Tropical Positivity and Determinantal Varieties

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Marie Brandenburg
joint work with Georg Loho and Rainer Sinn

Geometry meets Combinatorics in Bielefeld

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MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES

Overview

1. Tropicalization

2. Positive Tropicalization

3. Determinantal Varieties

tropical semiring $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot) = (\mathbb{R} \cup \{\infty\}, \min, +)$

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$$a \oplus b = \min(a, b)$$
$$a \odot b = a + b$$

tropicalization: transform algebraic varieties into polyhedral fans

complex Puisseux series $\mathcal{C} := \mathbb{C}\{\{t\}\}$

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$$\mathcal{C}^{2\times 2} \ni \tilde{A} = \begin{pmatrix} 1-t & 2t^{-3} \\ t^{1/2}+t^2 & 3 \end{pmatrix} \quad \operatorname{val}(\tilde{A}) = \begin{pmatrix} 0 & -3 \\ 1/2 & 0 \end{pmatrix}$$

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Tropicalization ⟩ Example

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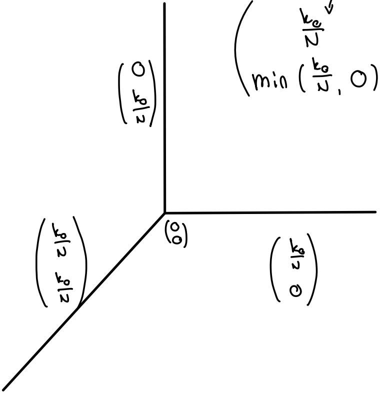

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Theorem (Fund. Th. of Algebraic Tropical Geometry)

$\operatorname{trop}(V(I)) = \{w \in \mathbb{R}^n \mid \operatorname{in}_w(I) \not\ni \textit{monomial}\}$

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Theorem (Speyer-Williams '05)

Let $w \in \mathrm{trop}(V(I))$. *Then* $w \in \mathrm{trop}^+(V(I))$

$\iff$ *all polynomials in* $\mathrm{in}_w(I)$ *have coefficients of both signs.*

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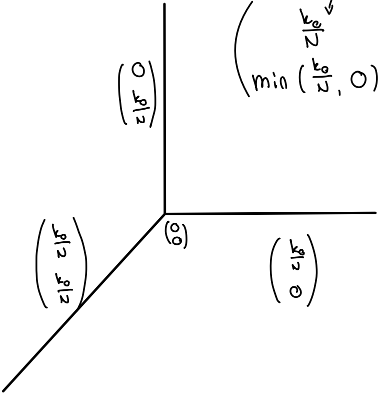

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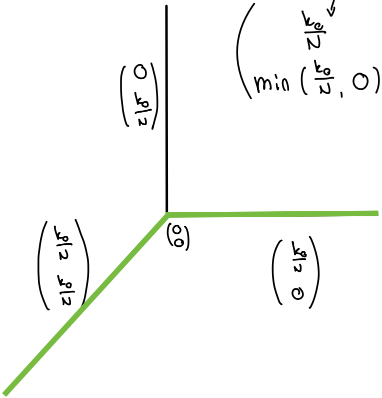

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Tropical determinantal variety

$$\mathrm{trop}(V(I_r))$$

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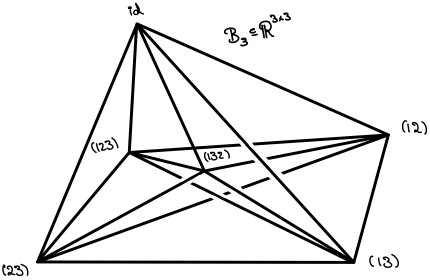

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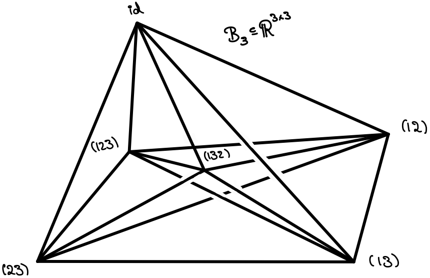

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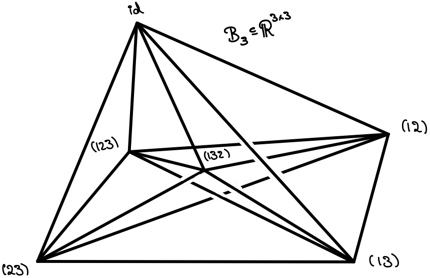

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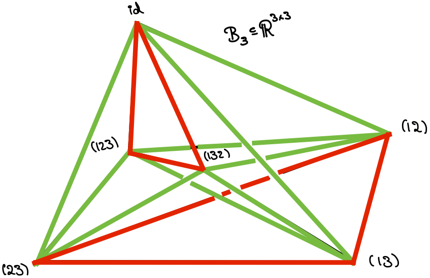

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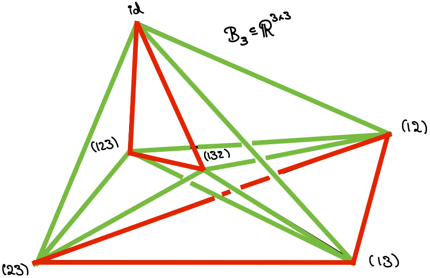

non-positive edges: $\{(\sigma, \pi) \mid \mathrm{sgn}(\pi\sigma) = 1\} \cong A_n$ alternating group

Point Configurations

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$A = \mathrm{val}(\tilde{A})$

$\rightarrow$ columns of $A \cong n$ points on $(r\text{-}1)$-dim'l tropical linear space in $\mathbb{TP}^{d-1} = \mathbb{R}^n / (\mathbb{R} + (1, \ldots, 1))$

Tropical Lines

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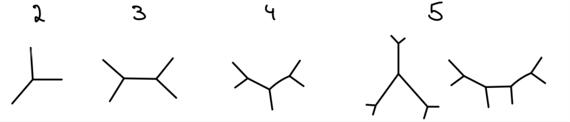

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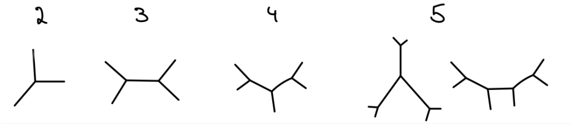

Theorem (follows from [Ardila '04])

Let $A \in \operatorname{trop}(V(I_2))$. *Then* $A \in \operatorname{trop}^+(V(I_2))$
$\iff$ *points form "consecutive chain" on tropical line*

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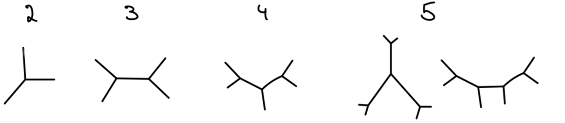

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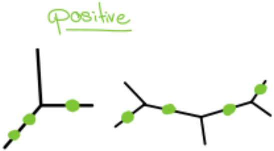

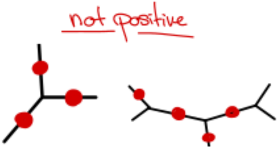

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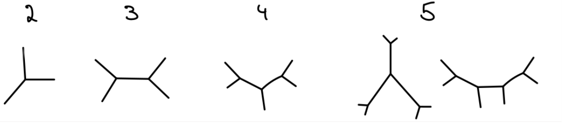

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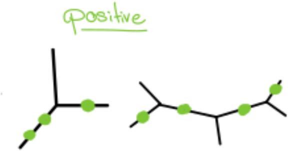
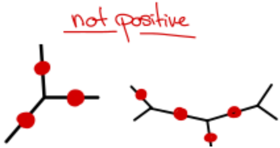
$\iff$ *points form "consecutive chain" on tropical line*

($\iff$ A *has Barvinok rank* 2*)*

($\iff$ *the ass. bicolored phylogenetic tree [Markwig-Yu'09] is a caterpillar tree)*





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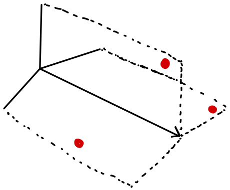
Definition.

A point configuration of 3 points form a starship on a tropical plane if they lie on 3 distinct 2-dimensional faces, which intersect in an unbounded 1-dimensional face.



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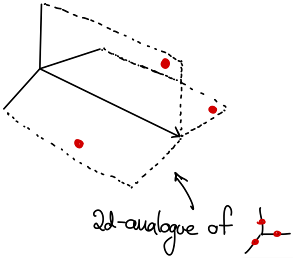

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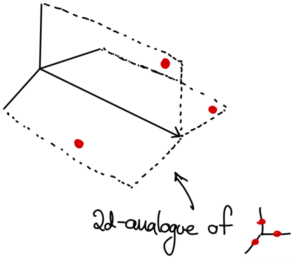
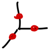


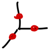
Theorem (B.-Loho-Sinn'22, "Starship Criterion")

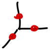
$A \in \mathrm{trop}^+(V(I_3))$

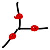
$\implies$ *the point configuration does not contain a starship*

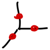
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Determinantal Varieties ⟩ Higher Ranks

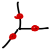
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

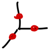
> Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

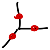
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

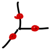
Recap

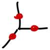
- Rank 2: positive $\implies$ no 

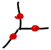
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^{+}(V(I_r))$ containing an analogue of .

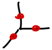
Recap

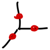
- Rank 2: positive $\implies$ no 
- Rank 3: positive $\implies$ no starship

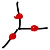
For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

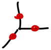
Recap

- Rank 2: positive $\implies$ no 
- Rank 3: positive $\implies$ no starship
- higher ranks: everything can happen

For ranks $r \geq 4$, higher-dimensional analogues of  may occur:

> Counterexamples for rank $r \geq 4$ of a positive point configuration $A \in \mathrm{trop}^+(V(I_r))$ containing an analogue of .

Recap

- Rank 2: positive $\implies$ no 
- Rank 3: positive $\implies$ no starship
- higher ranks: everything can happen

Thank you!