Conjugacy growth in groups, geometry and combinatorics

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Counting elements and conjugacy classes in groups
Finite groups

Let $S_3$ be the symmetric group on 3 objects.

$$S_3 = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1 \rangle$$

- Elements: words on generating set $\{a, b\}$:

$$\{1, a, b, ab, ba, aba\}$$
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How many elements? 6

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Growth of $G$: number of elements in $G$, depending on length.
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  \[
  \text{elements of length } = n : \quad a(n) := \#\{g \in G \mid |g|_X = n\}
  \]
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Growth of $G$: number of elements in $G$, depending on length.

- Standard growth functions. For all $n \geq 0$:

  elements of length $= n$: $a(n) := \# \{ g \in G \mid |g|_X = n \}$

  elements of length $\leq n$: $A(n) := \# \{ g \in G \mid |g|_X \leq n \}$. 
Counting conjugacy classes in groups

Conjugacy growth of $G$: number of conjugacy classes containing an element of length $n$ in $G$, for all $n \geq 0$. 
Counting conjugacy classes in groups

- **Conjugacy growth** of $G$: number of conjugacy classes containing an element of length $n$ in $G$, for all $n \geq 0$.

- $|g|_c$ is the shortest length of an element in the conjugacy class $[g]$ (w.r.t. $X$).
Counting conjugacy classes in groups

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- $|g|_c$ is the shortest length of an element in the conjugacy class $[g]$ (w.r.t. $X$).

- **Conjugacy growth functions**:

\[
c(n) := \# \{ [g] \in G \mid |g|_c = n \} \\
C(n) := \# \{ [g] \in G \mid |g|_c \leq n \}
\]
Examples: free groups – abelian and non-abelian
Cayley graph of $\mathbb{Z}^2$ with standard generators $a$ and $b$. 
$\mathbb{Z}^2$ with standard generators $a$ and $b$

\[ a(k) = 4k, \quad A(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1 \]
$\mathbb{Z}^2$ with standard generators $a$ and $b$

\[ a(k) = c(k) = 4k, \quad A(n) = C(n) = 1 + \sum_{k=1}^{n} 4k = 2n^2 + 2n + 1 \]
Free group $F(a, b)$ with generators $a$ and $b$

$$a(n) = 4 \cdot 3^{n-1}$$
Conjugacy growth in $F(a, b)$?

- $[aba] = \{aba, baa, aab, a^3ba^{-1}, \ldots \}$
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Not entirely correct: when powers are included, one shouldn't divide by $n$. 
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$c(n)$: take $\#$ of cyclically reduced (!) words of length $n$, and divide by $n$. 

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$$\implies c(n) \sim \frac{3^n}{n}$$
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$$\implies c(n) \sim \frac{3^n}{n}$$

Not entirely correct: when powers are included, one shouldn’t divide by $n$. 
Asymptotics of conjugacy growth in free groups

**Coornaert (2005):** For the free group $F_r$, the primitive (non-powers) conjugacy growth function is given by

$$c_p(n) \sim \frac{(2r - 1)^{n+1}}{2(r - 1)n} = K \frac{(2r - 1)^n}{n},$$

where $K = \frac{2r-1}{2(r-1)}$. 
Comparing standard and conjugacy growth
C(n) vs A(n)??

- Easy (no partial credit): $C(n) \leq A(n)$ and $C(n) = A(n)$ for abelian groups.
C(n) vs A(n)??

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- Medium:

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  \limsup_{n \to \infty} \frac{C(n)}{A(n)} = ?
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  **Conjecture (Guba-Sapir):** groups* of standard exponential growth have exponential conjugacy growth.
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- Hard:

**Conjecture (Guba-Sapir):** groups* of standard exponential growth have exponential conjugacy growth.

* Exclude the Osin or Ivanov type ‘monsters’!

- Easy/Hard: Compare standard and conjugacy growth rates.
Growth rates

The standard growth rate \( \limsup_{n \to \infty} \sqrt[n]{a(n)} \) of \( G \) wrt \( X \) is in fact a limit i.e.

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\alpha = \lim_{n \to \infty} \sqrt[n]{a(n)}.
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The conjugacy growth rate of \( G \) wrt \( X \) is

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\gamma = \limsup_{n \to \infty} \sqrt[n]{c(n)}.
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Growth rates

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\]

**Hull:** There are groups for which

\[
\liminf_{n \to \infty} \sqrt[n]{c(n)} < \limsup_{n \to \infty} \sqrt[n]{c(n)}, 
\]

that is, the limit does not exist.
Conjugacy vs. standard growth

<table>
<thead>
<tr>
<th></th>
<th>Standard growth</th>
<th>Conjugacy growth</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type</strong></td>
<td>pol., int., exp.</td>
<td>pol., int.*, exp.</td>
</tr>
<tr>
<td><strong>Quasi-isometry invariant</strong></td>
<td>yes</td>
<td>no**, but group invariant</td>
</tr>
<tr>
<td><strong>Rate of growth</strong></td>
<td>exists</td>
<td>exists (not always)</td>
</tr>
</tbody>
</table>

* Bartholdi, Bondarenko, Fink.

1st Recap

TWO types of counting in groups:

- growth of elements (called \textit{word growth} or \textit{standard growth}).
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‘Robust’ asymptotics.

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  ‘Robust’ asymptotics.

- growth of conjugacy classes.

  About 20-30 papers.
  ‘Less robust’ asymptotics.
Conjugacy growth: history and motivation
Conjugacy growth in geometry

Counting the primitive closed geodesics of bounded length on a compact manifold $M$ of negative curvature and exponential volume growth gives
Conjugacy growth in geometry

Counting the primitive closed geodesics of bounded length on a compact manifold $M$ of negative curvature and exponential volume growth gives via quasi-isometries.
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good exponential asymptotics for the primitive conjugacy growth of $\pi_1(M)$. 
Conjugacy growth in geometry

Counting the primitive closed geodesics of bounded length on a compact manifold $M$ of negative curvature and exponential volume growth gives

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- 1960s (Sinai, Margulis): $M$ = complete Riemannian manifolds or compact manifolds of pinched negative curvature;
- 1990s - 2000s (Knieper, Coornaert, Link): some classes of (rel) hyperbolic or CAT(0) groups.
Conjugacy growth asymptotics

Conjugacy growth asymptotics


Conjecture (Guba-Sapir): groups $\ast$ of standard exponential growth have exponential conjugacy growth.
Conjugacy growth asymptotics


- **Guba-Sapir (2010)**: asymptotics for various groups.

- **Conjecture (Guba-Sapir)**: groups* of standard *exponential* growth have exponential conjugacy growth.
Conjugacy growth asymptotics

- Breuillard-Cornulier (2010): (uniform) exponential conjugacy growth for soluble (non virtually nilpotent) groups.
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- Breuillard-Cornulier (2010): (uniform) exponential conjugacy growth for soluble (non virt. nilpotent) groups.

- Breuillard-Cornulier-Lubotzky-Meiri (2011): (uniform) exponential conjugacy growth for linear (non virt. nilpotent) groups.
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- Hull-Osin (2014): all acylindrically hyperbolic groups have exponential conjugacy growth.
The conjugacy growth series
The conjugacy growth series

Let $G$ be a group with finite generating set $X$.

- The **conjugacy growth series** of $G$ with respect to $X$ records the number of conjugacy classes of every length. It is

$$\tilde{\sigma}(G,X)(z) := \sum_{n=0}^{\infty} c(n)z^n,$$

where $c(n)$ is the number of conjugacy classes of length $n$. 
Conjugacy growth series in $\mathbb{Z}$, $\mathbb{Z}_2 \ast \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is:

$$\tilde{\sigma}(\mathbb{Z}, \{1, -1\})(z) = 1 + 2z + 2z^2 + \cdots = \frac{1 + z}{1 - z}.$$
Conjugacy growth series in $\mathbb{Z}$, $\mathbb{Z}_2 * \mathbb{Z}_2$

In $\mathbb{Z}$ the conjugacy growth series is:

$$\tilde{\sigma}(\mathbb{Z},\{1,-1\})(z) = 1 + 2z + 2z^2 + \cdots = \frac{1 + z}{1 - z}.$$ 

In $\mathbb{Z}_2 * \mathbb{Z}_2$ a set of conjugacy representatives is $1, a, b, ab, abab, \ldots$, so

$$\tilde{\sigma}(\mathbb{Z}_2 * \mathbb{Z}_2,\{a,b\})(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1 + 2z - 2z^3}{1 - z^2}.$$
A generating function $f(z)$ is

- **rational** if there exist polynomials $P(z), Q(z)$ with integer coefficients such that $f(z) = \frac{P(z)}{Q(z)}$;
- **algebraic** if there exists a polynomial $P(x, y)$ with integer coefficients such that $P(z, f(z)) = 0$;
- **transcendental** otherwise.
Rational series

If $f(z) = \frac{P(z)}{Q(z)}$ is a rational generating function for some sequence $a_n$, the roots of $Q(z)$ give the growth rate of $a_n$. 
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**Rational conjugacy growth series give conjugacy asymptotics.**

**Question:** For which groups are conjugacy growth series rational?
Growth series in groups: results and connections
Rationality

Being rational/algebraic/transcendental is not a group invariant!
Being rational/algebraic/transcendental is not a group invariant!

Theorem [Stoll, 1996]

The higher Heisenberg groups $H_r$, $r \geq 2$, have rational growth with respect to one choice of generating set and transcendental with respect to another.

$$H_2 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \div a, b, c, d, e \in \mathbb{Z} \right\}$$
Conjugacy in hyperbolic groups
Hyperbolic groups

Motivation: Most (finitely presented) groups are hyperbolic.

‘Definition’: Groups whose Cayley graph looks like the hyperbolic plane.

Examples: free groups, free products of finite groups, $SL(2, \mathbb{Z})$, virtually free groups *, surface groups, small cancellation groups, and many more.

* Virtually free = groups with a free subgroup of finite index.
RECALL: conjugacy growth series for $\mathbb{Z}$, $\mathbb{Z}_2 \ast \mathbb{Z}_2$

$\mathbb{Z}$, $\mathbb{Z}_2 \ast \mathbb{Z}_2$: virtually cyclic groups $\subseteq$ hyperbolic
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In $\mathbb{Z}_2 \ast \mathbb{Z}_2$ the conjugacy growth series is:

$$\tilde{\sigma}(\mathbb{Z}_2 \ast \mathbb{Z}_2, \{a, b\})(z) = 1 + 2z + z^2 + z^4 + z^6 \cdots = \frac{1 + 2z - 2z^3}{1 - z^2}.$$
The conjugacy growth series in free groups

Free groups ⊆ hyperbolic
The conjugacy growth series in free groups

Free groups ⊆ hyperbolic

- Rivin (2000, 2010): the conjugacy growth series of $F_k$ is not rational:

$$\tilde{\sigma}(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt,$$

where

$$\mathcal{H}(x) = 1 + (k - 1) \frac{x^2}{(1 - x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left( \frac{1}{1 - (2k - 1)x^d} - 1 \right).$$
Conjecture (Rivin, 2000)

If $G$ hyperbolic, then the conjugacy growth series of $G$ is rational if and only if $G$ is virtually cyclic.

Theorem (Antolín - C., 2017)
If $G$ is ‘large’ hyperbolic, the conjugacy growth series is transcendental.

Theorem (C. - Hermiller - Holt - Rees, 2016)
If $G$ is virtually cyclic, the conjugacy growth series of $G$ is rational.

NB: Both results hold for all symmetric generating sets of $G$. 
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Conjugacy growth was first studied in geometry
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Recently: results on rationality of standard and conjugacy growth series.

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FOR ALL GENERATING SETS!
Rationality of standard and conjugacy growth series

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<td>Rational (Benson ’83)</td>
<td>Rational (Evetts ’19)</td>
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FOR ALL GENERATING SETS!
Where are the geometry and the combinatorics?
Theorem (Antolín - C., 2017)

If $G$ is hyperbolic\(^1\), the conjugacy growth series is transcendental.

\(^1\)not virtually cyclic
Theorem. (Coornaert - Knieper 2007, Antolín - C. 2017)

Let $G$ be a ‘large’ hyperbolic group. There are constants $A, B, n_0 > 0$ such that

$$A \frac{\alpha^n}{n} \leq c(n) \leq B \frac{\alpha^n}{n}$$

for all $n \geq n_0$, where $\alpha$ is the word growth rate of $G$. 

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for all $n \geq n_0$, where $\alpha$ is the word growth rate of $G$.

MESSAGE:

The number of conjugacy classes of length $n$ is asymptotically the number of elements of length $n$ divided by $n$. 
The transcendence of the conjugacy growth series follows from the bounds

\[ A \frac{\alpha^n}{n} \leq c(n) \leq B \frac{\alpha^n}{n} \]
The transcendence of the conjugacy growth series follows from the bounds

\[ A \frac{\alpha^n}{n} \leq c(n) \leq B \frac{\alpha^n}{n} \]

together with

**Lemma (Flajolet: Trancendence of series based on bounds).**

Suppose there are positive constants \( A, B, h \) and an integer \( n_0 \geq 0 \) s.t.

\[ A \frac{e^{hn}}{n} \leq a_n \leq B \frac{e^{hn}}{n} \]

for all \( n \geq n_0 \). Then the power series \( \sum_{i=0}^{\infty} a_n z^n \) is not algebraic.
Corollary (Antolín - C.)

For any hyperbolic group $G$ with generating set $X$ the standard and conjugacy growth rates are the same:

$$\lim_{n \to \infty} \sqrt[n]{c(n)} = \gamma_G, X = \alpha_G, X.$$
Rivin’s conjecture for other groups: GEOMETRY

**Theorem** (Gekhtman and Yang, 2019)

Let $G$ be a non-elementary group with a finite generating set $S$. If $G$ has a **contracting element** with respect to the action on the corresponding Cayley graph, then the conjugacy growth series is transcendental.
Groups with contracting element

- Relatively hyperbolic groups,
- right-angled Coxeter groups,
- right-angled Artin groups,
- graphical small cancellation groups.
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- right-angled Coxeter groups,
- right-angled Artin groups,
- graphical small cancellation groups.

All have transcendental conjugacy growth series w.r.t. standard generating sets.
More groups and their conjugacy growth series

<table>
<thead>
<tr>
<th>Conjugacy Growth Series (^1)</th>
<th>Formula</th>
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<tbody>
<tr>
<td>Wreath products(^2)</td>
<td>Transcendental (Mercier '17)</td>
</tr>
<tr>
<td>Graph products(^3)</td>
<td>Transcendental (C.- Hermiler - Mercier '22)</td>
</tr>
<tr>
<td>BS(1,m)</td>
<td>Transcendental (C.- Evetts - Ho, '20)</td>
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</table>

\(^{1}\) w.r.t. standard gen. sets  
\(^{2}\) certain wreath products  
\(^{3}\) in most cases
Computing the conjugacy growth series
Counting cyclic representatives

- Let $L$ be a set of words, $a(k)$ the number of words of length $k$ in $L$, and
  
  \[ f_L(t) = \sum_{k \geq 1} a(k) t^k \]

  the generating function of $L$.

  Assume $L$ is closed under taking powers and cyclic permutations of words.
Counting cyclic representatives

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Assume $L$ is closed under taking powers and cyclic permutations of words.

- The generating function for the language $L\sim$ of cyclic representatives is

$$\int_0^z \frac{\sum_{k \geq 1} \phi(k) f_L(t^k)}{t} \, dt.$$
Rivin’s formula for free groups

- Take $L$ = the \textit{cyclically reduced words} in the free group, and $f_L$ its gen. function.
Rivin’s formula for free groups

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The generating function for $L\sim$ (the cyclic representatives) is

$$f_{L\sim}(z) = \int_0^z \frac{\sum_{d \geq 1} \phi(d) f_L(t^d)}{t} \, dt$$
Rivin’s formula for free groups

- Take $L$ = the cyclically reduced words in the free group, and $f_L$ its gen. function.

  The generating function for $L\sim$ (the cyclic representatives) is
  \[
  f_{L\sim}(z) = \int_0^z \sum_{d \geq 1} \frac{\phi(d)f_L(t^d)}{t} dt
  \]

- Rivin’s formula for conjugacy series of free groups

  \[
  \tilde{\sigma}(z) = \int_0^z \frac{\mathcal{H}(t)}{t} dt, \quad \text{where}
  \]
  \[
  \mathcal{H}(x) = 1 + (k - 1)\frac{x^2}{(1 - x^2)^2} + \sum_{d=1}^{\infty} \phi(d) \left( \frac{1}{1 - (2k - 1)x^d} - 1 \right).
  \]
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- SURPRISE:

$$f_{L\sim}(z) = \tilde{\sigma}(z)$$
Thm. (Rivin) For groups $G = \langle X \rangle$ and $H = \langle Y \rangle$, the conjugacy growth function of the direct product $G \times H$ w.r.t. $Z := X \cup Y$ is

$$\tilde{\sigma}_{G \times H} = \tilde{\sigma}_G \cdot \tilde{\sigma}_H.$$
Graph products, I

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**Thm. (Rivin)** Formula for the conjugacy growth series $\tilde{\sigma}$ of the free product $F_2 = \mathbb{Z} * \mathbb{Z} = \langle a, b \mid \rangle$ with $X = \{a, b\}^{\pm 1}$. 
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(Not rational!)

Goal. Compute the conjugacy growth series for raAg’s and graph products.
Graph products, II

**Def.** The graph product associated to a finite graph $\Gamma = (V, E)$ with vertex groups $G_u = \langle X_u \rangle$ for $u \in V$ is the group

$$G_V := \langle G_u \mid \{[g, h] = 1 \mid g \in G_u, h \in G_v, \bullet \overset{u}{\rightarrow} \overset{v}{\bullet} \in E \} \rangle$$

with generating set $X_V := \cup_{u \in V} X_u$.

- $\Gamma$ has no edges $\implies G_V = *_u G_u$ is the free product.
- $\Gamma$ is complete $\implies G_V = \times_u G_u$ is the direct product.

A **right-angled Artin group**, or **raAg**, is a graph product with every $G_u \cong \mathbb{Z}$.

A **right-angled Coxeter group**, or **raCg**, has each $G_u \cong \mathbb{Z}_2$. 


Graph product formula

**Theorem (C.- Hermiller-Mercier)**

Let $G_V$ be a graph product and $v \in V$. The conjugacy growth series of $G_V$ is given by

$$\tilde{\sigma}_V = \tilde{\sigma}_{V \setminus \{v\}} + \tilde{\sigma}_{Ct(v)}(\tilde{\sigma}_\{v\} - 1) + \sum_{S \subseteq Ct(v)} \tilde{\sigma}_S^M \mathbb{N} \left( \left( \frac{\sigma_{Ct(S) \setminus \{v\}}}{\sigma_{Ct(v) \cap Ct(S)}} - 1 \right) (\sigma_\{v\} - 1) \right).$$

Moreover, if $\{v\} \cup Ct(v) = V$, then $\tilde{\sigma}_V = \tilde{\sigma}_{Ct(v)} \tilde{\sigma}_v$. 
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\tilde{\sigma}_V = \tilde{\sigma}_{V \setminus \{v\}} + \tilde{\sigma}_{Ct(v)}(\tilde{\sigma}_v - 1) + \sum_{S \subseteq Ct(v)} \tilde{\sigma}_S^M N \left( \left( \frac{\sigma_{Ct(S) \setminus \{v\}}}{\sigma_{Ct(v) \cap Ct(S)}} - 1 \right) (\sigma_v - 1) \right).
$$

Moreover, if $\{v\} \cup Ct(v) = V$, then $\tilde{\sigma}_V = \tilde{\sigma}_{Ct(v)} \tilde{\sigma}_v$.

\[ N(f)(z) := \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\phi(k)}{kl} \left( f(z^k) \right)^l = \sum_{k=1}^{\infty} \frac{-\phi(k)}{k} \log(1 - f(z^k)). \]
Graph product formula

\[ \tilde{\sigma}_V = \tilde{\sigma}_{V \setminus \{v\}} + \tilde{\sigma}_{C_t(v)}(\tilde{\sigma}_{\{v\}} - 1) + \sum_{S \subseteq C_t(v)} \tilde{\sigma}_S^M \cdot \mathbb{N}\left(\left(\frac{\sigma_{C_t(S)\setminus\{v\}}}{\sigma_{C_t(v) \cap C_t(S)}} - 1\right)(\sigma_{\{v\}} - 1)\right). \]

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Final recap

- Conjugacy growth series for most groups studied so far are transcendental.
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- Source of transcendental-ness: counting cyclic representatives of a set introduces Euler’s $\varphi$, infinitely many poles etc.

- Conjecture (C, Evetts, Ho):
  The only finitely generated groups with rational conjugacy growth series are virtually abelian.
Questions

- Are the standard and conjugacy growth rates equal for all ‘natural’ groups?

  This holds for EVERY class of groups studied so far, but the proof is local.
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  This holds for EVERY class of groups studied so far, but the proof is local.

- Are there groups with algebraic conjugacy growth series?
Questions

▶ Are the standard and conjugacy growth rates equal for all ‘natural’ groups?

This holds for EVERY class of groups studied so far, but the proof is local.

▶ Are there groups with algebraic conjugacy growth series?

▶ How do the conjugacy growth series behave when we change generators?

Stoll: The rationality of the standard growth series depends on generators.
Thank you!