Conjugacy growth in groups, geometry and combinatorics

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Counting elements and conjugacy classes in groups

Finite groups

Let S_3 be the symmetric group on 3 objects.

$$S_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$$

- Elements : words on generating set $\{a, b\}$:

$$\{1, a, b, ab, ba, aba\}$$

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$$\texttt{elements of length} = n : \qquad a(n) := \sharp\{g \in G \mid |g|_X = n\}$$

$$\texttt{elements of length} \leq n : \qquad A(n) := \sharp\{g \in G \mid |g|_X \leq n\}.$$

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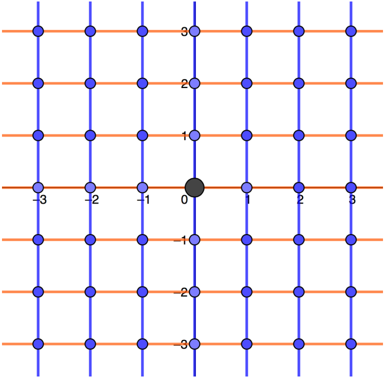
- $|g|_c$ is the shortest length of an element in the conjugacy class $[g]$ (w.r.t. X).
- Conjugacy growth functions:

$$c(n) := \sharp\{[g] \in G \mid |g|_c = n\}$$

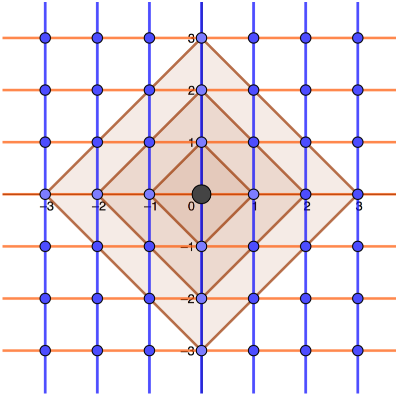
$$C(n) := \sharp\{[g] \in G \mid |g|_c \leq n\}$$

Examples: free groups – abelian and non-abelian

Cayley graph of $\mathbb{Z}^2$ with standard generators a and b



$\mathbb{Z}^2$ with standard generators a and b



$$a(k) = 4k, \quad A(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

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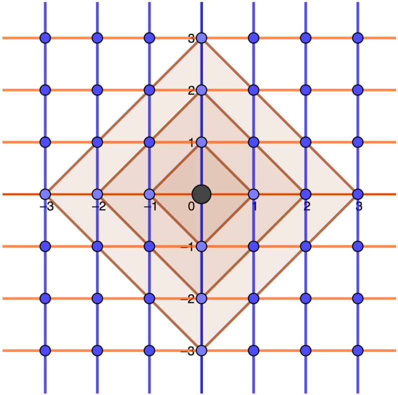
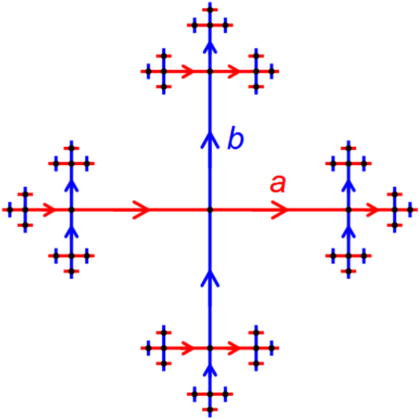


$$a(k) = c(k) = 4k, \quad A(n) = C(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1$$

Free group $F(a, b)$ with generators a and b



$$a(n) = 4 \cdot 3^{n-1}$$

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Not entirely correct: when powers are included, one shouldn't divide by n.

Asymptotics of conjugacy growth in free groups

Coornaert (2005): For the free group F_r, the primitive (non-powers) conjugacy growth function is given by

$$c_p(n) \sim \frac{(2r-1)^{n+1}}{2(r-1)n} = K\,\frac{(2r-1)^n}{n},$$

where $K = \frac{2r-1}{2(r-1)}$.

Comparing standard and conjugacy growth

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  $\star$ Exclude the Osin or Ivanov type 'monsters'!

- Easy/Hard: Compare standard and conjugacy growth rates.

Growth rates

The standard growth rate $\limsup_{n\to\infty} \sqrt[n]{a(n)}$ of G wrt X is in fact a limit i.e.

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Hull: There are groups for which

$$\liminf_{n\to\infty} \sqrt[n]{c(n)} < \limsup_{n\to\infty} \sqrt[n]{c(n)},$$

that is, the limit does not exist.

Conjugacy vs. standard growth

	Standard growth	Conjugacy growth
Type	pol., int., exp.	pol., int.*, exp.
Quasi-isometry invariant	yes	no**, but group invariant
Rate of growth	exists	exists (not always)

* Bartholdi, Bondarenko, Fink.

** Hull-Osin (2013): conjugacy growth not quasi-isometry invariant.

1st Recap

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- growth of elements (called *word growth* or *standard growth*).

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  'Robust' asymptotics.

- growth of conjugacy classes.

  About 20-30 papers.

  'Less robust' asymptotics.

Conjugacy growth: history and motivation

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- 1960s (Sinai, Margulis): $M =$ complete Riemannian manifolds or compact manifolds of pinched negative curvature;
- 1990s - 2000s (Knieper, Coornaert, Link): some classes of (rel) hyperbolic or CAT(0) groups.

Conjugacy growth asymptotics

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- Rivin (2000), Coornaert (2005): asymptotics for the free groups.
- Guba-Sapir (2010): asymptotics for various groups.
- **Conjecture (Guba-Sapir)**: groups* of standard exponential growth have exponential conjugacy growth.

Conjugacy growth asymptotics

- Breuillard-Cornulier (2010): (uniform) exponential conjugacy growth for soluble (non virt. nilpotent) groups.

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- Breuillard-Cornulier-Lubotzky-Meiri (2011): (uniform) exponential conjugacy growth for linear (non virt. nilpotent) groups.
- Hull-Osin (2014): all acylindrically hyperbolic groups have exponential conjugacy growth.

The conjugacy growth series

The conjugacy growth series

Let G be a group with finite generating set X.

- The conjugacy growth series of G with respect to X records the number of conjugacy classes of every length. It is

$$\widetilde{\sigma}_{(G,X)}(z) := \sum_{n=0}^{\infty} c(n) z^n,$$

where $c(n)$ is the number of conjugacy classes of length n.

Conjugacy growth series in $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is:

$$\widetilde{\sigma}_{(\mathbb{Z},\{1,-1\})}(z) = 1 + 2z + 2z^2 + \cdots = \frac{1+z}{1-z}.$$

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\widetilde{\sigma}_{(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})}(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1+2z-2z^3}{1-z^2}.$$

Rational, algebraic, transcendental

A generating function $f(z)$ is

- rational if there exist polynomials $P(z)$, $Q(z)$ with integer coefficients such that $f(z) = \frac{P(z)}{Q(z)}$;
- algebraic if there exists a polynomial $P(x, y)$ with integer coefficients such that $P(z, f(z)) = 0$;
- transcendental otherwise.

Rational series

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Rational conjugacy growth series give conjugacy asymptotics.

Question: For which groups are conjugacy growth series rational?

Growth series in groups: results and connections

Rationality

Being rational/algebraic/transcendental is not a group invariant!

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Theorem [Stoll, 1996]

The higher Heisenberg groups $H_r, r \geq 2$, have rational growth with respect to one choice of generating set and transcendental with respect to another.

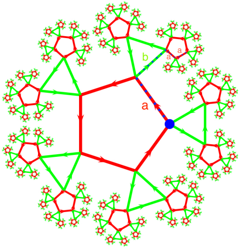
$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, d, e \in \mathbb{Z} \right\}$$

Conjugacy in hyperbolic groups

Hyperbolic groups

Motivation: Most (finitely presented) groups are hyperbolic.

'Definition': Groups whose Cayley graph looks like the hyperbolic plane.



Examples: free groups, free products of finite groups, $SL(2, \mathbb{Z})$, virtually free groups *, surface groups, small cancellation groups, and many more.

* Virtually free = groups with a free subgroup of finite index.

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In $\mathbb{Z}_2 * \mathbb{Z}_2$ the conjugacy growth series is:

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The conjugacy growth series in free groups

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- Rivin (2000, 2010): the conjugacy growth series of F_k is not rational:

$$\widetilde{\sigma}(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}$$

$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

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If G hyperbolic, then the conjugacy growth series of G is rational if and only if G is virtually cyclic.

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Theorem (Antolín - C., 2017)

If G is 'large' hyperbolic, the conjugacy growth series is transcendental.

$\Leftarrow$

Theorem (C. - Hermiller - Holt - Rees, 2016)

If G is virtually cyclic, the conjugacy growth series of G is rational.

NB: Both results hold for all symmetric generating sets of G.

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Recently: results on rationality of standard and conjugacy growth series.

	Standard Growth Series	Conjugacy Growth Series
Hyperbolic	Rational (Cannon, Gromov, Thurston)	Transcendental (C.- Antolín' 17)
Virtually abelian		

FOR ALL GENERATING SETS!

Rationality of standard and conjugacy growth series

	Standard Growth Series	Conjugacy Growth Series
Hyperbolic	Rational (Cannon, Gromov, Thurston)	Transcendental (Antolín - C. '17)
Virtually abelian	Rational (Benson '83)	Rational (Evetts '19)

FOR ALL GENERATING SETS!

Where are the geometry and the combinatorics?

Theorem (Antolín - C., 2017)

If G is hyperbolic[1], the conjugacy growth series is transcendental.

[1] not virtually cyclic

Idea of proof: GEOMETRY

Theorem. (Coornaert - Knieper 2007, Antolín - C. 2017)

Let G be a 'large' hyperbolic group. There are constants $A, B, n_0 > 0$ such that

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

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MESSAGE:

The number of conjugacy classes of length n is asymptotically the number of elements of length n divided by n.

End of the proof: COMBINATORICS

The transcendence of the conjugacy growth series follows from the bounds

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The transcendence of the conjugacy growth series follows from the bounds

$$A\frac{\alpha^n}{n} \leq c(n) \leq B\frac{\alpha^n}{n}$$

together with

Lemma (Flajolet: Trancendence of series based on bounds).

Suppose there are positive constants $A, B, \mathbf{h}$ and an integer $n_0 \geq 0$ s.t.

$$A\frac{e^{\mathbf{h}n}}{n} \leq a_n \leq B\frac{e^{\mathbf{h}n}}{n}$$

for all $n \geq n_0$. Then the power series $\sum_{i=0}^{\infty} a_n z^n$ is not algebraic.

Consequence of Rivin's (confirmed) Conjecture

Corollary (Antolín - C.)

For any hyperbolic group G with generating set X the standard and conjugacy growth rates are the same:

$$\lim_{n\to\infty} \sqrt[n]{c(n)} = \gamma_{G,X} = \alpha_{G,X}.$$

Rivin's conjecture for other groups: GEOMETRY

Theorem (Gekhtman and Yang, 2019)

Let G be a non-elementary group with a finite generating set S. If G has a contracting element with respect to the action on the corresponding Cayley graph, then the conjugacy growth series is transcendental.

Groups with contracting element

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All have transcendental conjugacy growth series w.r.t. standard generating sets.

More groups and their conjugacy growth series

	Conjugacy Growth Series[1]	Formula
Wreath products[2]	Transcendental (Mercier '17)	✓
Graph products[3]	Transcendental (C.- Hermiler - Mercier '22)	✓
BS(1,m)	Transcendental (C.- Evetts - Ho, '20)	✓

[1] w.r.t. standard gen. sets

[2] certain wreath products

[3] in most cases

Computing the conjugacy growth series

Counting cyclic representatives

- Let L be a set of words, $a(k)$ the number of words of length k in L, and $f_L(t) = \sum_{k \geq 1} a(k) t^k$ the generating function of L.

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- The generating function for the language $L_\sim$ of cyclic representatives is

$$\int_0^z \frac{\sum_{k\geq 1} \phi(k) f_L(t^k)}{t}\, dt.$$

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- Rivin's formula for conjugacy series of free groups

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$$\mathcal{H}(x) = 1 + (k-1)\frac{x^2}{(1-x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left(\frac{1}{1-(2k-1)x^d} - 1 \right).$$

- SURPRISE:

$$f_{L_\sim}(z) = \widetilde{\sigma}(z)$$

Graph products, I

Thm. (Rivin) For groups $G = \langle X \rangle$ and $H = \langle Y \rangle$, the conjugacy growth function of the direct product $G \times H$ w.r.t. $Z := X \cup Y$ is

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(Not rational!)

Goal. Compute the conjugacy growth series for raAg's and graph products.

Graph products, II

Def. The **graph product** associated to a finite graph $\Gamma = (V, E)$ with vertex groups $G_u = \langle X_u \rangle$ for $u \in V$ is the group

$$G_V := \langle G_u \mid \{[g, h] = 1 \mid g \in G_u, h \in G_v, \ \overset{u}{\bullet}\!\!-\!\!\overset{v}{\bullet} \in E\} \rangle$$

with generating set $X_V := \cup_{u \in V} X_u$.

- Γ has no edges $\implies$ $G_V = *_u G_u$ is the free product.
- Γ is complete $\implies$ $G_V = \times_u G_u$ is the direct product.

A **right-angled Artin group**, or **raAg**, is a graph product with every $G_u \cong \mathbb{Z}$.

A **right-angled Coxeter group**, or **raCg**, has each $G_u \cong \mathbb{Z}_2$.

Graph product formula

Theorem (C.- Hermiller-Mercier)

Let G_V be a graph product and $v \in V$. The conjugacy growth series of G_V is given by

$$\tilde{\sigma}_V = \tilde{\sigma}_{V\setminus\{v\}} + \tilde{\sigma}_{\mathrm{Ct}(v)}(\tilde{\sigma}_{\{v\}} - 1) + \sum_{S\subseteq \mathrm{Ct}(v)} \tilde{\sigma}_S^{\mathcal{M}} \; \mathrm{N}\left(\left(\frac{\sigma_{\mathrm{Ct}(S)\setminus\{v\}}}{\sigma_{\mathrm{Ct}(v)\cap \mathrm{Ct}(S)}} - 1\right)(\sigma_{\{v\}} - 1)\right).$$

Moreover, if $\{v\} \cup \mathrm{Ct}(v) = V$, then $\tilde{\sigma}_V = \tilde{\sigma}_{\mathrm{Ct}(v)}\tilde{\sigma}_v$.

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$$\mathrm{N}(f)(z) := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\phi(k)}{kl}\left(f(z^k)\right)^l = \sum_{k=1}^{\infty} \frac{-\phi(k)}{k} Log(1 - f(z^k)).$$

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Final recap

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- **Conjecture (C, Evetts, Ho)**:

  The only finitely generated groups with rational conjugacy growth series are virtually abelian.

Questions

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  This holds for EVERY class of groups studied so far, but the proof is local.

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- Are there groups with algebraic conjugacy growth series?

- How do the conjugacy growth series behave when we change generators?

  Stoll: The rationality of the standard growth series depends on generators.

Thank you!