,\dots,n\})$ holds for all $j \in \{1,2,\dots,n\}$.

(This observation is due to Jan Schröer.)

Exercise 2.5. Let $Q = (Q_0, Q_1, s, t)$ the quiver with vertices $Q_0 = \{1,2,3\}$ and arrows $1 \to 2, 2 \to 3$ and $1 \to 3$. Describe the mutation class of Q.

Exercise 2.6. Use Keller's applet to prove that the following quivers are mutation to quivers whose underlying diagram is a tree (and hence they are mutation equivalent to acyclic quivers).





(This observation was pointed out by Andrew Hubery.)

Exercise 2.7. Let $\mathcal{A}(\mathbf{x}, Q)$ be the cluster algebra attached to the seed $x_1 \to x_2 \leftarrow x_3$ of type A_3 from Example 2.3.15 (c). Then the set of cluster variables

$$\chi(\mathbf{x}, Q) = \left\{ x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{1+x_1x_3}{x_2}, \frac{1+x_2}{x_3}, \frac{1+x_2+x_1x_3}{x_1x_2}, \frac{1+x_2+x_1x_3}{x_2x_3}, \frac{(1+x_2)^2+x_1x_3}{x_1x_2x_3} \right\},$$

is by definition a generating set of the cluster algebra. In this exercise we want to convince ourselves that this generating set is not minimal with respect to inclusion at all.

(a) Let us introduce the abbreviations $u = x_1$, $v = x_3$, $w = \frac{1+x_2}{x_1}$, $t = \frac{1+x_2+x_1x_3}{x_2x_3}$. Write every cluster variable in $\chi(\mathbf{x}, Q)$ as a polynomial expression in u, v, w, t.

(b) Prove that the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is isomorphic to the coordinate ring of a 3-dimensional hypersurface in 4-dimensional space.

Exercise 2.8. In this exercise we wish to establish a left and right symmetry in the definition of a skew-symmetrizable matrix. Let n be a positive integer. Prove that an $n \times n$ matrix $\widetilde{B}$ with integer entries is skew-symmetrizable if and only if there exists a diagonal matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ with positive integer entries such that BD is skew-symmetric.

Chapter 3

Examples of cluster algebras

3.1 Sequences and Diophantine equations attached to cluster algebras

A *Diophantine* equation is a polynomial equation in one or several variables over the integers that we wish to solve also over the integers. As the tenth in the famous list of problems David Hilbert asks for an algorithm to solve every Diophantine equation. Matiyasevich's theorem asserts that no algorithm exists. Thus, particular methods to solve Diophantine equations are particularly interesting. In this section we wish to illustrate how cluster theory can help to solve particular Diophantine equations.

3.1.1 Sequences associated with cluster algebras

Let $\mathcal{A}(\mathbf{x}, Q)$ be a cluster algebra of rank $n \in \mathbb{N}$ associated with a quiver Q with vertices $Q_0 = \{1, 2, \ldots, n\}$. Let us consider the vertex $1 \in Q_0$. Proposition 2.2.3 asserts that $\mu_1(\mu_1(Q)) \cong Q$. By chance, we sometimes have $\mu_1(Q) \cong Q$. In this section we wish to construct a sequence of distinguished cluster variables in this situation.

Suppose that the mutation $\mu_1(Q) = Q' = (Q_0, Q'_1, s', t')$ is isomorphic to the original quiver $Q = (Q_0, Q_1, s, t)$. Then, by definition the isomorphism is given by two bijections $\sigma\colon Q_0 \to Q_0$ and $\tau\colon Q_1 \to Q'_1$. For simplicity suppose that the bijection σ is the cyclic permutation $(123 \ldots n)$. Let us denote the cluster variable $x'_1 = \mu_1(x_1)$ by the symbol x_{n+1}. A cyclic reordering transforms the mutated cluster $\mu_1(\mathbf{x}, Q) = (x'_1, x_2, \ldots, x_n)$ into the cluster $(x_2, x_3, \ldots, x_n, x_{n+1})$. We see that the mutated seed $\mu_1(\mathbf{x}, Q)$ is isomorphic to the seed $((x_2, x_3, \ldots, x_{n+1}), Q))$ with the *same* quiver Q. If we iterate this process, we get a sequence of cluster variables $(x_i)_{i \in \mathbb{N}^+}$. Moreover, there is a single Laurent polynomial $P \in \mathbb{Z}[X_1, X_2, \ldots, X_n]$ such that the recursive formula $x_{n+i} = P(x_i, x_{i+1}, \ldots, x_{i+n-1})$ holds true for all indices $i \geq 1$. Using this formula we can extend the sequence to a sequence $(x_i)_{i \in \mathbb{Z}}$. Note that the x_i with $i \leq 0$ are also cluster variables of the cluster algebras $\mathcal{A}(\mathbf{x}, Q)$.

The above discussion is very theoretical. Do such situations actually exist? In the following sections we will see that desired isomorphisms $\mu_k(Q) \cong Q$ occasionally exist and we will study the associated sequences for these examples.

3.1.2 Cluster algebras attached to quivers with 2 vertices

Let us study such sequences for some examples. First of all, all cluster algebras of rank 2 have this properties. Consider the quiver $Q(b)$ from Example 2.2.6 with two vertices $1, 2$ and $b \geq 1$ arrows $1 \to 2$. As we have already mentioned in a previous section, we have an isomorphism $\mu_1(Q) \cong Q$

of quiver, so that we can construct a sequence of cluster variables as above. The first elements in the sequence are the initial cluster variables $x_1, x_2 \in \mathcal{F}$. We can use the following recursion formulae to compute the elements x_i with $i \geq 2$ and $i \leq 0$, respectively:

$$x_{i+2} = \frac{1 + x_{i+1}^b}{x_i}, \qquad x_{i-2} = \frac{1 + x_{i-1}^b}{x_i}.$$

More symmetrically, we can write the recursion as $x_{i+1}x_{i-1} = x_i^b + 1$ for all $i \in \mathbb{Z}$.

Let us look at the cases $b = 1, 2$ in more detail. We have already studied the case $b = 1$, in which Q is a quiver of type A_2. The computations in Example 2.3.15 show that the sequence $(x_i)_{i \in \mathbb{Z}}$ becomes a 5-periodic sequence

$$\ldots, x_1, x_2, \frac{1 + x_2}{x_1}, \frac{1 + x_1 + x_2}{x_1 x_2}, \frac{1 + x_1}{x_2}, x_1, x_2, \frac{1 + x_2}{x_1}, \ldots$$

of Laurent polynomials in x_1 and x_2, with integer coefficients. Hence the cluster algebra is of finite type. Now let us consider the case $b = 2$. The quiver $Q(2)$ is known as the *Kronecker quiver*, because Leopold Kronecker classified the representations of the quiver. Some clusters are shown in Figure 3.1 and some cluster variables in the sequence $(x_i)_{i \in \mathbb{Z}}$ are the following:

$$\ldots, \frac{1 + x_1^2}{x_2}, x_1, x_2, \frac{1 + x_2^2}{x_1}, \frac{1 + 2x_2 + x_2^2 + x_1}{x_1^2 x_2}, \frac{1 + 3x_2 + 3x_2^2 + x_2^3 + 2x_1 + 2x_1 x_2 + x_1^2}{x_1^3 x_2^2}, \ldots$$

By Exercise 1.2 the sequence is not periodic so that the cluster algebra is of infinite type. Nevertheless the sequence several remarkable properties. As we have mentioned in Section 2.2.6, it is useful to study a sequence by its *invariants*. Surprisingly, the sequence $(x_i)_{i \in \mathbb{Z}}$ admits an invariant.

Proposition 3.1.1. Let Q be the Kronecker quiver and $(x_i)_{i \in \mathbb{Z}}$ the sequence of cluster variables of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$. Then the rational expression

$$T(i) = \frac{1 + x_i^2 + x_{i+1}^2}{x_i x_{i+1}} \in \mathcal{F}$$

is independent of $i \in \mathbb{Z}$ and hence an invariant.

Proof. Let $i \in \mathbb{Z}$ be an integer. Then the elementary calculations

$$\frac{1 + x_i^2 + x_{i+1}^2}{x_i x_{i+1}} = \frac{x_i^2 + x_i x_{i+2}}{x_i x_{i+1}} = \frac{x_i + x_{i+2}}{x_{i+1}} = \frac{x_i x_{i+2} + x_{i+2}^2}{x_{i+1} x_{i+2}} = \frac{1 + x_{i+1}^2 + x_{i+2}^2}{x_{i+1} x_{i+2}}.$$

imply that $T(i) = T(i+1)$ for all $i \in \mathbb{Z}$. Hence the expression remains unchanged when we pass from $i \in \mathbb{Z}$ to $i + 1$ and we have $T(i) = T(j)$ for all $i, j \in \mathbb{Z}$ by mathematical induction. □

Let us denote this element by $T = T(i) \in \mathcal{F}$. Note that T is actually in element in the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$, because we can write the element as $T = x_0 x_3 - x_1 x_2$. Moreover, the above calculation shows that $T x_i = x_{i-1} + x_{i+1}$ for all $i \in \mathbb{Z}$. In other words, the non-linear exchange relation $x_{i+1} x_{i-1} = x_i^2 + 1$ for all $i \in \mathbb{Z}$ degenerates to the linear recurrence relation $x_{i+1} + x_{i-1} = T x_i$ for all $i \in \mathbb{Z}$. As further consequence we can conclude on the one hand that the cluster algebra $\mathcal{A}(\mathbf{x}, Q) \cong \mathbb{Q}[x_1, x_2, T] = \mathbb{Q}x_0, x_1, x_2, x_3$ is finitely generated and isomorphic to the coordinate ring $\mathbf{Q}[X_1, X_2, Y]/(X_1^2 + X_2^2 + 1 - X_1 X_2 Y)$ of a hypersurface . On the other hand we can conclude that all cluster variables are Laurent polynomials in x_1 and x_2.

If we specialize $x_1 = x_2 = 1$, then the positive part sequence of cluster variables specializes to the sequence $(f_i)_{i \in \mathbb{N}}$ from the solution of Exercise 1.2 which is defined by the starting values $f_0 = f_1 = 1$ and the recursion $f_{i+1} + f_{i-1} = 3f_i$. (And we had noticed the degeneration of the

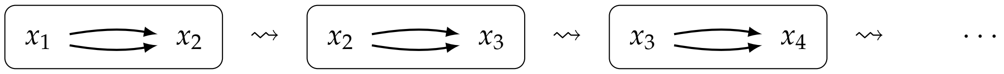


Figure 3.1: The sequence of cluster variables for the Kronecker quiver

non-linear exchange relation to a linear exchange relation already in the solution of that exercise.) Proposition 3.1.1 shows that the specialized clusters (f_i, f_{i+1}) for all $i \in \mathbb{N}$ are integer solutions to the quadratic Diophantine equation $a^2 + b^2 + 1 = 3ab$. The next proposition asserts that we can characterize the sequence by this property.

Proposition 3.1.2. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ be a pair of natural numbers such that $a^2 + b^2 + 1 = 3ab$. Then there exists a natural number $i \in \mathbb{N}$ such that $(a, b) = (f_i, f_{i+1})$ or $(b, a) = (f_i, f_{i+1})$.

Proof. We prove the proposition by mathematical induction on $\max(a, b)$. It is clear that if $a = 0$ or $b = 0$, then the pair $(a, b) \in \mathbb{N} \times \mathbb{N}$ can not be a solution of the equation $a^2 + b^2 + 1 = 3ab$. Therefore, the pair $(1, 1)$ is the only solution with $a, b \leq 1$. Furthermore, it is the only solution with $a = b$.

Now let us suppose that $(a, b) \in \mathbb{N} \times \mathbb{N}$ is a solution where at least one of the entries is strictly larger than 1. Without loss of generality we can assume that $a < b$ so that $b > 1$. Let us fix a. Then $X^2 + a^2 + 1 - 3aX = 0$ is a quadratic equation with root b. Let b' be the unique other root of this quadratic equation. By Viete's Theorem we have $b' = 3a - b$ and $bb' = a^2 + 1$. The pair (b', a) is again a solution with positive integers. We claim that $\max(b', a) < \max(b, a)$. We have $a < b$ by assumption and $b' < b$, because otherwise the chain of inequalities $a^2 + 1 = bb' \geq b^2 \geq (a + 1)^2$ would be true, which is impossible. By induction hypothesis there exists a natural number $i \in \mathbb{N}$ such that $(b', a) = (f_{i-1}, f_i)$. We conclude that $(a, b) = (\frac{a^2+1}{b}, b) = (f_i, f_{i+1})$. □

In other words, the *cluster transformations* $\mu_1 \colon (a, b) \mapsto (\frac{1+b^2}{a}, b)$ and $\mu_2 \colon (a, b) \mapsto (a, \frac{1+a^2}{b})$ generate, starting from the initial solution $(1, 1)$, all positive solutions of the Diophantine equation $a^2 + b^2 + 1 = 3ab$. More precisely, the pairs $(\mu_1 \circ \mu_2 \circ \mu_1 \ldots)(1, 1)$ yield the positive solutions (a, b) with $a > b$ and the pairs $(\mu_2 \circ \mu_1 \circ \mu_2 \ldots)(1, 1)$ yield the positive solutions (a, b) with $a < b$.

3.1.3 Cluster algebras of rank 2

Let us amplify the discussion in Section 3.1.1. We replace quivers, which correspond to skew-symmetric exchange matrices, by skew-symmetrizable matrices. A non-zero skew-symmetrizable 2×2 matrix has the form $B = \left(\begin{smallmatrix} 0 & -b \\ c & 0 \end{smallmatrix}\right)$ for some non-zero integers b and c with the same sign. Without loss of generality let us assume that $b, c > 0$ or $b = c = 0$. Let $\mathbf{x} = (x_1, x_2)$ be an initial cluster. Then the cluster variables form a sequence $(x_n)_{n \in \mathbb{Z}}$. The exchange relation from some cluster (x_{n-1}, x_n) to the neighboring cluster (x_n, x_{n+1}) is

$$x_{n+1}x_{n-1} = \begin{cases} x_n^b + 1, & \text{if n is even;} \\ x_n^c + 1, & \text{if n is odd;} \end{cases}$$

for all $n \in \mathbb{Z}$. Thus, without loss of generality we may assume $c \geq b$. For brevity, we denote the corresponding cluster algebra by $\mathcal{A}(b, c)$.

In the case $(b, c) = (1, 2)$ the sequence $(x_n)_{n \in \mathbb{Z}}$ is periodic with period 6, as an elementary calculation shows (compare also Section 1.1):

$$\ldots, x_1, x_2, \frac{x_2^2 + 1}{x_1}, \frac{x_1 + x_2^2 + 1}{x_1 x_2}, \frac{x_1^2 + 2x_1 + 1 + x_2^2}{x_1 x_2^2}, \frac{x_1 + 1}{x_2}, x_1, x_2, \ldots$$

In a similar spirit, the sequence $(x_n)_{n\in\mathbb{Z}}$ is periodic with period 8 in the case $(b,c) = (1,3)$ (compare also Exercise 1.1):

$$\ldots, x_1, x_2, \frac{x_2^3+1}{x_1}, \frac{x_1+x_2^3+1}{x_1x_2}, \frac{x_1^3+3x_1^2+3x_1+1+2x_2^3+3x_1x_2^3+x_2^6}{x_1^2x_2^3},$$
$$\frac{x_1^2+2x_1+1+x_1^3}{x_1x_2^3}, \frac{x_1^3+3x_1^2+3x_1+1+x_2^3}{x_1x_2^3}, \frac{x_1+1}{x_2}, x_1, x_2, \ldots$$

The next proposition asserts that these two example together with the cluster algebras of type A_2 are the only instances of coefficient-free cluster algebras of rank 2 with only finitely many cluster variables.

Proposition 3.1.3. The cluster algebra $\mathcal{A}(b,c)$ admits only finitely many cluster variables if and only if $bc \in \{0,1,2,3\}$.

Proof. The example show that $\mathcal{A}(b,c)$ admits only finitely many cluster variables if $bc \in \{0,1,2,3\}$. For the reverse direction, suppose that $bc \geq 4$. We specialize $x_1 = x_2 = 1$ and show that the sequence of specialized cluster variables $(x_{2n})_{n\in\mathbb{N}^+}$ is strictly increasing, hence the corresponding non-specialized cluster variables are pairwise different. We prove the statement by mathematical induction. By definition, we have $x_3 = 2$ and so $x_4 = 2^c + 1 > 1 = x_2$. Assume that the positive integer n is even. By definition, the exchange relations

$$x_{n-2}x_n = x_{n-1}^c + 1, \qquad x_{n-1}x_{n+1} = x_n^b + 1, \qquad x_nx_{n+2} = x_{n+1}^c + 1$$

hold. It follows that $x_{n-1}^c x_{n+1}^c = (x_n^b+1)^c$ and $(x_{n-2}x_n - 1)(x_nx_{n+2} - 1) = (x_n^b+1)^c$. So the inequality $x_{n+2} > x_n$ is equivalent to the inequality $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_n^b + 1)^c$. Inequality $bc \geq 4$ implies that we either have $b, c \geq 2$ or we have $b = 1$ and $c \geq 4$. As the first case, suppose that $b, c \geq 2$. It suffices to show that $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_n^2 + 1)^2$, which is equivalent to the obvious inequality $x_{n-2}x_n^3 < x_n^4 + 3x_n^2 + x_{n-2}x_n$. As the second case, suppose that $b = 1$ and $c \geq 4$. It suffices to show that $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_n + 1)^4$, which is equivalent to the obvious inequality $x_{n-2}x_n^3 < x_n^4 + 4x_n^3 + 7x_n^2 + 4x_n + x_{n-2}x_n$. □

3.1.4 An example of rank 3

For a second example, let $n \in \mathbb{N}$ be a natural number. Let us consider the quiver $Q = (Q_0, Q_1, s, t)$ with vertices $Q = \{1, 2, \ldots, n\}$ where we introduce exactly one arrow $i \to j$ whenever $i < j$. Then Q is an acyclic quiver and the vertices are in topological order. Then it is easy to see that the quiver $\mu_1(Q)$ is isomorphic to Q and an isomorphism is given the permutation $(12\ldots n)$. Let $(x_i)_{i\in\mathbb{Z}}$ be the associated sequence of cluster variables.

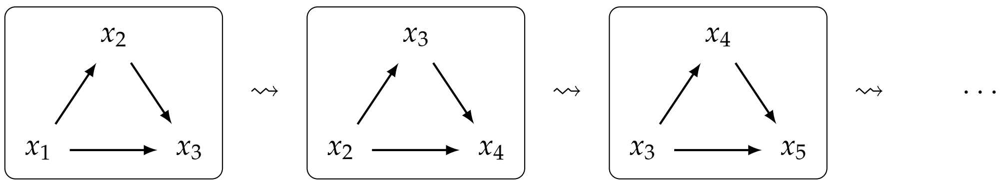


Figure 3.2: Another sequence of cluster variables

For simplicity let us assume that $n = 3$. In this case the sequence $(x_i)_{i\in\mathbb{Z}}$ is given by the initial cluster $(x_1, x_2, x_3) \in \mathcal{F}^3$ and the recursion formula

$$x_{i+3} = P(x_i, x_{i+1}, x_{i+2}) = \frac{1 + x_{i+1}x_{i+2}}{x_i}$$

for $i \in \mathbb{Z}$. Figure 3.2 illustrates the sequence of clusters. The situation is in some respects very different from the Kronecker case. For example, as we mutate every cluster at only two vertices, we cannot expect every cluster variables to appear in the sequence. On the other hand the sequence shares certain features with the cluster algebra of the Kronecker quiver. Surprisingly, the sequence also admits an invariant. More precisely:

Proposition 3.1.4. The following rational expression is independent of $i \in \mathbb{Z}$:

$$T = T(i) = \frac{x_{i-1} + x_{i+1} + x_i(x_{i-1}^2 + x_{i+1}^2)}{x_{i-1}x_ix_{i+1}}.$$

Proof. Let $i \in \mathbb{Z}$ and $(r,s,t) = (x_{i-1}, x_i, x_{i+1})$. Put $x_{i+2} = \frac{1+st}{r} = r'$. Then we have

$$\frac{s + r' + t[s^2 + (r')^2]}{str'} = \frac{sr^2 + (1+st)r + t[r^2s^2 + (1+st)^2]}{rst(1+st)} = \frac{r + t + sr^2 + st^2}{rst},$$

which implies the claim. □

Remark 3.1.5. For all $i \in \mathbb{Z}$ we have

$$Tx_{i+1} - x_{i-1} = \frac{x_{i-1} + x_{i+1} + x_ix_{i+1}^2}{x_{i-1}x_i} = \frac{1 + x_{i+1}x_{i+2}}{x_i} = x_{i+3}.$$

As in the case of the Kronecker quiver, the non-linear recurrence relation $x_{i+3} = P(x_1, x_2, x_3)$ for all $i \in \mathbb{Z}$ degenerates to the linear recurrence relation $x_{i+2} + x_{i-2} = Tx_i$ for all $i \in \mathbb{Z}$. Especially, all occuring terms are Laurent polynomials in x_1, x_2 and x_3.

As above, let us specialize $x_1 = x_2 = x_3 = 1$. The table in Figure 3.3 lists the first elements in the sequence. The linear recurrence relation implies that all elements are natural numbers. Because T specializes to 4, all specialized clusters (x_i, x_{i+1}, x_{i+2}) for $i \in \mathbb{N}^+$ are solution to the quadratic Diophantine equation $a + c + b(a^2 + c^2) = 4abc$. For example, the triples $(1,1,1)$, $(1,1,2)$, $(1,2,3)$ and $2,3,7$ are solutions to this equation. However, not every solution of the equation comes from such a triple as the solution $(6,1,2)$ shows: the sequence $(x_i)_{i\in\mathbb{N}}$ is weakly increasing, so we have $x_i \le x_{i+1} \le x_{i+2}$ for all natural numbers $i \ge 1$.

i	1	2	3	4	5	6	7	8
x_i	1	1	1	2	3	7	11	26

Figure 3.3: The sequence of specialized cluster variables

3.1.5 Somos sequences and cluster algebras

Let $k \ge 2$ be an integer. The *k-Somos sequence* is a sequence $(a_i)_{i\in\mathbb{N}^+}$ of rational numbers defined by starting values $a_1 = a_2 = \ldots, a_k = 1$ and the recursion

$$a_{i+k} = \frac{1}{a_i} \sum_{r=1}^{\lfloor k/2 \rfloor} a_{i+k-r}a_{k+r}.$$

for all natural numbers $i \ge 1$. For instance, the k-Somos sequence is constant for $k \in \{2,3\}$. The first non-trivial cases are $k = 4$, in which case recursion becomes $a_{i+4}a_i = a_{i+3}a_{i+1} + a_{i+2}^2$, and $k = 5$, in which case recursion becomes $a_{i+5}a_i = a_{i+4}a_{i+1} + a_{i+3}a_{i+2}$. The following table lists elements of these Somos sequences.

	i	1	2	3	4	5	6	7	8	9	10
Somos 4	a_i	1	1	1	1	2	3	7	23	59	314
Somos 5	a_i	1	1	1	1	1	2	3	5	11	37

Figure 3.4: Somos sequences

Figure 3.5: The quivers for two Somos sequences

Note that all elements are in the table natural numbers – an observation truely linked to the Laurent phenomenon. The sequences are named after Michael Somos, who encounters the sequence in the context of elliptic theta functions, cf. Gale [Gal]. Several authors have then introduce further Somos sequences. Malouf [Mal] proves that the elements of the 4-Somos sequence are integers. Fomin-Zelevinsky [FZ5] prove that the k-Somos sequence is integral for $k = 3, 4, 5, 6$ and we will have a look at their argument when we will discuss the Laurent phenomenon. The k-Somos sequence has non-integer elements for $k = 7, 8, 9, \ldots$

Are the sequences associated with some cluster algebra? Yes, indeed they are and Figure 3.5 displays the associated quivers. The quiver is featured in Exercise 2.2. In both cases we have $\mu_1(Q) \cong Q$ and the attached sequences of cluster variables specialize to the Somos sequences. Higher Somos sequence do not admit an interpretation as sequences of cluster variables. In the case $k = 6$ for example, the Somos exchange relation is trinomial whereas the cluster exchange relation is binomial.

3.1.6 The Markov equation and cluster algebras

Let Q be the quiver with vertices $Q_0 = \{1, 2, 3\}$ and exactly two arrows $1 \rightrightarrows 2$, exactly two arrows $2 \rightrightarrows 3$ and exactly two arrows $3 \rightrightarrows 1$ from Figure 2.1. Here, mutation at *every* vertex yields an ismorphic quiver, i.e. we have $\mu_1(Q) \cong \mu_2(Q) \cong \mu_3(Q) \cong Q$. We conclude that for every cluster (x_1, x_2, x_3) of $\mathcal{A}(\mathbf{x}, Q)$ the three possible mutations have the same form

$$\mu_1\colon (x_1, x_2, x_3) \mapsto \left(\frac{x_2^2 + x_3^2}{x_1}, x_2, x_3\right),$$
$$\mu_2\colon (x_1, x_2, x_3) \mapsto \left(x_1, \frac{x_1^2 + x_3^2}{x_2}, x_3\right),$$
$$\mu_3\colon (x_1, x_2, x_3) \mapsto \left(x_1, x_2, \frac{x_1^2 + x_2^2}{x_3}\right).$$

Again, we can study the cluster algebra by its invariant. It is easy to check that the three

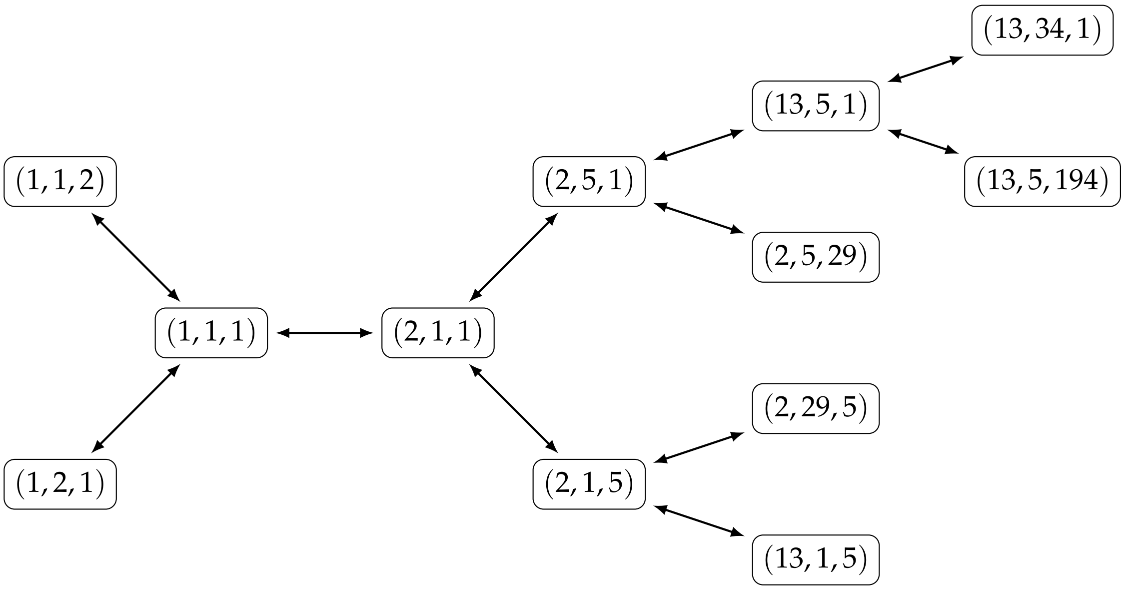


Figure 3.6: Solutions to the Markov equation

mutations do not change the expression

$$T = \frac{x_1^2 + x_2^2 + x_3^3}{x_1 x_2 x_3} \in \mathcal{F}.$$

The invariant specializes to $T = 3$ when we specialize $x_1 = x_2 = x_3 = 1$. The three mutations yield solutions to the Diophantine equation $a^2 + b^2 + c^2 = 3abc$. This equation is known as the *Markov equation* as Markov [Ma] introduces the equation. Beineke-Brüstle-Hille [BBH] notice the importance of the Markov equation for cluster algebras of rank 3. Using similar arguments as in the Kronecker case we can show the following.

Proposition 3.1.6. All solutions of the Markov equation can be obtained from the trivial solution by a sequence of cluster transformations.

Proof. Let $(a, b, c) \in \mathbb{N}^3$ be a solution of the equation $a^2 + b^2 + c^2 = 3abc$. We prove by mathematical induction on $\max(a, b, c)$ that there exists a sequence $(i_1, i_2, \ldots, i_r)$ of indices in $\{1, 2, 3\}$ of length $r \geq 0$ such that $(a, b, c) = (\mu_{i_1} \circ \mu_{i_2} \circ \ldots \circ \mu_{i_r})(1, 1, 1)$.

Note that no solutions with $a = 0, b = 0$ or $c = 0$ exists. Then the claim is true for $\max(a, b, c) = 1$. Let us now assume that $(a, b, c) \in \mathbb{N}^3$ is a solution with $\max(a, b, c) \geq 2$. Without loss of generality we may assume that $a \leq b \leq c$. Note that $a \neq b$ and $b \neq c$ in this case, so that we have $a < b < c$. We use the same trick as in the Kronecker case. Let $c' \in \mathbb{Q}$ be the second solution of the quadratic equation $f(X) = X^2 - 3abX + a^2 + b^2$. Viete's theorem implies $c' = 3ab - c$ and $cc' = a^2 + b^2$, so that c' is again a positive integer. The triple (a, b, c) is again a solution in positive integers. Note that $f(b) = 2b^2 + a^2 - 3ab^2 < 0$ unless $a = b = 1$, in which case we get solutions $(1, 1, 1)$ and $(1, 1, 2)$ of required form. We conclude $f(b) = (b - c)(b - c') < 0$ and $b < c'$, so that $\max(a, b, c') < \max(a, b, c)$. The claim now follows easily by applying the induction hypothesis to the triple (a, b, c'). □

We will refer to the cluster algebras featured in this section also as *Kronecker cluster algebra*, *Somos cluster algebras* and *Markov cluster algebra*.

3.2 The Kronecker cluster algebra

Let Q be the Kronecker quiver $1 \rightrightarrows 2$ and $\mathbf{x} = (x_1, x_2) \in \mathcal{F}$ a cluster of length 2. We have seen that the element $T = \frac{1+x_1^2+x_2^2}{x_1x_2} \in \mathcal{A}(\mathbf{x}, Q)$ plays a special role. In the following sections we wish to study the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ more closely, because cluster algebras of rank 2 are a good prototype for the whole theory.

3.2.1 Homogeneous linear recurrence relations

In Section 3.1.2 we have seen that the cluster variables of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ form a sequence obeying a linear recurrence relation. In this section we wish to recollect some material on linear recurrences. We work over an algebraically closed, but otherwise general base field k.

Let $n \geq 1$ be a positive integer and let $\mu = (\mu_0, \mu_1, \ldots, \mu_n) \in k^{n+1}$ be a sequence of scalars with $\mu_0, \mu_n \neq 0$. We denote by $A(\mu)$ the set of all sequences $\mathbf{a} = (a_i)_{i\in\mathbb{Z}}$ of elements in k such that the relation $\mu_0 a_i + \mu_1 a_{i+1} + \ldots + \mu_n a_{i+n} = 0$ holds for all integers $i \in \mathbb{Z}$. The set $A(\mu)$ is closed under addition and scalar multiplication and hence forms a linear subspace of the k-vector space $k^{\mathbb{Z}}$ of all sequences of elements in k. Moreover, it is closed under shifts. We define a map $\phi\colon A(\mu) \to k^n$ by setting $\phi(\mathbf{a}) = (a_1, a_2 \ldots, a_n)$.

Proposition 3.2.1. The map $\phi\colon A(\mu) \to k^n$ is an isomorphism of vector spaces.

Proof. The map ϕ is clearly k-linear. We prove that it is surjective and injective. Let $(a_1, a_2, \ldots, a_n) \in k^n$ be a vector. We extend the vector to sequence $\mathbf{a} \in A(\mu)$ by setting recursively

$$\begin{aligned} a_{i+n} &= -\frac{1}{\mu_n}(\mu_0 a_i + \mu_1 a_{i+1} + \ldots + \mu_{n-1} a_{i+n-1}), && \text{for } i \geq 1; \\ a_i &= -\frac{1}{\mu_0}(\mu_1 a_{i+1} + \mu_2 a_{i+2} + \ldots + \mu_n a_{i+n}), && \text{for } i \leq 0. \end{aligned}$$

By construction $\mathbf{a} \in A(\mu)$ and $\phi(\mathbf{a}) = (a_1, a_2, \ldots, a_n)$, so the map ϕ is surjective. Suppose that $\phi(\mathbf{a}) = 0$ for some $\mathbf{a} \in A(\mu)$. By an induction argument the recursion implies that $a_i = 0$ for all integers $i \in \mathbb{Z}$. Hence the map ϕ is also injective. □

Especially, the k-dimension of the vector space $A(\mu)$ is equal to n. The *characteristic polynomial* of μ is the polynomial $P_\mu = \mu_0 + \mu_1 X + \mu_2 X^2 + \ldots + \mu_n X^n \in k[X]$. Let λ be a root of the characteristic polynomial (in the algebraically closed field k) with multiplicity r. Note that $\lambda \neq 0$. First of all, note that the sequence $(\lambda^i)_{i\in\mathbb{Z}}$ is an element in $A(\mu)$. Moreover, for every $s \in \{1, 2, \ldots, r-1\}$ the element λ is also a zero of the s-th derivative $P_\mu^{(s)}$ of the characteristic polynomial. Hence, the sequence $\big((i+1)(i+2)\ldots(i+s-1)\lambda^i\big)_{i\in\mathbb{Z}}$ is an element in $A(\mu)$. We deduce that the space $A_\lambda = \{p(i)\lambda^i \colon p \in k[X] \text{ s. th. } \deg(p) < r\}$ is a subspace of $A(\mu)$ of dimension r. As the sum of the dimensions of the A_λ, for all roots λ of P_μ, is equal to n and as their contains only the 0, we get a direct sum decomposition $A(\mu) = \oplus_\lambda A(\lambda)$

For example, we can derive an explicit formula for the Fibonacci numbers $(F_n)_{n\in\mathbb{N}}$ with $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ by calculating the zeros of the characteristic polynomial. In fact, the roots of the polynomial $X^2 - X - 1$ are closely related to the *golden section*. We get

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

In particular, the formula implies that the quotient F_{n+1}/F_n converges to the golden section in the limit $n \to \infty$.

3.2.2 A combinatorial interpretation of the coefficients

In Section 3.1.2 we have seen that the cluster variables of the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ are Laurent polynomials in $\mathbf{Z}[x_1^{\pm}, x_2^{\pm}]$. In this section we wish to provide a combinatorial interpretation of the coefficients.

First note that the cluster $(x_1, x_2) \in \mathcal{F}^2$ is a $\mathcal{F}$-linear combination of $(1, 0)$ and $(0, 1)$. Thus it is enough to study the simpler sequence with starting values $(0, 1)$ instead of (x_1, x_2). Let us define a sequence $(s_n)_{n\in\mathbb{Z}}$ by $s_0 = 1$, $s_1 = T$ and $s_{n+1} = Ts_n - s_{n-1}$ for all $n \in \mathbb{Z}$. Note that $s_{-1} = 0$. The elements 1, T, $T^2 - 1$, $T^3 - 2T$, $T^4 - 3T^2 + 1, \ldots$ of the sequence are called *Chebychev polynomials*, or more precisely normalized Chebychev polynomials of the second kind.

For natural numbers $n, p, q \in \mathbb{N}$ let $c_{n,p,q} \in \mathbb{N}$ be the number of subsets of $\{1, 2, \ldots, n\}$ that contain exactly p odd, exactly q even and no consecutive elements. For example, we have $c_{7,2,1} = 3$, because the possible subsets are $\{1, 3, 6\}$, $\{1, 4, 7\}$ and $\{2, 5, 7\}$. A famous exercise asserts that the number of all subsets without consecutive numbers (and no condition on the number of odd and even numbers) is a Fibonacci number. The next statement is due to Caldero-Zelevinsky [CZ, Theorem 5.2].

Proposition 3.2.2. Let $n \geq 0$ be a natural number. Then the element $s_n \in \mathcal{F}$ is equal to the sum

$$s_n = \frac{1}{x_1^n x_2^n} \left(\sum_{p=0}^{n} \sum_{q=0}^{n} c_{2n,p,q} x_1^{2q} x_2^{2p} \right).$$

Proof. We prove the proposition by strong mathematical induction. The statement is clearly true for $n = 0$ and $n = 1$. Now assume that $n \geq 2$. By induction hypothesis, the coefficient $x_1^{2q-n} x_2^{2p-n}$ in the Laurent polynomial $Ts_{n-1} - s_{n-2}$ is equal to $c_{2n-2,p,q} + c_{2n-2,p,q-1} + c_{2n-2,p-1,q} - c_{2n-4,p,q}$. Now let us partition the subsets A of $\{1, 2, \ldots, n\}$ that contain exactly p odd, exactly q even and no consecutive elements into three classes. No such subset contains both $2n$ and $2n - 1$.

1. Assume that $2n \notin A$ and $2n - 1 \notin A$.

   In this case, the set A is a subset of $\{1, 2, \ldots, 2n - 2\}$ with exactly p odd, exactly q even and no consecutive elements. The number of such subsets is equal to $c_{2n-2,p,q}$.

2. Assume that $2n \notin A$ and $2n - 1 \in A$.

   It follows that $2n - 2 \notin A$. In this case, the A is a union of $\{2n - 1\}$ and a subset A' with exactly $p - 1$ odd, exactly q even and no consecutive elements. The number of such subsets is equal to $c_{2n-2,p-1,q}$. But not every such subset A' yields an admissible set A. The subsets A' that do not yield admissible sets A are precisely the sets with $2n - 2 \in A'$. For those subsets we have $2n - 3 \notin A'$, so that A' is the union of $\{2n - 2\}$ and a subset A'' of $\{1, 2, \ldots, 2n - 4\}$ with exactly $p - 1$ odd, exactly $q - 1$ even and no consecutive elements. Every such subset A'' yields an admissible subset A' and the number of such subsets is equal to $c_{2n-4,p-1,q-1}$. Altogether we get $c_{2n-2,p-1,q} - c_{2n-4,p-1,q-1}$ sets A.

3. Assume that $2n \in A$ and $2n - 1 \notin A$.

   In this case, the set A is the union of $\{2n\}$ and a subset A' of $\{1, 2, \ldots, 2n - 2\}$ with exactly p odd, exactly $q - 1$ even and no consecutive elements. The numbers of such subsets is equal to $c_{2n-2,p,q-1}$ and every such subset A' yields an admissible set A.

We deduce that the coefficient of monomial $x_1^{2q-n} x_2^{2p-n}$ in the Laurent polynomial s_n is equal to the number $c_{2n-2,p,q} + c_{2n-2,p-1,q} + c_{2n-2,p,q-q} - c_{2n-4,p-1,q-1} = c_{2n,p,q}$. $\square$

In this recursion formula we only use number $c_{n,p,q}$ for even numbers n. In a similar spirit, there are recursion formulae for the numbers $c_{n,p,q}$ that involve both even and odd numbers. For instance, we can prove the formula $c_{2n+1,p,q} = c_{2n,p,q} + c_{2n-1,p-1,q}$ for all natural numbers $n, p, q \in \mathbb{N}$ by distinguishing whether the largest number $2n+1$ is an elements in the considered set or not. Similarly, we have $c_{2n+2,p,q} = c_{2n+1,p,q} + c_{2n,p,q-1}$ for all natural numbers $n, p, q \in \mathbb{N}$. It turns out that the recursions are useful for deriving a formula for the cluster variables.

Proposition 3.2.3. Let $n \geq 1$ be a natural number. Then the cluster variable x_n is equal to the sum

$$x_{n+2} = \frac{1}{x_1^n x_2^{n-1}} \left(\sum_{p=0}^{n} \sum_{q=0}^{n} c_{2n-1,p,q} x_1^{2q} x_2^{2p} \right).$$

Proof. We have $x_1 = x_2 s_{-1} - x_1 s_0$ and $x_2 = x_2 s_0 - x_1 s_{-1}$, so that we have $x_{n+2} = x_2 s_n - x_1 s_{n-1}$ for all $n \in \mathbb{Z}$. We conclude

$$\begin{aligned} x_{n+2} &= \frac{1}{x_1^n x_2^n} \left(\sum_{p=0}^{n} \sum_{q=0}^{n} c_{2n,p,q} x_1^{2q} x_2^{2p+1} \right) - \frac{1}{x_1^{n-1} x_2^{n-1}} \left(\sum_{p=0}^{n-1} \sum_{q=0}^{n-1} c_{2n-2,p,q} x_1^{2q+1} x_2^{2p} \right) \\ &= \frac{1}{x_1^n x_2^{n-1}} \left(\sum_{p=0}^{n} \sum_{q=0}^{n} c_{2n,p,q} x_1^{2q} x_2^{2p} - \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} c_{2n-2,p,q} x_1^{2(q+1)} x_2^{2p} \right) \\ &= \frac{1}{x_1^n x_2^{n-1}} \sum_{p=0}^{n} \sum_{q=0}^{n} \left(c_{2n,p,q} - c_{2n-2,p,q-1} \right) x_1^{2q} x_2^{2p} = \frac{1}{x_1^n x_2^{n-1}} \sum_{p=0}^{n} \sum_{q=0}^{n} c_{2n-1,p,q} x_1^{2q} x_2^{2p} \end{aligned}$$

Here, we use the convention $c_{n,p,q} = 0$ whenever one of the indices is negative, which is consistent with all recursion formulae. The statement follows. □

We can obtain a similar formula for the cluster variables x_n with $n \leq 0$ by switching the roles of x_1 and x_2. Especially, all coefficients are non-negative numbers, because they count the number of elements in certain sets. It is a common strategy to find a combinatorial interpretation for proving positivity of givem integer numbers. A more general conjecture of Fomin-Zelevinsky [FZ] asserts the following:

Conjecture 3.2.4 (Positivity conjecture)**.** Let $\mathcal{A}(\mathbf{x}, Q)$ be an arbitrary cluster algebra attached to a seed $(\mathbf{x}, Q)$ of rank n. Then every cluster variable u of $\mathcal{A}\mathbf{x}, Q)$ is an element the semiring $\mathbb{N}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Laurent polynomials with non-negative coefficients in the initial cluster $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

There is an explicit formula for the coefficients in term of *binomial coefficients*. Recall that for natural number $n, k \in \mathbb{N}$ the binomial coefficient $\binom{n}{k}$ counts the number of subsets of $\{1, 2, \ldots, n\}$ with exactly k elements. We adopt the convention $\binom{n}{k} = 0$ when $k < 0$ and that $\binom{n}{0} =$ for $n < 0$. With a similar argument as above, by distinguishing whether n is in the considered subset or not, we obtain the recursion formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for all natural numbers $k \in \mathbb{N}$ and all positive integers $n \in \mathbb{N}^+$. Therefore, the binomial coefficients fit into *Pascal's triangle*.

Binomial coefficients occur in various combinatorial contexts. The name comes from the interpretation of the numbers as coefficients in the binomial expansion $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-1}$ for all $n \geq 0$. For another example, the binomial coefficient $\binom{a+b}{a}$ counts the number of shortest lattice paths from the point $(0,0) \in \mathbb{N}^2$ to the point $(a,b) \in \mathbb{N}^2$.

Proposition 3.2.5. Let $m \geq 1$. The number of subsets of $\{1, 2, \ldots, m\}$ that contain exactly p odd, exactly q even and no consecutive elements is given by the formulae

$$c_{m,p,q} = \begin{cases} \binom{n-p}{q} \binom{n-q}{p}, & \text{if } m = 2n \text{ is even;} \\ \binom{n-p}{q} \binom{n+1-q}{p}, & \text{if } m = 2n+1 \text{ is odd;} \end{cases}$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Figure 3.7: Pascal's triangle

1	4	10	20	35	56
1	3	6	10	15	21
1	2	3	4	5	6
1	1	1	1	1	1

Figure 3.8: The number of shortest lattice paths

for all $p \in \{0, 1, \ldots, \lfloor \frac{m+1}{2} \rfloor\}$ and all $q \in \{0, 1, \ldots, n\}$.

Proof. We prove the statement by strong mathematical induction on m. It is true for $m = 1$ due to the convention $\binom{-1}{0} = 1$. Now assume that $m \geq 2$. Using the induction hypothesis we deduce that, depending on whether m is even or odd,

$$\begin{aligned}
c_{2n,p,q} &= c_{2n-1,p,q} + c_{2n-2,p,q-1} = \tbinom{n-1-p}{q}\tbinom{n-q}{p} + \tbinom{n-1-p}{q-1}\tbinom{n-q}{p} = \tbinom{n-p}{q}\tbinom{n-q}{p}, \\
c_{2n+1,p,q} &= c_{2n,p,q} + c_{2n-1,p-1,q} = \tbinom{n-p}{q}\tbinom{n-q}{p} + \tbinom{n-p}{q}\tbinom{n-q}{p-1} = \tbinom{n-p}{q}\tbinom{n+1-q}{p},
\end{aligned}$$

which proves the statement. □

The author is not aware of a direct combinatorial proof. Nonetheless the proposition gives an explicit formula for the coefficients that appear when we write the cluster variables as Laurent polynomials in the initial cluster variables.

3.2.3 Sketch of the derivation of another formula for the coefficients

There is a different way to compute the elements s_n for $n \geq 0$, by the methods introduced in the last section. The roots of the characteristic polynomial $X^2 - TX + 1$ attached to the linear recurrence relation are the two elements

$$\lambda_{1,2} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4}\right) \in \overline{\mathcal{F}}.$$

The sequences $(s_n)_{n\in\mathbb{Z}}$ and $(x_n)_{n\in\mathbb{Z}}$ are then linear combinations of the sequences $(\lambda_1^n)_{n\in\mathbb{Z}}$ and $(\lambda_2^n)_{n\in\mathbb{Z}}$. In fact, for s_n we get combination (which we only need to check for $n = -1, 0$):

$$s_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}.$$

Using the explicit formula for the eigenvalues and the binomial theorem we can derive the formula $s_n = \sum_{s=0}^{n}(-1)^s\binom{n-s}{s}T^{n-2s}$ for the Chebychev polynomials s_n for all $n \geq 0$. Both expressions must be equal, so we have proved the identity

$$\frac{1}{x_1^n x_2^n}\sum_{p=0}^{n}\sum_{q=0}^{n}\binom{n-p}{q}\binom{n-q}{p}x_1^{2q}x_2^{2p} = \sum_{s=0}^{n}(-1)^s\binom{n-s}{s}\left(\frac{1+x_1^2+x_2^2}{x_1x_2}\right)^{n-2s}.$$

Again, the author is not aware of a direct proof of the identity.

3.3 Cluster algebras of type A

In this section we wish to provide to models for cluster algebras of type A. The first model is the coordinate ring of the Grassmann variety, where the Plücker relations play the role of the exchange relations. The second model arises from the triangulations of regular polygon which we studied in Section 2.2.3. Here, the Ptolemy relations play the role of the exchange relations.

3.3.1 Generalities on Grassmann varieties

Let d and n be natural numbers such that $d \leq n$. The *Grassmannian* $\mathrm{Gr}_d(k^n)$ is the set of all d-dimensional subspaces of the n-dimensional vector space k^n. It is named after the mathematician and linguist Hermann Grassmann.

In particular, the projective space $\mathbb{P}^n(k)$ is the Grassmannian $\mathrm{Gr}_1(k^{n+1})$ of lines in the vector space k^{n+1}. A line is spanned by a non-zero vector $(x_0, x_1, \ldots, x_n) \in k^{n+1}$. Two such vectors $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ span the same line if and only if they are linearly dependent, i.e. there exists some scalar $\lambda \in k$ such that $x = \lambda y$. In this case we will write $x \sim y$. We deduce that $\mathbb{P}(k) = (k^n\backslash\{0\})/\sim$. We denote the equivalence class of $x = (x_0, x_1, \ldots, x_n) \in k^{n+1}\backslash\{0\}$ in $\mathbb{P}^n(k)$ by $(x_0 : x_1 : \ldots : x_n)$.

In general, note that a d-dimensional vector space is spanned by d linearly independent vectors $(x_{i1}, x_{i2}, \ldots, x_{in}) \in k^n$ for $i \in \{1, 2, \ldots, d\}$. The vectors constitute the row vectors of a $d \times n$ matrix X of full rank with entries in k. Two such matrices $X, Y \in \mathrm{Mat}_{d\times n}(k)$ define the same subspace if and only if there exists an invertible matrix $d \times d$ matrix A such that $X = AY$. In this case we will write $X \sim Y$, which defines an equivalence relation.

For every subset $I \subseteq \{1, 2, \ldots, n\}$ and every $X \in \mathrm{Mat}_{d\times n}(k)$ we define the *Plücker coordinate* $p_I(X)$ as the $d \times d$ minor of the matrix X on columns I. The coordinates are named after Julius Plücker. Altogether, there are $N = \binom{n}{d}$ Plücker coordinates, and so the collection $(p_I(X))$ defines a point the affine space k^N (after choosing an ordering of the indices I). For example, a 2-dimensional subspace V in $\mathbb{R}^3$ is spanned by two vectors. In this case, there are three Plücker coordinates. The coordinates are essentially the coordinates of the cross product $x \times y$, which uniquely determines the subspace V, because it is perpendicular to V with respect to the standard scalar product. It turns out that this is not true in general.

Next we wish to explain why the Grassmannian $\mathrm{Gr}_d(k^n)$ is in general a projective variety. First of all, note that for a matrix X of full rank $(p_I(X)) \in k^N$ cannot be zero, because not all the $d \times d$ vanish. Moreover, if $X = AY$ for some $A \in \mathrm{Gl}_d(k)$, then $p_I(X) = \det(A)p_I(Y)$ for all subsets I. It follows that $(p_I(X)) \sim (p_I(Y))$ in this case, so that every d-dimensional subspace $V \subseteq k^n$ defines an element $\phi(V) \in \mathbb{P}^{N-1}(k)$. We call the map the *Plücker embedding*. The next proposition justifies the name.

Proposition 3.3.1 (Plücker embedding)**.** The map $\phi\colon \operatorname{Gr}_d(k^{n+1}) \to \mathbb{P}^{N-1}(k)$ is injective.

Proof. Suppose that the subspaces V, W satisfy $\phi(V) = \phi(W)$. Let $X, Y \in \operatorname{Mat}_{d\times n}(k)$ be matrices that define these subspaces. Then by definition we have $p_I(X) = p_I(Y)$ for all subsets $I \in \{1, 2, \ldots, n\}$. There exists at least one such subset with $p_I(X) \neq 0$, and hence $p_I(Y) \neq 0$. By an appropriate change of bases we may assume that the column vectors of X and Y attached to the indices I are for both matrices the standard basis vectors $e_1, e_2, \ldots, e_d \in k^d$. As both basis changes are induced by (possibly different) matrices with the same determinant, the relations $p_I(X) = p_I(Y)$ remain true for all subsets $I \in \{1, 2, \ldots, n\}$. The $d \times d$ minors of X and Y and on columns $(I\backslash\{i\}) \cup \{j\}$ are equal to x_{ij} and y_{ij}, respectively, for all $i \in \{1, 2, \ldots, d\}$ and $j \in \{1, 2, \ldots, n\}$. We can conclude that $X = Y$ and $V = W$. □

On the contrary, the map ϕ is not surjective in general, but we can describe the image of the map explicitly. To formulate a precise statement, let us introduce some notation. Let $X \in \operatorname{Mat}_{d\times n}(k)$ be a matrix with rows $x_1, x_2, \ldots, x_n \in k^d$. For a (not necessarily increasing) sequence $\mathbf{i} = (i_1, i_2, \ldots, i_d)$ of elements in $\{1, 2, \ldots, n\}$ let $p_{\mathbf{i}}(X)$ be the determinant of the matrix with columns $x_{i_1}, x_{i_2}, \ldots, x_{i_d}$. Of course, $p_{\mathbf{i}}(X)$ is zero if it contains an element twice and it changes the sign when we swap two elements. We extend this notation to arbitrary points in k^{N-1}. Let $(p_I) \in k^{N-1}$ and let $\mathbf{i}$ be a sequence of elements in $\{1, 2, \ldots, n\}$ of length d. In this situation we define $p_{\mathbf{i}}$ to be zero, if the sequence contains an elements twice, and to be $(-1)^r p_I$, where I is the set of elements in $\mathbf{i}$ and r is the numbers of swaps needed to transform $\mathbf{i}$ into an increasing sequence, otherwise.

Now let $I, J \subseteq \{1, 2, \ldots, n\}$ be two subsets of size $d-1$ and $d+1$, respectively, with elements $i_1 < i_2 < \ldots < i_{d-1}$ and $j_1 < j_2 < \ldots < j_{d+1}$. For every natural number $r \in \{1, 2, \ldots, d+1\}$ let $\mathbf{i}(r)$ be the sequence $(i_1, i_2, \ldots, i_{d-1}, j_r)$ and $\mathbf{j}(r)$ be the sequence $(j_1, j_2, \ldots, \widehat{j_r}, \ldots, j_{d+1})$.

Proposition 3.3.2 (Plücker relations)**.** An element $(p_I) \in \mathbb{P}^{N-1}(k)$ lies in the image of the Plücker embedding ϕ if and only if the alternating sum

$$\sum_{r=1}^{d+1} (-1)^r p_{\mathbf{i}(r)} p_{\mathbf{j}(r)} = 0$$

vanishes for all subsets $I, J \subseteq \{1, 2, \ldots, n\}$ of size $d-1$ and $d+1$, respectively.

Proof. For the first direction we prove that the Plücker relations hold for all tuples (p_I) in the image of the map, i.e. they hold for all tuples $(p_I(X))$ where $X = (x_{i,j}) \in \operatorname{Mat}_{d\times n}(k)$ is a matrix of full rank. Write $X = (x_1, x_2, \ldots, x_n)$ as a sequence of column vector. Suppose that $i_1 < i_2 < \ldots < i_{d-1}$ and $j_1 < j_2 < \ldots < j_{d+1}$ are sequences as above. We want to show that

$$\sum_{r=1}^{d+1} (-1)^r \det(x_{i_1}, x_{i_2}, \ldots, x_{i_{d-1}}, x_{j_r}) \det(x_{j_1}, x_{j_2}, \ldots, \widehat{x_{j_r}}, \ldots, x_{j_{d+1}}) = 0.$$

We expand the determinant in the first factor of each summand along the last column according to Laplace's rule. Denote the $(d-1) \times (d-1)$ matrix obtained from $(x_{i_1}, x_{i_2}, \ldots, x_{i_{d-1}})$ by deleting the s-th row by $X_{\mathbf{i}}^{(s)}$. Then the left hand side of the last equation is equal to

$$\sum_{r=1}^{d+1} \sum_{s=1}^{d} (-1)^{r+s+d} x_{s,j_r} \det(X_{\mathbf{i}}^{(s)}) \det(x_{j_1}, x_{j_2}, \ldots, \widehat{x_{j_r}}, \ldots, x_{j_{d+1}}).$$

Now switch the order of summation. The sum may be be seen as a Laplace expansion of the $(d+1)\times(d+1)$ matrix $X_{\mathbf{j}}^{(s)}$ that we obtain from $(x_{j_1}, x_{j_2}, \ldots, x_{j_{d+1}})$ by adding a copy of the s-th row vector on the top of the matrix. The determinant of this matrix vanishes, because it has two equal rows. We conclude that the whole sum must be zero.

For the reverse direction, suppose that some element $(p_I) \in \mathbb{P}^{N-1}(k)$ satisfies the Plücker relations. We construct a preimage. The subspace

$$V = \left\{ v = (v_r) \in k^n \colon \sum_{r=1}^{d+1} (-1)^r p_{j_1, j_2, \ldots, \widehat{j_r}, \ldots, j_{d+1}} v_{i_r} = 0 \text{ for all } 1 \le j_1 < j_2 < \ldots < j_{d+1} \le n \right\}$$

can be shown to have dimension d. By construction we have $\phi(V) = (p_I)$. □

For more detailed information on the Pücker embedding see Kleiman-Laksov [KL].

As a consequence, every Grassmannian $\mathrm{Gr}_d(k^n)$ is a projective algebraic subvariety of the projective space $\mathbb{P}^{N-1}(k)$. Using the description of the Grassmannian as orbits of the action of the group $\mathrm{Gl}_d(k)$ (of dimension d^2) on the set of matrices of full rank (which form a dense subset of the space $\mathrm{Mat}_{d\times n}(k)$ of dimension dn) one can show that the dimension of $\mathrm{Gr}_d(k^n)$ is $nd - d^2 = (n-d)d$.

3.3.2 Grassmannians and cluster algebras

Let n be a natural number. We consider the Grassmannian $\mathrm{Gr}_2(k^{n+3})$ of planes in the space k^{n+1}. Then the Plücker coordinates are indexed by sequences $1 \le a < b \le n+3$ of length 2. We will write P_{ab} instead of $P_{(a,b)}$. Let us work out the Plücker relations explicitly. Possible subsets I consist of one element, say $I = \{a\}$ for some $1 \le a \le n+3$ and possible subsets I consist of three elements, say $J = \{b, c, d\}$ for some $1 \le b < c < d \le n+3$. The Plücker relation becomes $P_{ab}P_{cd} - P_{ac}P_{bd} + P_{ad}P_{bc} = 0$.

The set of relations is very redundant. We show that it suffices to impose the Plücker relations with $1 \le a < b < c < d \le n+3$. Note that if $a = b$, then this relation becomes the trivial relations $P_{ac}P_{ad} = P_{ac}P_{ad}$ which we can neglect. The same is true for $a = c$ or $a = d$. If a lies between b and c, then the relation is essentially equal to the Plücker relation for $I = \{b\}$ and $J = \{a, c, d\}$, so we can neglect relations arising for these choices as well. The same is true when a lies between c and d or when a is larger than d.

Let us specialize $k = \mathbb{C}$. We conclude that the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_2(k^{n+3})]$ of the Grassmannian is isomorphic to the algebra

$$\mathcal{A} = \mathbb{C}[X_{ab} \colon 1 \le a, b \le n+3] / (X_{ab}X_{cd} - X_{ac}X_{bd} + X_{ad}X_{bc} \colon 1 \le a < b < c < d \le n+3).$$

If we think of the element X_{ab} of being attached to the segment M_{ab} of the regular $(n+3)$-gon from Section 2.2.3 that connects the vertices a and b. Then M_{ab} is a side of the polygon if a and b are consecutive numbers and a diagonal otherwise. We deduce that the Plücker relation precisely becomes the exchange relation for the cluster algebra attached to the quiver of the triangulation. We deduce that the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_2(k^{n+3})]$ carries the structure of a cluster algebra. Every triangulation T defines a quiver $Q(T)$ with n mutable and $n+3$ frozen vertices, whose mutation class contains all quivers of type A. It also defines a collection of cluster and frozen variables. Thus, it defines a seed and a cluster algebra. By the above discussion, the cluster algebra is naturally isomorphic to the homogeneous coordinate ring of the Grassmannian.

Moreover, If we specialize all frozen variables to 1, we get a geometric model for the cluster algebra of type A without frozen vertices.

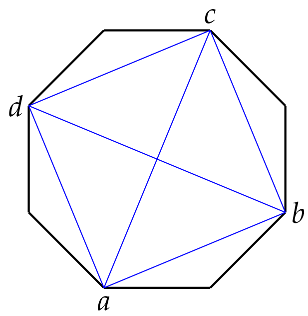


Figure 3.9: A quadrangle in a triangulation

Let us summarize the combinatorial model for the cluster algebra:

Cluster variables	$\leftrightarrow$	Diagonals
Frozen variables	$\leftrightarrow$	Sides
Clusters	$\leftrightarrow$	Triangulations
Mutations	$\leftrightarrow$	Flips
Exchange relations	$\leftrightarrow$	Plücker relations

In this context, the relations are also called *Ptolemy relations*. Ptolemy's theorem is classical theorem from Euclidean geometry. Assume that A, B, C, D are points in the Euclidean plane that lie in tis order on a circle. The theorem asserts that $|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$. For a proof, let us identify the Euclidean plane and the complex plane. For arbitrary complex numbers, the identity $(A-C)(B-D) = (A-B)(C-D) + (A-D)(B-C)$ holds. We deduce $|AC| \cdot |BD| \leq |AB| \cdot |CD| + |AD| \cdot |BC|$. Equality holds if and only if the complex numbers 0, $(A-B)(C-D)$ and $(A-D)(B-C)$ are collinear, i.e. the quotient of the numbers is a positive real number. The quotient is known as the the *cross ratio*. We conclude that we have equality especially in the case when A, B, C, D lie in this order on the circle, in which case the polar angles of the quotients $(A-B)/(A-D)$ and $(C-D)/(B-C)$ are equal.

Unfortunately, Euclidean geometry yields no direct model of cluster algebras. If the points A, B, C, D lie on a circle, then the distances also satisfy the relation $|AB| \cdot |BC| \cdot |AC| + |BC| \cdot |CD| \cdot |BD = |AB| \cdot |DA| \cdot |BD| + |BC| \cdot |CD| \cdot |BD|$. Fomin-Shapiro-Thurston [FST] and Fomin-Thurston [FT] give a model via non-Euclidean geometry by the concept of lambda lengths.

3.3.3 Triangulations of polygons and Schiffler's expansion formula

Ralf Schiffler gives an explicit formula for arbitrary cluster variables in this cluster algebra as Laurent polynomials in an initial cluster. We wish to present the formula and follow Schiffler's article in our exposition.

Let n be a natural number. For the rest of the section we fix a triangulation $\{T_1, T_2, \ldots, T_n\}$ of the regular polygon P_{n+3} by n diagonals. Moreover, we denote the sides of the $(n+3)-gon$ by $T_{n+1}, T_{n+2}, \ldots, T_{2n+3}$. We denote the set of all T_i with $\{1, 2, \ldots, 2n+3\}$ by the letter T. Let us extend the notion of crossing diagonals to side. We say that two sides or diagonals M and M' of P_{n+3} are *crossing* if they are different and the intersection of M and M' contains no point in the interior of P_{n+3}. The fundamental definition is the following.

Definition 3.3.3 (Schiffler path)**.** Let a and b be different vertices of P_{n+3}. A *Schiffler path* from a to b is a sequence $(a_0, a_1, \ldots, a_r)$ of vertices of P_{n+3} such that the following properties hold:

(T1) The path starts in a and ends in b, i.e. we have $a_0 = a$ and $a_r = b$.

(T2) For every pair (a_{k-1}, a_k) with $k \in \{1, 2, \ldots, r\}$ of consecutive vertices there exists an $T_{i_k} \in T$ that connects a_{k-1} and a_k.

(T3) No edge occurs twice, i.e. we have $i_j \neq i_k$ if $j \neq k$.

(T4) The length r of the path is an odd number.

(T5) The element T_{i_k} crosses M_{ab} if k is even.

(T6) If $1 \leq j < k \leq r$ and the elements T_{i_j} and T_{i_k} both cross the segment M_{ab}, then the crossing point of T_{i_j} with M_{ab} is closer to a than the crossing point of T_{i_j} with M_{ab}.

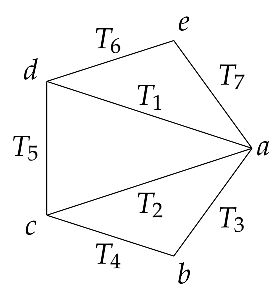
Definition 3.3.4 (Path monomials)**.** With a Schiffler path $\alpha = (a_0, a_1, \ldots, a_r)$, that uses the edges $T_{i_1}, T_{i_2}, \ldots, T_{i_r}$, we associate the monomial

$$x(\alpha) = \prod_{k \text{ odd}} x_{i_k} \prod_{k \text{ even}} x_{i_k}^{-1} \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_{2n+3}^{\pm 1}].$$

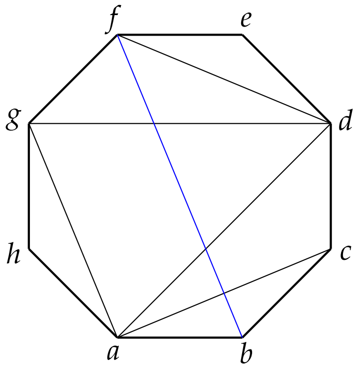
Note that condition $(T4)$ is superflous. For two different vertices a and b of P_{n+3} we denote the set of all Schiffler paths from a to b by the symbol $\mathcal{P}(a, b)$. Moreover, we define the element $X(a, b) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_{2n+3}^{\pm 1}]$ to be the sum of the monomials $X(\alpha)$ for all Schiffler paths α from vertex a to vertex b.

Example 3.3.5. (a) Let a and b be vertices of P_{n+3} such that $M_{ab} \in T$. As no element in T crosses M_{ab}, a Schiffler must have length 1. Thus, there is only one Schiffler path from a to b whose only edge is the segment M_{ab} itself.

(b) For a triangulation of a regular pentagon we obtain the following Schiffler paths. As we have noticed above, there is only one Schiffler path from a to d, so that $X(a, d) = x_1$. Similarly, $X(a, e) = x_2$ and the sides of the polygon yields the variables x_3, x_4, x_5, x_6, x_7. What are the Schiffler paths from b to d? There is no such path of length 1 as the segment M_{bd} does not belong to the triangulation. The segment crosses only the element $T_2 \in T$. So there are two Schiffler paths, namely (b, c, a, d) with monomial $x_1 x_2^{-1} x_4$ and (b, a, c, d) with monomial $x_2^{-1} x_3 x_5$. We conclude that $X(b, d) = (x_1 x_4 + x_3 x_5)/x_2$ and similarly $X(c, e) = (x_2 x_6 + x_5 x_7)/x_1$. Finally, the Schiffler paths from b to e are (b, c, a, e), (b, a, d, e) and (b, a, c, d, a, e), so that $X(b, e) = (x_1 x_4 x_7 + x_2 x_3 x_6 + x_3 x_5 x_7)/(x_1 x_2)$.



(c) Let T be the following triangulation of an octogon. The Schiffler paths from b to f are (b, a, d, f), (b, a, c, d, g, f), (b, c, a, g, d, f) and (b, a, c, d, a, g, d, f).



Note that the definition s symmetric. More precisely, if $(a_0, a_1, \dots, a_r)$ is Schiffler path from a to b, then $(a_r, a_{r-1}, \dots, a_0)$ is a Schiffler path from b to a. Especially, we deduce that $X(a,b) = X(b,a)$ for all different vertices a, b of P_{n+3}.

Let $\mathcal{A}$ be the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_d(k^{n+3})]$ of the Grassmannian, which is a cluster algebra. For every $i \in \{1, 2, \dots, 2n+3\}$ we identity the element T_i of the triangulation with the corresponding vertex in the ice quiver $Q(T)$. We view $Q(T)$ as an initial ice quiver. If T_i connects vertices a and b, then we identify the element x_i (which is equal to $X(a,b)$ by the example above) with the cluster or frozen variable $x_{ab} \in \mathcal{A}$.

Theorem 3.3.6 (Schiffler)**.** For all different vertices a and b we have $X(a,b) = X_{ab}$.

As corollaries, we obtain the Laurent phenomenon and the positivity of the coefficients. In fact, all non-zero coefficients are equal to 1, as the following argument shows. Suppose that α and β are Schiffler paths from vertex a to vertex b. We have to show that $x(\alpha) \neq x(\beta)$. Suppose that $x(\alpha) \neq x(\beta)$. Comparing the denominators, we see the set of diagonals that cross M_{ab} is the same for both paths. By $(T6)$ the crossing diagonals appear in the same order. But then the sequences of vertices are the same for both paths, so that $\alpha = \beta$.

Proof. We prove the theorem by mathematical induction on the number of elements $T_i \in T$ that cross the segment M_{ab}. There are no such crossings if and only if the segment M_{ab} belongs to T, in which case the statement is true by the above discussion.

Now assume that the segment M_{ab} crosses at least one element in T. Among all the diagonals that cross M_{ab} we consider the diagonal $T_i \in T$ whose crossing point is closest to the vertex a. The diagonal T_i connects two vertices, which we may call c and d. Note that the four vertices a, b, c, d are pairwise different and hence form a quadrilateral. We introduce the abbreviations $M = M_{ab}$, $L = M_{bc}$ and $L' = M_{bd}$. Moreover, by m we denote the intersection point of the diagonals of the quadrilateral $abcd$. Finally, we denote by P_a and P_b the two polygons that we obtain from P_{n+3} by cutting along T_i such that $a \in P_a$ and $b \in P_b$.

We make a series of observations. Firstly, we claim that M_{ac} and M_{ad} belong to T. For a proof, note that the diagonal T_i (as any diagonal) borders exactly two triangles of the triangulation, one of which lies in P_a and one of which lies in P_b. The third vertex of the triangle in P_a must be a, because otherwise we would get a crossing point of M and an element in T that is closer to a than m. Therefore, $M_{ad} = T_j$ and $M_{ac} = T_{j'}$ for some indices $j, j' \in \{1, 2, \dots, 2n+3\}$.

Secondly, every $T_k \in T$ that crosses L must also cross M, because it cannot leave the triangle bcm on the side mc. We conclude that the number of elements $T_k \in T$ that cross the segment L is smaller than the number of elements $T_k \in T$ that cross the segment M. The same statement is true for L'. By induction hypothesis we can assume that $X(c,b) = X_{cb}$ and $X(d,b) = X_{db}$. Thus it suffices to prove $X(a,b)x_i = X(c,b)x_j + X(d,b)x_{j'}$.

Thirdly, the converse of the previous observation is false. There are diagonals $T_k \in T$ that that cross M, but do not cross L (as the diagonal T_j shows for example). However, every such diagonal

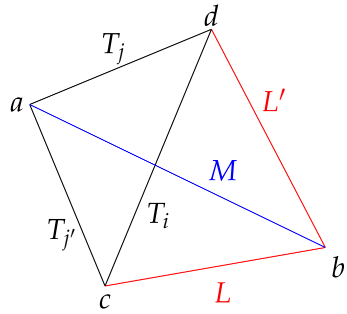


Figure 3.10: The crucial step in Schiffler's proof

T_k is adjacent to vertex c. Similarly, every diagonal that crosses M, but does not cross L', is adjacent to vertex d.

Fourthly, every Schiffler path from a to b must start with the edge T_j or with the edge $T_{j'}$. For a proof, note that the first step T_k of a Schiffler path from a to b starts in part P_a and, as it does not cross T_i, it completely lies in the interior or on the boundary of P_a. If the endpoint of T_k would be different from c and d, then the second step would also completely lie in the interior or on the boundary of P_a and could not cross the line M.

Fifthly, the Schiffler paths from a to b that start with the pair (T_j, T_i) are in bijection with the Schiffler paths from c to b that do not use the edge T_i. If $(a_0, a_1, \ldots, a_r)$ is a Schiffler path from c to b that does not use T_i, then $(a, d, c, a_1, \ldots, a_r)$ is a Schiffler path from a to b, because condition $(T5)$ follows from the second observation and the other conditions follow by construction. Conversely, if $(a, d, c, a_1, \ldots, a_r)$ is a Schiffler path from a to b, then $(c, a_1, \ldots, a_r)$ is a Schiffler path from c to b, because by condition $(T6)$ only the first step uses edge adjacent to c.

Sixthly, the Schiffler paths from a to b that start with the edge T_j and do not use the edge T_i are in bijection with the Schiffler paths from c to b that do use the edge T_i. If $(a_0, a_1, \ldots, a_r)$ is a Schiffler path from c to b that uses T_i, then $a_0 = c$ and $a_1 = d$ and $(a, a_1, \ldots, a_r)$ is a Schiffler path from a to b because of condition $(T5)$. It does not contain the edge T_i. Conversely, if $(a, d, a_2, \ldots, a_r)$ is a Schiffler path from a to b that starts with the edge T_j and does not use the edge T_i, then $(c, d, a_2, \ldots, a_r)$ is a Schiffler path from c to b, because by condition $(T6)$ only the first step uses edge adjacent to c.

Seventhly, we conclude that the Schiffler paths from a to b that start with the edge T_j are in bijection with the Schiffler paths from c to b. By construction, the sum of the corresponding monomials taken over all these paths is $X(c, b)x_j x_i^{-1}$. Analogously, the Schiffler paths that start with the edge $T_{j'}$ yield the term $X(d, b)x_{j'} x_i^{-1}$. We deduce that $X(a, b)x_i = X(c, b)x_j + X(d, b)x_{j'}$. □

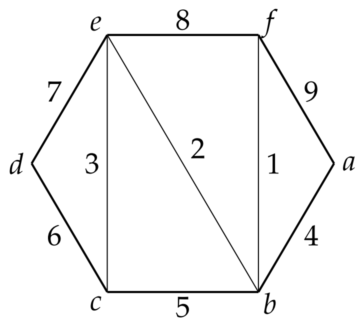
3.4 Exercises

Exercise 3.1. Let $\mathcal{A}(\mathbf{x}, Q)$ be the Markov cluster algebra. In this exercise we want to show that the invariant $T = \frac{x_1^2+x_2^2+x_3^2}{x_1 x_2 x_3} \in \mathcal{F}$ does *not* lie in the cluster algebra.

(a) Prove that the Markov cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ becomes an $\mathbb{N}$-graded algebra when we assign each cluster variable the degree 1.

(b) Prove that $T \notin \mathcal{A}(\mathbf{x}, Q)$.

Exercise 3.2. Consider the following triangulation of the regular hexagon.



(a) For all possible vertices $a_1, a_2 \in \{a, b, c, d, e, f\}$ determine all Schiffler paths from a_1 to a_2.

(b) For all possible vertices $a_1, a_2 \in \{a, b, c, d, e, f\}$ compute $X(a_1, a_2)$.

(c) How are the results related to the cluster variables in Figure 2.14?

Chapter 4

The Laurent phenomenon

4.1 The proof of the Laurent phenomenon

In this section we wish to prove Laurent phenomenon [FZ, Theorem 3.1]. In the literature there exist two different proofs. Next to Fomin-Zelevinsky's original proof, Berenstein-Fomin-Zelevinsky [BFZ3] provided a different proof via so-called lower and upper bounds. We will present the original proof, as it works in greater generality and

For the setup, let $m \geq n \geq 1$ be natural numbers and let $\mathbf{x} = (x_1, x \ldots, x_m)$ be an extended cluster with cluster variables $x_1, x_2, \ldots, x_n$ and frozen variables $x_{n+1}, x_{n+2}, \ldots, x_m$ in an ambient field $\mathcal{F}$. Moreover, let $\tilde{B}$ be an $m \in n$ exchange matrix. Then the pair $(\mathbf{x}, B)$ is a seed.

Theorem 4.1.1 (Fomin-Zelevinsky)**.** Every cluster variable of $\mathcal{A}(\mathbf{x}, \widetilde{B})$ is an element in the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm}, x_{n+1}, x_{n+2}, \ldots, x_m]$.

Especially, the two cluster algebras satisfy $\mathcal{A}(\mathbf{x}, \widetilde{B}) \subseteq k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm}, x_{n+1}, x_{n+2}, \ldots, x_m]$ and $\mathcal{A}(\mathbf{x}, \widetilde{B})^{inv} \subseteq k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_m^{\pm 1}]$. As any mutation equivalent seed $(\mathbf{y}, \widetilde{B}') \sim (\mathbf{x}, \widetilde{B})$ defines the same cluster algebra, every cluster variable is also a Laurent polynomial in the elements of an arbitrary cluster $\mathbf{y}$ of $\mathcal{A}(\mathbf{x}, \widetilde{B})$.

4.1.1 Warmups

In this section we wish to provide some motivation for the proof, which will become technical at some point. The aim of the first proposition is to show the Laurent phenomenon fails as soon a you begin to modify input data. So the class of dynamical systems that fulfill Laurentness is very rigid (which might count as a retroactive justification for the definition of a cluster algebra at least in rank 2).

Proposition 4.1.2. Assume that $(a_n)_{n\in\mathbb{Z}}$ is a sequence of positive integers. Let x_1 x_2 algebraically independent elements in an ambient field $\mathcal{F}$. Define a sequence $(x_n)_{n\in\mathbb{Z}}$ of elements in $\mathcal{F}$ by the recursion $x_{n-1}x_{n+1} = x_n^{a_n} + 1$. Then $\ldots = a_{-2} = a_0 = a_2 = a_4 = \ldots$ and $\ldots = a_{-1} = a_1 = a_3 = \ldots$

Proof. Suppose u, v, w, x, y are consecutive elements in the sequence $(x_n)_{n\in\mathbb{Z}}$. By defintition, there are positive integers a, b, c such that

$$w = \frac{v^a + 1}{u}, \qquad x = \frac{w^b + 1}{v} = \frac{(v^a + 1)^b + u^b}{u^b v}, \qquad y = \frac{x^c + 1}{w} = \frac{\left[(v^a + 1)^b + u^b\right]^c + u^{bc}v^c}{u^{bc-1}v^b(v^a + 1)}$$

By assumption, $y \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$. This means that the polynomial $v^a + 1 \in \mathbb{Z}[u, v]$ must divide the polynomial $\left[(v^a + 1)^b + u^b\right]^c + u^{bc}v^c$. The binomial expansion only yields one term that is not divisible by $v^a + 1$. We conclude that the polynomial $u^{bc} + u^{bc}v^c$ must be divisible by $v^a + 1$, so that $v^c + 1$ is divisible by $v^c + 1$. Similarly, we can conclude that $v^a + 1$ is divisible by $v^c + 1$ when we write u as a rational expression in x and y. It follows that $a = c$, which implies the claim by an easy inductive argument. □

Second, the Laurent phenomenon clearly implies that we all cluster variables become integers when we specialize all cluster and frozen variables in a single cluster to 1. As a toy model for the whole prove we now prove integrality of the elements of the 4-Somos sequence from Section 3.1.5. Recall that we have defined the 4-Somos sequence $(x_n)_{n\in\mathbb{N}^+}$ by initial values $x_1 = x_2 = x_3 = x_4 = 1$ and the recursion $x_{n+4}x_n = x_{n+3}x_{n+1} + x_{n+2}^2$ for $n \geq 1$. The elements are specialized cluster variables of the cluster algebra attached to the quiver in Figure 3.5. We follow Gale [Gal] in our exposition.

Proposition 4.1.3. For all $n \geq 1$ we have $x_n \in \mathbb{Z}$.

Proof. First, we claim that if a, b, c, d are consecutive members in the sequence, then a, b, c, d are pairwise coprime. We prove the claim by mathematical induction. It it clearly true for the initial four elements as they are all equal to 1. Now suppose that a, b, c, d, e are consecutive elements such that a, b, c, d are pairwise coprime. We have to prove that b, c, d, e are pairwise coprime, i.e. we have to show that $\gcd(b, e) = \gcd(c, e) = \gcd(d, e) = 1$. The claim follows easily from the exchange relation $ae = bd + c^2$. If prime number p would divide both b and e, then it would also divide c which is impossible. The other two claim follow similarly.

Now we provide a proof of the actual statement, which we also do by mathematical induction. It is easy to see that the initial eight elements are integers as we only divide by 1. Suppose that $a, b, c, d, e, f, g, h, i$ are consecutive elements such that a, b, c, d, e, f, g, h are integers. We have to prove that i is an integer. We consider the exchange relations $ae = bd + c^2$, $bf = ce + d^2$, $cg = df + e^2$, $dh = eg + f^2$ and $ei = fh + g^2$. In other words, we have to show that $fh + g^2$ is divisible by e. Let us reduce all numbers modulo e. By the above claim we know that $b, c, d \in (\mathbb{Z}/e\mathbb{Z})^\times$ are invertible. Using the recursions we can now write f, g, h as expressions in b, c, d:

$$f \equiv \frac{d^2}{b}, \qquad g \equiv \frac{d^3}{bc}, \qquad h \equiv \frac{d^3}{b^2} \qquad \pmod{e}.$$

So we obtain the congruence $fh + g^2 \equiv \frac{d^5}{b^3c^2}(c^2 + bd) \pmod{e}$. Surprisingly, the numerator $c^2 + bd$ also appears in the equation $ae = bd + c^2$, which implies that $fh + g^2$ id divisible by e. □

Readers who have understood the arguments in this section, will be able to understand the whole proof. Cum grano salis, we replace the divisibility arguments over the integers in the above proof by divisibility arguments in (Laurent) polynomial rings. An upshot of the proof is the following:

(A) Cluster variables obtained from the cluster $\mathbf{x}$ by a sequence of mutations at pairwise different indices are Laurent polynomials in $\mathbf{x}$ by construction (compare w and x in the first warmup and e, f, g, h in the second warmup).

(B) The first cancellation happens when we mutate the cluster $\mathbf{x}$ by sequence that uses an index twice (compare y in the first warmup and i in the second warmup). For instance, the first non-trivial step in the proof is to show that the mutation $(\mu_i\mu_j\mu_i)(\mathbf{x})$ (for indices $i \neq j$) produces Laurent polynomials.

(C) The Laurentness of $(\mu_i\mu_j\mu_i)(\mathbf{x})$ (for all indices $i \neq j$) implies Laurentness of all cluster variables by an induction argument. In the general case, when the cluster variables do not form a sequence but are parametrized by a regular tree, we organize the induction by a so-called caterpillar.

(D) For the induction step we show certain coprimality results in the ring of (Laurent) polynomials.

4.1.2 Divisibility in commutative algebra

For the second step in the proof we recollect some basic concepts from commutative algebra. Let A be a commutative and unital k-algebra. Let us also assume that A is an integral domain.

We say that an element $y \in A$ *divides* the element $x \in A$ if there exists an element $z \in A$ such that $x = yz$. In this case, we also say that x is *divisible* by y. For example, every element is divisible by 1 and by itself. A non-zero element $p \in A$ is called a *prime element* if the following implication holds: If p divides the product xy, then p divides x or p divides y. For example, a prime number p is a prime element in the ring $\mathbb{Z}$ of integers.

An element $x \in A$ is called a *unit* if there exists an element $y \in A$ such that $xy = 1$. We denote the set of all units by $A^\times$. Two elements x and y of A are called *associated* if there exists a unit $\lambda \in A^\times$ such that $x = \lambda y$.

We call an element $x \in A$ *reducible* if there exist two elements $y, z \in A \backslash A^\times$ such that $x = yz$. It is called *irreducible* otherwise. Note that every prime element $p \in A$ is irreducible: Assume that p is a prime element and $p = yz$ for some $y, z \in A$. Then p divides the product yz and hence p divides y or p divides z. Without loss of generality suppose that p divides y. Then there exists some $x \in A$ such that $y = px$. In this case $p = pxz$. As the algebra A does not contain zero divisors, the get $1 = xz$, so that the factor z is a unit. The converse is not true.

The algebra A is called a *unique factorization domain* if the following implication holds: If $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$ are irreducible elements with $x_1 x_2 \cdots x_r = y_1 y_2 \cdots y_s$, then $r = s$ and there exists a permutation $\sigma \in S_r$ such that $x_i = y_{\sigma(i)}$ for all $i \in \{1, 2, \ldots, r\}$. *Gauß's Lemma* asserts that the polynomial ring $A = k[X_1, X_2, \ldots, X_n]$ in $n \geq 1$ variables is a unique factorization domain. The units in this ring are $A^\times = k^\times$ for all fields k. It follows that the Laurent polynomial ring $A = k[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}]$ is a unique factorization domain with units $A^\times = \{X_1^{a_1} X_2^{a_2} \cdot \ldots \cdot X_n^{a_n} \colon \mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n\}$. For a non-example, the ring $A = \mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain, because $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ and $A^\times = \{\pm 1\}$.

For the rest of the section assume that A is a unique factorization domain. Then every irreducible element is also a prime element. An element $d \in A$ is called a *greatest common divisor* of x and y if d divides every element that divides both x and y. In a unique factorization domain, a greatest common divisor always exists and all greatest common divisors are associated to each other. More precisely, if $x_1 x_2, \cdots, x_r$ are pairwise non-associated and irreducible elements, then greatest common divisor of the monomials $x_1^{a_1} x_2^{a_2} \cdot \ldots \cdot x_r^{a_r}$ and $x_1^{b_1} x_2^{b_2} \cdot \ldots \cdot x_r^{b_r}$ is equal to $x_1^{m_1} x_2^{m_2} \cdot \ldots \cdot x_r^{m_r}$ where $m_i = \min(a_i, b_i)$ for all $i \in \{1, 2, \ldots, r\}$. We denote a greatest common divisor of two elements $x, y \in A$ by $\gcd(x, y)$. Two elements $x, y \in A$ are called *coprime* if a greatest common is equal to 1.

4.1.3 Three mutations

Suppose that i and j are different mutable indices. We wish prove that the cluster $(\mu_i \circ \mu_j \circ \mu_j)(\mathbf{x})$ consists of Laurent polynomials in the variables of the initial cluster. To avoid subindices, we abbreviate $u = x_i$, $v = x_j$, $w = \mu_i(x_i)$, $x = (\mu_j \circ \mu_i)(x_j)$ and $y = (\mu_i \circ \mu_j \circ \mu_j)(x_i)$. Then, the four clusters $\mathbf{x}$, $\mu_i(\mathbf{x})$, $(\mu_j \circ \mu_i)(\mathbf{x})$ and $(\mu_i \circ \mu_j \circ \mu_j)(\mathbf{x})$ all have $n-2$ cluster variables and $m-n$ frozen variables in common, and the other two elements are (u, v), (v, w), (w, x) and (x, y), respectively.

Let $\mathcal{L} = k[x_r^{\pm 1} \colon 1 \leq r \leq m, r \neq i, j]$ be the Laurent polynomial ring in the common cluster and frozen variables. We claim that $w, x, y \in \mathcal{L}[u^{\pm 1}, v^{\pm 1}]$. The claim is obviously true for w and y. The non-trivial step is to show that $y \in \mathcal{L}[u^{\pm 1}, v^{\pm 1}]$. First of all, note that this is true if the entry

b_{ij} (and hence also b_{ji}) is equal to zero, because in this case $y = u$. Without loss of generality we may assume that $b_{ij} < 0$ (and hence $b_{ji} > 0$). For further arguments we also consider the mutation $B' = \mu_i(B)$. For brevity, we put $b'_{ij} = b$ and $b_{ji} = -c$. It follows that b and c are positive integers.

By the exchange relations, we have $uw = v^c L + M$, $vx = w^b N + P$ and $wy = x^c R + S$ for some monomials $L, M, N, P, Q, R, S \in \mathcal{L}^\times$. Using the exchange relations we can express w, x and y as rational functions in u and v with coefficients in $\mathcal{L}$. Explicitly, we obtain

$$\begin{aligned} w &= \frac{v^c L + M}{u}, \\ x &= \frac{w^b N + P}{v} = \frac{(v^c L + M)^b N + u^b P}{u^b v}, \\ y &= \frac{x^c Q + R}{w} = \frac{\left[(v^c L + M)^b N + u^b P\right]^c Q + R u^{bc} v^c}{(v^c L + M) u^{bc-1} v^c}. \end{aligned}$$

We see that w and x are automatically Laurent polynomials in $\mathcal{L}[u^{\pm 1}, v^{\pm 1}]$. For the containedness $y \in \mathcal{L}[u^{\pm 1}, v^{\pm 1}]$, it is enough to show that the polynomial $v^c L + M \in k[x_k \colon 1 \leq k \leq m]$ divides the polynomial in the numerator. We expand the term in the numerator via the binomial theorem. We see that it is enough to show that the polynomial $v^c L + M$ divides the polynomial $u^{bc} v^c R + u^{bc} P^c Q$. We conclude that it is enough to show that $v^c L + M$ divides $v^c R + P^c Q$. We claim that L divides R (inside the polynomial ring $k[x_k \colon 1 \leq k \leq m]$) and that the identity $P^c Q L = M R$ holds. The claims finish the argument.

To this end, we describe P, Q, L, M and R more explicitly. Let us partition the vertex set $Q_0 \backslash \{i, j\}$ into four classes. More precisely, we put

$$\begin{aligned} \mathcal{D} &= \{D \in Q_0 \backslash \{i,j\} \colon b'_{D,i} \geq 0, b'_{D,j} < 0\}, & \mathcal{E} &= \{E \in Q_0 \backslash \{i,j\} \colon b'_{E,i} \geq 0, b'_{E,j} \geq 0\}, \\ \mathcal{F} &= \{F \in Q_0 \backslash \{i,j\} \colon b'_{F,i} < 0, b'_{F,j} < 0\}, & \mathcal{G} &= \{G \in Q_0 \backslash \{i,j\} \colon b'_{G,i} \geq 0, b'_{G,j} < 0\}. \end{aligned}$$

For an element $D \in \mathcal{D}$ we abbreviate $d = |b'_{D,i}|$ and $d' = |b'_{D,j}|$. For elements in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ we proceed analogously. Moreover, it will be convenient to refine the partition by setting $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$ where $\mathcal{G}_1 = \{g \in G \colon g \geq cg'\}$ and $\mathcal{G}_2 = \{g \in G \colon g < cg'\}$. We can now read off the monomials L, M and P from the quivers $Q(B')$. We have

$$L = \prod_{D \in \mathcal{D}} D^d \cdot \prod_{E \in \mathcal{E}} E^e, \qquad M = \prod_{F \in \mathcal{F}} F^f \cdot \prod_{G \in \mathcal{G}} G^g, \qquad P = \prod_{E \in \mathcal{E}} E^{e'} \cdot \prod_{G \in \mathcal{G}} G^{g'}.$$

To describe the monomials Q and R let us consider the quiver of the matrix $\mu_j(\widetilde{B}')$. The partition of the vertex set enables us to describe the relevant entries in that matrix explicitly. We obtain

$$Q = \prod_{F \in \mathcal{F}} F^f \cdot \prod_{G_1 \in \mathcal{G}_1} G_1^{g_1 - cg_1'}, \qquad R = \prod_{D \in \mathcal{D}} D^d \cdot \prod_{E \in \mathcal{E}} E^{e + ce'} \prod_{G_2 \in \mathcal{G}_2} G_2^{cg_2' - g_2}.$$

We conclude that L divides R and that the identity $P^c Q L = M R$ holds.

4.1.4 The proof of the Laurent phenomenon

Now we combine the case of three mutations with coprimality statements to obtain a proof of the Laurent phenomenon. The crucial step is to compute certain greatest common divisors. We use the same notation as above and we suppose that $b, c \geq 1$. Then we claim that inside the Laurent polynomial ring $\mathcal{L}[u^{\pm 1}, v^{\pm 1}]$ we have $\gcd(w, x) = \gcd(w, y) = 1$. For the first claim note that the elements $u, u^b v, u^b P \in \mathcal{L}[u^{\pm 1}, v^{\pm 1}]$ are actually units. Therefore, we can conclude that $\gcd(w, x) = \gcd(v^c L + M, (v^c L + M)^b N + u^b P) = \gcd(V^c L + M, u^b P) = 1$. For the second claim,

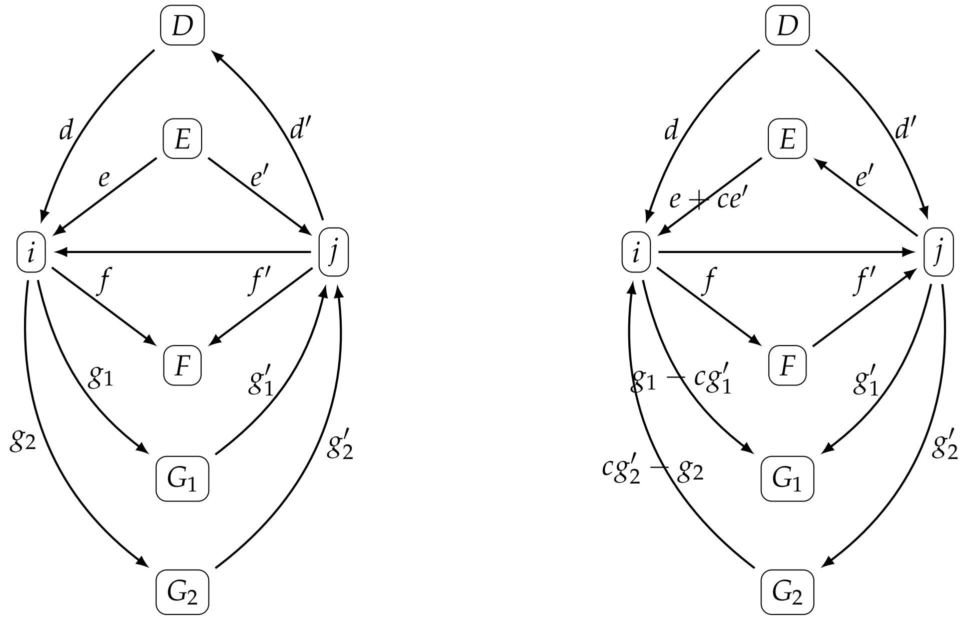


Figure 4.1: Five kinds of vertices

let us investigate the quotient of the the polynomial in the numerator of y and the polynomial $v^cL + M$. If $b > 1$, then the quotient is equal to $u^{bc}RL^{-1} = u^{bc}P^cQM^{-1}$ plus a multiple of $v^cL + M$. In this case $\gcd(w, y) = \gcd(wv^cL + M, u^{bc}RL^{-1}) = 1$, because $u^{bc}RL^{-1} \in \mathcal{L}[u^{\pm 1}, v^{\pm 1}]$ is a unit. For $b = 1$ we obtain $\gcd(w, y) = \mathrm{u^bP + cN} = 1$, because a common divisor would be an element in $\mathcal{L}$ but L and M are coprime in $\mathcal{L}$.

4.2 Exercises

Exercise 4.1. Let Q be the quiver with vertices $Q_0 = \{1, 2, 3\}$ and arrows $Q_1 = \{1 \to 2, 2 \to 3, 1 \to 3\}$. In Section 3.1.4 we attached to Q the sequence $(x_n)_{n\in\mathbb{Z}}$ defined by the initial values $x_1 = x_2 = x_3 = 1$ and the recursion $x_{n+3}x_n = 1 + x_{n+2}x_{n+1}$ for $n \geq 1$. Prove that all elements are integers.

Chapter 5

Solutions to exercises

Exercise 1.1. We verify the relation by direct computation. The vectors $(r,s) \in \mathbb{R}^2$ for the monomials $a^r b^s$ (with $r,s \geq 0$) are $(1,0)$, $(1,1)$, $(2,3)$, $(1,2)$, $(1,3)$ and $(0,1)$. These are precisely the coordinates of the positive roots for G_2 in the basis of simple roots, as shown in Figure 1.3.

$$
\begin{array}{ccccccc}
\left(\frac{a^3+3a^2+3a+1+b^3}{ab^3}, \frac{a+1}{b}\right) & \xrightarrow{F_3} & \left(\frac{a+1}{b}, a\right) & \xrightarrow{F_1} & (a,b) & \xrightarrow{F_3} & \left(b, \frac{b^3+1}{a}\right) \\
\uparrow F_1 & & & & & & \downarrow F_1 \\
\left(\frac{a^2+2a+1+b^3}{ab^2}, \frac{a^3+3a^2+3a+1+b^3}{ab^3}\right) & & & & & & \left(\frac{b^3+1}{a}, \frac{a+b^3+1}{ab}\right) \\
\nwarrow F_3 & & & & & & \swarrow F_3 \\
& \left(\frac{a^3+3a^2+3a+1+2b^3+3ab^3+b^6}{a^2b^3}, \frac{a^2+2a+1+b^3}{ab^2}\right) & & \xleftarrow{F_1} & & \left(\frac{a+b^3+1}{ab}, \frac{a^3+3a^2+3a+1+2b^3+3ab^3+b^6}{a^2b^3}\right) &
\end{array}
$$

Exercise 1.2. Define a sequence $(f_k)_{k\in\mathbb{N}}$ by the initial elements $f_0 = f_1 = 1$ and the recursion $f_{k+1} = 3f_k - f_{k-1}$ for $k \geq 1$. The following table lists the first elements of the sequence.

k	0	1	2	3	4	5	6
f_k	1	1	2	5	13	34	89

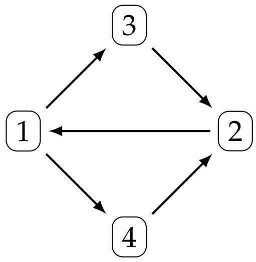
Put $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 3 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. By an induction argument it is easy to see that for all $k \geq 0$ we have

$$M^k \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} f_k \\ f_{k+1} \end{pmatrix}, \quad \text{and thus} \quad M^k \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{pmatrix}.$$

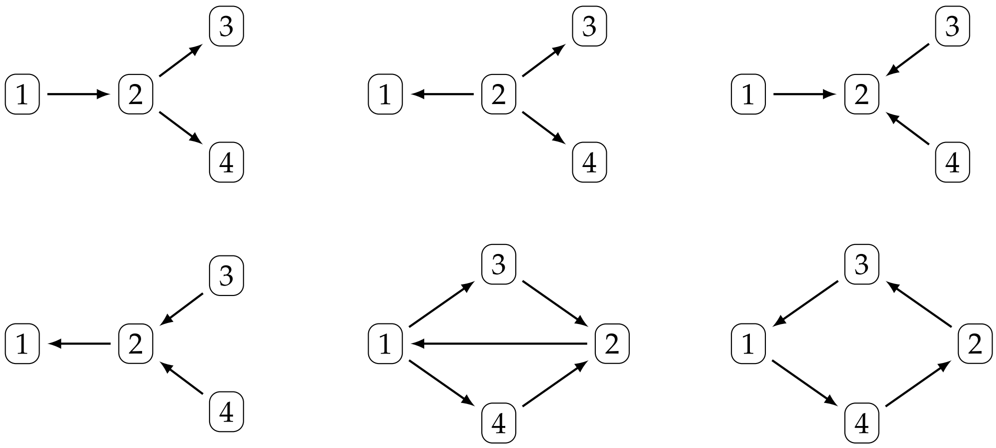
We conclude that $f_k f_{k+2} - f_{k+1}^2 = \det(M)^k \det(N) = 1$ which implies $F_2(f_k, f_{k+1}) = (f_{k+1}, f_{k+2})$ for all $k \geq 0$. By an induction argument it is easy to see that $F_2^k(1,1) = (f_k, f_{k+1})$ for all $k \geq 0$. As the sequence $(f_k)_{k\in\mathbb{N}}$ is strictly increasing, we have $F_2^k(1,1) \neq (1,1)$ for all $k \geq 1$ which implies $F_2^k \neq \mathrm{id}$ for all $k \geq 1$. □

Remark: It is easy to see that the sequence $(f_k)_{k\in\mathbb{N}}$ is every other *Fibonacci number*. The integrality of the sequence is an instance of the Laurent phenomenon.

Exercise 2.1 (a) We compute $\mu_2(Q)$ according to the mutation rules M1-M4. My M1 we reverse the incoming arrow $1 \to 2$ and by M2 we reverse the two outgoing arrows $2 \to 3$ and $2 \to 4$. By M3 the two paths $1 \to 2 \to 3$ and $1 \to 2 \to 4$ yield arrows $1 \to 3$ and $1 \to 4$. A cancellation of 2-cycles is not necessary. The result is the following quiver $\mu_2(Q)$:

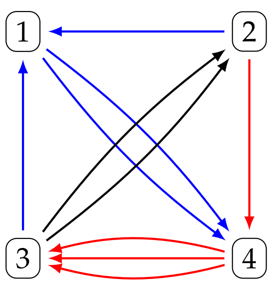


(b) The set of isomorphism classes of the following six quivers is closed under mutation. Thus, it is the mutation class of Q. Especially, the mutation class contains all isomorphism classes of the $2^3 = 8$ orientations of the underlying diagram guaranteed by Proposition 2.2.8.

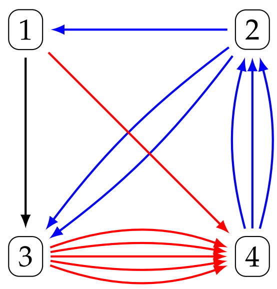


Exercise 2.2 (a) We calculate $\mu_1(Q)$ and $\mu_2(Q)$ according to the mutation rules M1-M4 by partitioning the arrows in direct predecessors and direct successors (which we colour blue), arrows between direct predecessors and direct successors (which we colour red) and remaning arrows (which we colour black). The results are the following two quivers.

$\mu_1(Q)$:



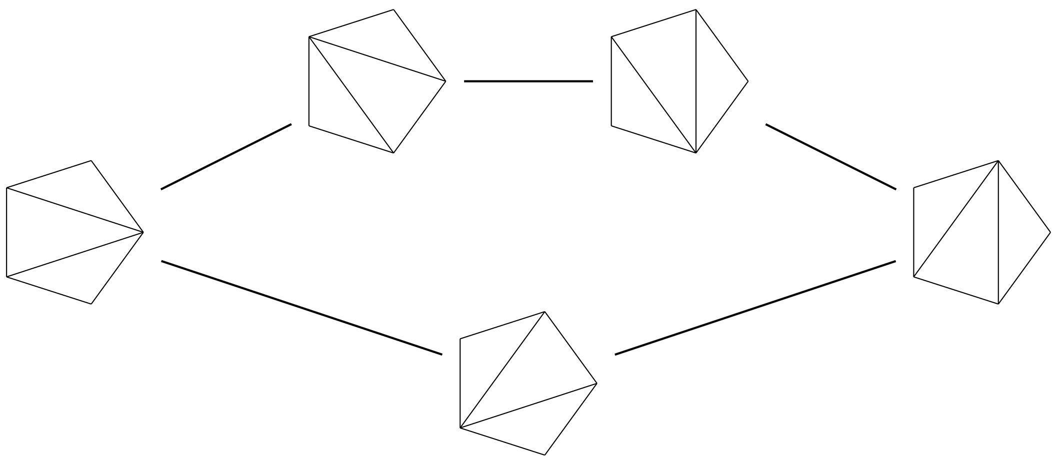
$\mu_2(Q)$:



By Proposition 2.2.3 we have an isomorphism $\mu_2(\mu_2(Q)) \cong Q$, which we could also verify by a direct computation.

(b) Surprisingly, we have $\mu_1(Q) \cong Q$. Thus Q, $\mu_1(Q)$ and $\mu_2(\mu_2(Q))$ are isomorphic to each other, whereas $\mu_2(Q)$ lies in a different isomorphism class.

Exercise 2.3. The regular pentagon admits $C_3 = 5$ triangulations. Each triangulation consists of two diagonals and thus we can mutate it at two places. The following graph indicates which triangulations are related to each other by a flip. By chance, the graph is again a pentagon.



Exercise 2.4 (a) Let $B'' = (b''_{ij})$ be the $n \times n$ integer matrix with entries

$$b''_{ij} = b_{ij} + \tfrac{1}{2}(b_{ik}|b_{kj}| + |b_{ik|}b_{kj}).$$

Note that the matrices B' and B'' have the same rank, because we can obtain one from the other by a sign change in k-th row and column. So it is enough to prove that $\operatorname{rank}(B) = \operatorname{rank}(B'')$. But we obtain B'' from B by the following operation. In the first step, we add to every column vector $c_j = (b_{ij})_{i \in \{1,2,\ldots,n\}} \in \mathbb{Z}^n$ with $j \neq i$ a scalar multiple of the k-th column vector $c_k = (b_{ik})_{i \in \{1,2,\ldots,n\}} \in \mathbb{Z}^n$. Here, the scalar is equal to $\frac{1}{2}|b_{kj}|$. These operations do change the rank of the matrix. Moreover, these operations leave the k-th row vector $r_k = (b_{kj})_{j \in \{1,2,\ldots,n\}i} \in \mathbb{Z}^n$ invariant. In the second step, we add a scalar multiple of $r_k = \in \mathbb{Z}^n$ to every row vector $r_j = (b_{ij})_{j \in \{1,2,\ldots,n\}i} \in \mathbb{Z}^n$ with $j \neq i$. These operation leave the rank of the matrix invariant as well. $\square$

Remark: In general, the matrix B is not of full rank. For instance, if n is odd, then by skew symmetry $\det(B) = \det(-B^T) = (-1)^n \det(B^T) = -\det(B)$. So $\det(B) = 0$ and B is not of full rank in this case.

(b) Let $j \in \{1, 2, \ldots, n\}$ and let $g_j = \gcd(b_{ij} \colon i \in \{1, 2, \ldots, n\})$ and $g'_j = \gcd(b'_{ij} \colon i \in \{1, 2, \ldots, n\})$ the greatest common divisors of the j-th column of B and B', respectively. The definition readiliy implies that $g_j | b_{ij}$ and $g_j | b_{ik} b_{kj}$ for every $i \in \{1, 2, \ldots, n\}$. We conclude that g_j divides b'_{ij} for every $i \in \{1, 2, \ldots, n\}$ so that $g_j | g'_j$. We apply the same argument to the mutation $B = \mu_k(B')$ to get $g'_j | g_j$. Both relations can only hold when $g_j = g'_j$. $\square$

Exercise 2.5. The mutation contains one other element, namely the isomorphism class of the quiver $Q' = (Q'_0, Q'_1, s', t')$ with vertices $Q'_1 = \{1, 2, 3\}$ and one arrow $2 \to 1$, one arrow $3 \to 2$ and two arrows $1 \rightrightarrows 3$. $\square$

Exercise 2.6. As experiments with the applet show, a possible sequence of mutations is $\mu_1(Q)$ in the first case and $(\mu_4 \circ \mu_5 \circ \mu_1 \circ \mu_5 \circ \mu_2)(Q)$ in the second case. $\square$

Exercise 2.7 (a) The four elements u, v, w and t are trivially polynomial expression in themselves. As a first step it is both easy to guess and to check that

$$x_2 = uw - 1, \qquad \frac{1 + x_1 x_3}{x_2} = vt - 1, \qquad \frac{(1 + x_2)^2 + x_1 x_3}{x_1 x_2 x_3} = wt - 1.$$

Using these expressions we can write the last two cluster variables as polynomial expressions:

$$\frac{1+x_2}{x_3} = tx_2 - x_1 = uwt - u - t, \qquad \frac{1+x_2+x_1x_3}{x_1x_2} = \frac{1+x_2}{x_1}\cdot\frac{1+x_1x_3}{x_2} - x_3 = vwt - v - w.$$

(b) The elements u, v, w in $\mathbb{Q}(x_1, x_2, x_3) = \mathbb{Q}(u, v, w)$ are algebraically independent over $\mathbb{Q}$. The element $t \in \mathbb{Q}(u, v, w)$ is equal to $t = \frac{uw+uv}{(uw-1)v}$. Therefore, we have an isomorphism $\mathcal{A}(\mathbf{x}, Q) \cong \mathbb{Q}[U, V, W, T]/(UVWT - UV - UW - VT)$.

Exercise 2.8. Suppose that B is skew symmetrizable. By definition, there exists a diagonal matrix with positive integer diagonal entries such DB is skew-symmetric, i.e. $DB = -(DB)^T = -B^TD$. Then we have $BD^{-1} = -D^{-1}B^T$. Let D' be scalar multiple of the inverse D^{-1} that has positive integer entries. Then BD' is skew symmetric. The reverse direction follows similarly.

Exercise 3.1. All exchange relations between clusters (x, y, z) and (x', y, z) are of the form $xx' = y^2 + z^2$, which is homogeneous of degree 2 when we assign each cluster variable the degree 1. Hence this assignment induces a grading on the whole cluster algebra with index set $\mathbb{N}$. Assume that $T \in \mathbb{A}(\mathbf{x}, Q)$. Then $Tx_1x_2x_3 = x_1^2 + x_2^3 + x_3^2$ is on the one hand a homogeneous element of degree 2, on the other hand a linear combination of homogeneous elements of degree at least 3, which is a contradiction. □

Exercise 3.2. For all vertices a_1, a_2 will get denote by $X_0(a_1, a_2) \in \mathcal{F}$ the element that we obtain from $X(a_1, a_2)$ by specializing $x_4 = x_5 = \ldots = x_9 = 1$. For all vertices a_1, a_2 such that the segment M_{a_1,a_2} is equal to some $T_i \in T$ we have $X(a_1, a_2) = X_0(a_1, a_2) = x_i$, as there can only one Schiffler path. Especially, we have $X(b, f) = x_1$, $X(b, e) = x_2$ and $X(c, e) = x_3$. For the other pairs of vertices we obtain:

$$\mathcal{P}(a,e) = \{(abfe), (afbe)\}, \quad X(a,e) = \frac{x_4x_8}{x_1} + \frac{x_2x_9}{x_1}, \quad X_0(a,e) = \frac{1+x_2}{x_1},$$

$$\mathcal{P}(c,f) = \{(cbef), (cebf)\}, \quad X(c,f) = \frac{x_5x_8}{x_2} + \frac{x_1x_3}{x_2}, \quad X_0(c,f) = \frac{1+x_1x_3}{x_2},$$

$$\mathcal{P}(b,d) = \{(bced), (becd)\}, \quad X(b,d) = \frac{x_5x_7}{x_3} + \frac{x_2x_6}{x_3}, \quad X_0(b,d) = \frac{1+x_2}{x_3},$$

$$\mathcal{P}(a,c) = \{(abec), (afbc), (abfebc)\}, \quad X(a,c) = \frac{x_3x_4}{x_2} + \frac{x_5x_9}{x_1} + \frac{x_4x_5x_8}{x_1x_2}, \quad X_0(a,c) = \frac{x_1x_3+x_2+1}{x_1x_2},$$

$$\mathcal{P}(d,f) = \{(dcef), (debf), (decbef)\}, \quad X(d,f) = \frac{x_6x_8}{x_3} + \frac{x_1x_7}{x_2} + \frac{x_5x_7x_8}{x_2x_3}, \quad X_0(d,f) = \frac{x_2+x_1x_3+1}{x_2x_3}$$

$$\mathcal{P}(a,d) = \{(abed), (afbecd), (abfecd), (afbced), (abfebced)\},$$
$$X(a,d) = \frac{x_4x_7}{x_2} + \frac{x_2x_6x_9}{x_1x_3} + \frac{x_4x_6x_8}{x_1x_3} + \frac{x_5x_7x_9}{x_1x_3} + \frac{x_4x_5x_7x_8}{x_1x_2x_3},$$
$$X_0(a,d) = \frac{x_1x_3 + x_2^2 + 2x_2 + 1}{x_1x_2x_3}.$$

The specialized elements $X_0(a_1, a_2) \in \mathcal{F}$ for different and non-adjacent vertices a_1, a_2 are the cluster variables in Figure 2.14.

Remark: The set $\mathcal{P}(a, d)$ contains a Schiffer path, namely $(afbecd)$, that also crosses the segment M_{ad} in an odd step. Thus, the converse of condition $(T5)$ may fail. Moreover, it shows that coefficients for the specialized cluster variables can be greater than 1.

Bibliography

[Art] M. Artin: *Algebra*, Birkhäuser Verlag (1993).

[ASS] I. Assem, D. Simson, A. Skowronski, *Elements of the Representation Theory of Associative Algebras*, London Mathematical Society (2006).

[BBH] A. Beineke, T. Brstle, L. Hille, *Cluster-Cyclic Quivers with three Vertices and the Markov Equation*, Algebras and Representation Theory **14**, no.1 (2011), 97–112. arXiv:0612213.

[BFZ3] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Mathematical Journal **126** (2005), no. 1, 1–52. arXiv:math/0305434.

[Bos] S. Bosch, *Algebra*, Springer (1999).

[Ca] A. Cayley, *On Gauss's Pentagramma Mirificum*, Philosophical Magazine, **XLIL** (1871), 311–312.

[CC] Ph. Caldero, F. Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Commentarii Mathematici Helvetici **81** (2006), no. 6, 595–616. arXiv:math/0410187.

[Co1] H. S. M. Coxeter, *Discrete groups generated by reflections*, Annals of Mathematics **35** (1936), no. 3, 588–621.

[Co2] H. S. M. Coxeter, *The complete enumeration of finite groups of the form* $r_i^2 = (r_i r_j)^{k_{ij}} = 1$, Journal of the London Mathematical Society **10** (1935), 21–25.

[CZ] Ph. Caldero, A. Zelevinsky, *Laurent expansions in cluster algebras via quiver representations*, Moscow Mathematical Journal **6** (2006), no. 3, 411–429. arXiv:math/0604054.

[FR] S. Fomin, N. Reading, *Root systems and generalized associahedra*, Lecture notes for the IAS/Park City graduate summer school in geometric combinatorics, IAS/Park City Mathematics Series **13**, American Mathematical Society, Providence, RI (2007), 63–131. arXiv:math/0505518.

[FST] S. Fomin, M. Shapiro, D. Thurston, *Cluster algebras and triangulated surfaces. I. Cluster complexes*, Acta Mathematica **201** (2008), no. 1, 83–146. arXiv:math/0608367.

[FT] S. Fomin, D. Thurston, *Cluster algebras and triangulated surfaces. Part II: lambda lengths*, Preprint arXiv:1210.5569.

[FZ] S. Fomin, A. Zelevinsky, *Cluster algebras I: Foundations*, Journal of the American Mathematical Society **15** (2002), no. 2, 497–529. arXiv:math/0104151.

[FZ2] S. Fomin, A. Zelevinsky, *Cluster algebras. II. Finite type classification*, Inventiones Mathematicae **154** (2003), no. 1, 63–121. arXiv:math/0208229

[FZ4] S. Fomin, A. Zelevinsky, *Cluster algebras IV: Coefficients*, Compositio Mathematica **143** (2007), no. 1, 112–164. arXiv:math/0602259.

[FZ5] S. Fomin, A. Zelevinsky, *The Laurent phenomenon*, Advances in Applied Mathematics *28* (2002), no. 2, 119–144.

[Ga] C. F. Gauß, *Collected Works*, Volume III, 481–490.

[Gal] D. Gale, *Mathematical Entertainments: The Strange and Surprising Saga of the Somos Sequences*. Mathematical Intelligencer **13** (1991), 40–42.

[Ho] A. Hone, *Laurent Polynomials and Superintegrable Maps*, SIGMA **3** (2007), 022. arXiv:math/0702280.

[KL] S. Kleiman and D. Laksov, *Schubert Calculus*, The American Mathematical Monthly **79** (1972), no. 10, 1061–1082.

[Mal] J. Malouf, *An integer sequence from a rational recursion*, Discrete Mathematics **110** (1992), 257–261.

[Ma] A. Markov, *Sur les formes quadratiques binaires indéfinies*, Mathematische Annalen, Band XVII (1882), 379–399.

[SS] G. Scheja, U. Storch, *Lehrbuch der Algebra*, B. G. Teubner (1980).

[Sch] R. Schiffler, *A cluster expansion formula (A_n case)*, Electronic Journal of Combinatorics **15** (2008), #R641. arXiv:math/0611956.

[Sta] R. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press (1999).

[Ze] A. Zelevinsky, *Semicanonical basis generators of the cluster algebra of type $A_1^{(1)}$*, Electronic Journal of Combinatorics **14** (1), Research Paper 4 (2007). arXiv:math/0606775.