A quantum cluster algebra of Kronecker type and the dual canonical basis

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The talk concerns the **Kronecker quiver** which we denote by $Q$.

![Kronecker quiver](image)

**Figure:** The Kronecker quiver
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Recall the following facts about representations of $Q$:

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Figure: The preinjective component of the AR quiver
The definition of the quantized universal enveloping algebra $U_q(\mathfrak{n})$ and Lusztig’s $T$-automorphisms

The representation theory of $Q$ is truly linked with Lie theory. Therefore recall the following notions:

1. Consider a Kac-Moody Lie algebra $\mathfrak{g}$ of type $A^{(1)}_1$ and let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be its triangular decomposition.

2. The quantized universal enveloping algebra $U_q(\mathfrak{n})$ is the $\mathbb{Q}(q)$-algebra generated by two elements $E_0$ and $E_1$ subject to the relations

   $$E_i^2 E_j - [3] E_i E_j E_i + [3] E_i E_j E_i - E_j E_i^2 = 0,$$

   for $i \neq j$ where $[3] = q^2 + 1 + q^{-2}$.

3. Lusztig defined $T$-automorphisms $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ for $i = 0, 1$.

4. The Weyl group $W$ is generated by two elements $s_0$ and $s_1$ that act on the simple roots $\alpha_1$ and $\alpha_2$ of $\mathfrak{g}$ by $s_0(\alpha_0) = -\alpha_0$, $s_0(\alpha_1) = 2\alpha_0 + \alpha_1$, $s_1(\alpha_0) = \alpha_0 + 2\alpha_1$, and $s_1(\alpha_1) = -\alpha_1$. 
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\begin{align*}
\alpha_0, \\
s_0(\alpha_1) &= 2\alpha_0 + \alpha_1, \\
s_0 s_1(\alpha_0) &= 3\alpha_0 + 2\alpha_1, \\
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\end{align*}
Objects attached to a Weyl group element of length 4

Consider the AR quiver from above. We focus on the two injective modules and their \( \tau \)-translates. The \textit{dimension vectors} (under the correspondence with the positive roots according to \textit{Kac’s theorem}) can be described by initial subsequences of \( w = s_0 s_1 s_0 s_1 \in W: \)

- \( \alpha_0 \quad \sim \quad u_0 = E_0 \in U_q(n), \)
- \( s_0(\alpha_1) = 2\alpha_0 + \alpha_1 \quad \sim \quad u_1 = T_0(E_1) \in U_q(n), \)
- \( s_0 s_1(\alpha_0) = 3\alpha_0 + 2\alpha_1 \quad \sim \quad u_2 = T_0 T_1(E_0) \in U_q(n), \)
- \( s_0 s_1 s_0(\alpha_1) = 4\alpha_0 + 3\alpha_1 \quad \sim \quad u_3 = T_0 T_1 T_0(E_1) \in U_q(n). \)
The subalgebra $U_q(w)$ and its structure

As observed by Leclerc the elements satisfy straightening relations

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\begin{align*}
    u_i u_{i+1} &= q^{-2} u_{i+1} u_i, & (0 \leq i \leq 2), \\
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The straightening relations enable us to write everything in \( U_q(w) \) as a linear combination of elements of the form

\[
u[a] = u_3^{a_3} u_2^{a_2} u_1^{a_1} u_0^{a_0}, \quad (a = (a_3, a_2, a_1, a_0) \in \mathbb{N}^4).
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In fact, \( \{ u[a] : a \in \mathbb{N}^4 \} \) is a \( \mathbb{Q}(q) \)-basis of \( U_q(w) \), a PBW basis.
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In fact, \( \{ u[a] : a \in \mathbb{N}^4 \} \) is a \( \mathbb{Q}(q) \)-basis of \( U_q(w) \), a PBW basis. For every \( a \) there is some \( E[a] = q^b u_3^{a_3} u_2^{a_2} u_1^{a_1} u_0^{a_0} \) that is adjoint to \( u[a] \) w.r.t. Kashiwara’s bilinear form; \( \{ E[a] : a \in \mathbb{N}^4 \} \) is called dual PBW basis.
The dual canonical basis of $U_q(w)$

Let $\sigma$ be the antiinvolution fixing $E_0$ and $E_1$ s.t. $\sigma(q) = q^{-1}$. By a theorem of Leclerc there is a unique $\mathbb{Q}(q)$-basis $\{ B[a] : a \in \mathbb{N}^4 \}$ of $U_q(w)$ such that for every $a \in \mathbb{N}^4$ the following two conditions hold:

- $B[a] - E[a] \in \bigoplus_{b \in \mathbb{N}^4} q\mathbb{Z}[q]E[b]$;
- $\sigma(B[a]) = q^N B[a]$ for some $N \in \mathbb{Z}$.
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The basis is known as the dual canonical basis. Examples of dual canonical basis elements are:

- the generators $u_0, u_1, u_2, u_3$;
- $p_0 = u_2u_0 - q^2u_1^2$ and $p_1 = u_3u_1 - q^2u_2^2$. 
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The integral form $U_q(w)_{\mathbb{Z}}$ of $U_q(w)$ turns out to be a $q$-deformation of Geiß-Leclerc-Schröer’s cluster algebra $\mathcal{A}(\mathcal{C}_w)$. 
The cluster algebra

Cluster algebras have been introduced by Fomin-Zelevinsky. Let us look at the cluster algebra $\mathcal{A}(C_w)$ Geiß-Leclerc-Schröer attached to $w$. 
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Start with an initial cluster consisting of two frozen variables $P_0$ and $P_1$ and two mutable variables $U_0$ and $U_1$. 
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Mutation exchanges $U_0$ with another variable $U_2$ such that

$$U_0 U_2 = U_1^2 + P_0.$$
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The cluster algebra $\mathcal{A}(C_w) \subset \mathbb{Q}(U_0, U_1)$ is the $\mathbb{Q}$-algebra generated by the cluster variables $\ldots, U_{-1}, U_0, U_1, U_2, U_3, U_4, \ldots$ and $P_0$ and $P_1$. 

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The Laurent phenomenon and linear exchange relations

Put $P_0 = P_1 = 1$. Exchange relations become $U_{n-1} U_{n+1} = U_n^2 + 1$ (for $n \in \mathbb{Z}$); the cluster algebra $\mathcal{A}(C_w)$ degenerates to the coefficient-free cluster algebra $\mathcal{A}$ which was studied by Caldero-Zelevinsky.
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Put $U_0 = U_1 = 1$. The sequence becomes 2, 5, 13, 34, 89, etc. Every term in the sequence turns out to be a natural number. In fact, the sequence is every other Fibonacci number. The integrality is an instance of the Laurent phenomenon: Every cluster variable is a Laurent polynomial in $U_0$ and $U_1$. 

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2 The nonlinear exchange relation from above may be replaced by the linear three-term recursion

$$U_{n+1} + U_{n-1} = TU_n$$

where $T = \frac{U_0^2 + U_1^2 + 1}{U_0U_1}$ (for $n \in \mathbb{Z}$). For example, $U_{n+1} + U_{n-1} = 3U_n$ (for $n \in \mathbb{Z}$) in the case $U_0 = U_1 = 1$. 
The algebra $U_q(w)_\mathbb{Z}$ is a quantum cluster algebra in the sense of Berenstein-Zelevinsky. It degenerates to Geiß-Leclerc-Schröer’s cluster algebra $\mathcal{A}(C_w)$ in the classical limit $q = 1$.

Figure: The initial cluster with frozen (blue) and mutable (red) variables
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Further results are:

1. The dual canonical basis element $B[n + 1, 0, 0, n]$ (for $n \in \mathbb{N}$) specializes to the cluster variable $U_{n+3}$ of the cluster algebra $\mathcal{A}(\mathcal{C}_w)$ of Kronecker type from above in the classical limit $q = 1$.  

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2. Two adjacent quantized cluster variables $B[n + 1, 0, 0, n]$ and $B[n, 0, 0, n - 1]$ are $q$-commutative.

3. We give recursions of Chebychev type for the quantized cluster variables $B[n, 0, 0, n]$ and $B[n + 1, 0, 0, n]$ that allow to compute the elements explicitly.
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4. The dual canonical basis elements $B[n,0,0,n]$ and $B[n+1,0,0,n]$ specialized at $q = 1$ become elements in Caldero-Zelevinsky’s semicanonical basis of $\mathcal{A}(\mathcal{C}_w)$.
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7. We give some expansions of products, e.g., $B[1,0,0,1]B[n,0,0,n-1] = q^{1-4n}B[n+1,0,0,n] + q^{-4n}B[n,1,1,n-1]$. 

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