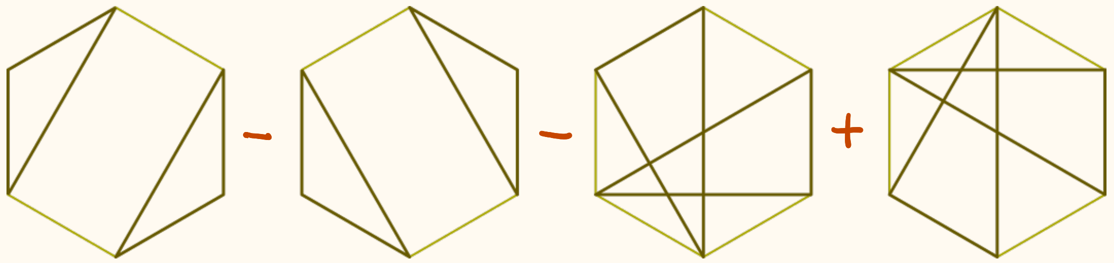
Grassmannians over Rings and Subpolygons



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Motivation / Overview

$$\begin{tikzcd}
\text{Cluster algebras} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Weyl groupoids} \arrow[dl, leftrightarrow] \\
& \text{Arithmetic matroids} &
\end{tikzcd}$$

$$\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -2 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 2 & 3 & -2 & -3 & 2 & 2 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 2 & -2 & -3 & 4 & 4 & -3 & -4 & 2 & 1 & 3
\end{pmatrix}$$

Grassmannian and Plücker coordinates

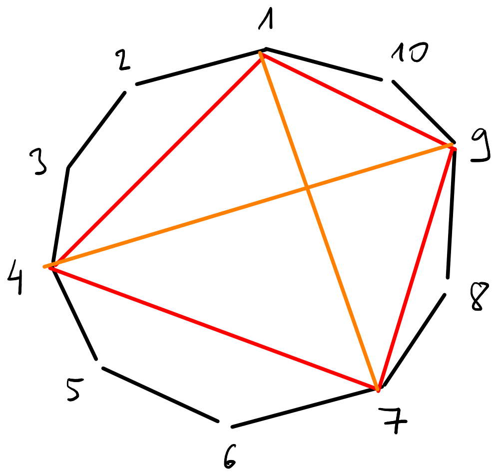
$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kn} \end{pmatrix} \in \mathbb{Q}^{k \times n}, \qquad k \leq n.$$

matrix / arrangement of hyperplanes / subspace
(columns) (rows)

$1 \leq i_1 < \cdots < i_k \leq n$: $\quad X_{i_1,\ldots,i_k} = (x_{\mu,i_\nu})_{\mu,\nu} \in \mathbb{Q}^{k \times k}$ (square submatrix)

$p_{i_1,\ldots,i_k} := \det(X_{i_1,\ldots,i_k})$ Plücker coordinates of the subspace of $\mathbb{Q}^n$ generated by the rows of X, a point on the Grassmannian $G_{\mathbb{Q}}(k,n)$.

Example $k=2$

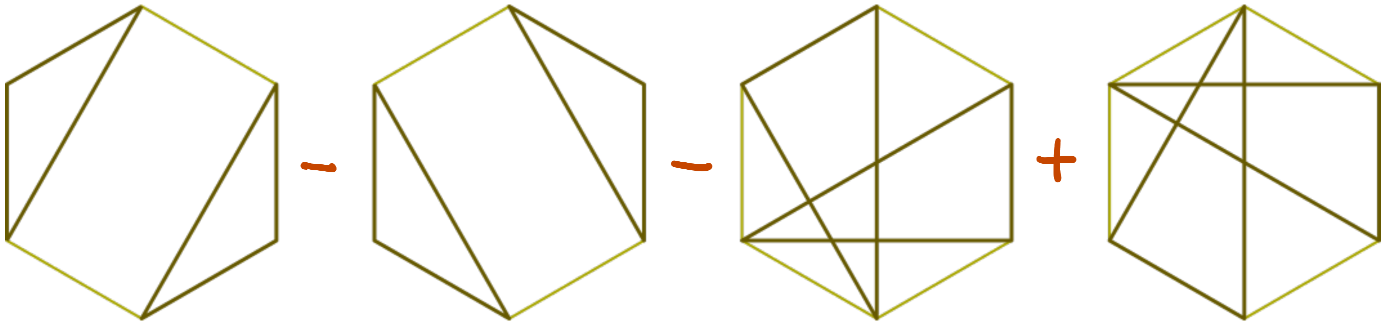


$k=2, \quad n=10$

visualize p_{ij} as diagonal (i,j) in an n-gon.

$$p_{1,7}\, p_{4,9} = p_{1,4}\, p_{7,9} + p_{1,9}\, p_{4,7}$$

Example $k = 3$



Representations and specializations

Plücker relations:

$$\forall\ i_1,\dots,i_{k-1},\ j_0,\dots,j_k:\quad \sum_{r=0}^{k} (-1)^r\, p_{i_1,\dots,i_{k-1},j_r}\, p_{j_0,\dots,\hat{j_r},\dots,j_k} = 0.$$

Coordinate ring of $Gr_{\mathbb{Q}}(k,n)$:

$$A_{k,n} := \mathbb{Q}\left[t_{i_1,\dots,i_k} \mid 1 \le i_1 < \dots < i_k \le n\right] / (\text{Plücker relations for } t_{*,\dots,*})$$

Specialization: homomorphism $\psi : A_{k,n} \longrightarrow \mathbb{Q}$.

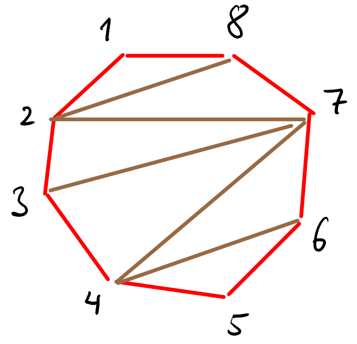
Representation: matrix $X \in \mathbb{Q}^{k \times n}$:

$$\forall\ 1 \le i_1 < \dots < i_k \le n \quad \psi(t_{i_1,\dots,i_k}) = \det(X_{i_1,\dots,i_k})$$

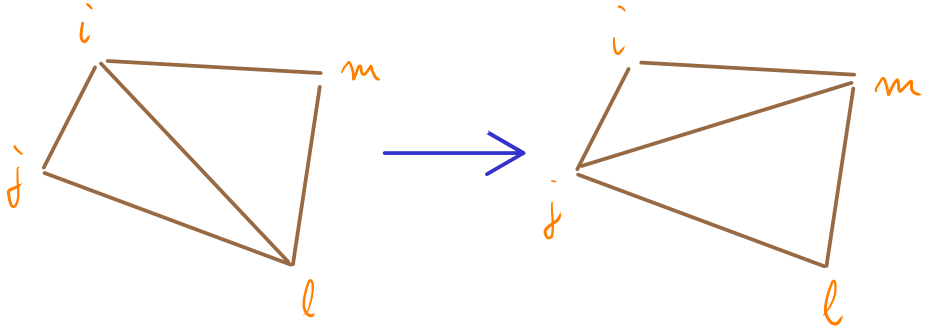
Cluster structure $k=2$

$A_{2,n}$ as a cluster algebra: $t_{i,j}$ are variables
$t_{i,i+1}$ are frozen variables (are never mutated)

cluster: triangulation of a convex n-gon by non-intersecting diagonals.

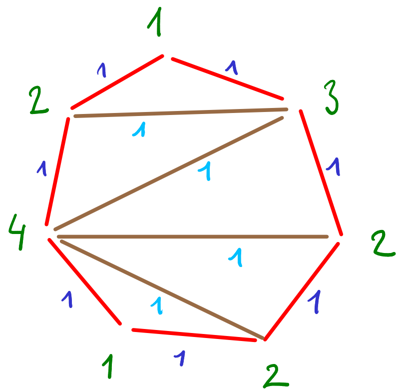


variables of a cluster determine all other variables via mutation.
↑
Plücker relation.



Conway-Coxeter frieze patterns

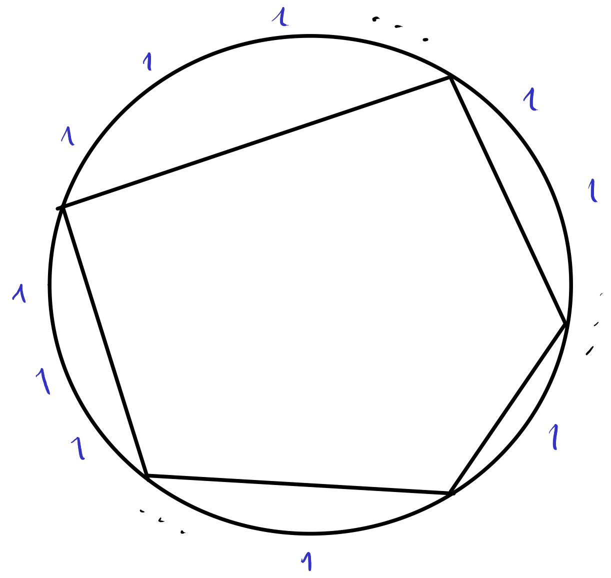
Frieze pattern : specialization of $\mathcal{A}_{2,n}$ that maps all variables of a cluster to 1. (in particular, frozen variables to 1)



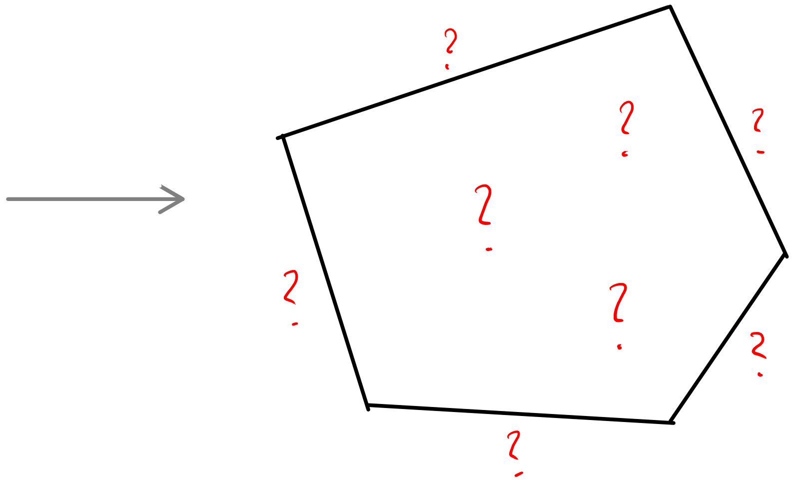
$$
\begin{array}{cccccccccccccc}
0 & 1 & 1 & 1 & 3 & 2 & 1 & 0 & & & & & & \\
 & 0 & 1 & 2 & 7 & 5 & 3 & 1 & 0 & & & & & \\
 & & 0 & 1 & 4 & 3 & 2 & 1 & 1 & 0 & & & & \\
 & & & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & & & \\
 & & & & 0 & 1 & 2 & 3 & 7 & 4 & 1 & 0 & & \\
 & & & & & 0 & 1 & 2 & 5 & 3 & 1 & 1 & 0 & \\
 & & & & & & 0 & 1 & 3 & 2 & 1 & 2 & 1 & 0
\end{array}
$$

representation X

Subpolygons



Conway-Coxeter frieze



specialization to $\mathbb{Z}_{>0}$.

Triangles in frieze patterns

Theorem (C., Holm, Jorgensen, '20)

$(a,b,c) \in \mathbb{N}^3$ appears as labels of a triangle in some Conway-Coxeter frieze if and only if:

(1) $\gcd(a,b) = \gcd(b,c) = \gcd(a,c)$

(2) $\nu_2(a) = \nu_2(b) = \nu_2(c) = 0$ or $|\{\nu_2(a), \nu_2(b), \nu_2(c)\}| > 1$.

Subpolygons in frieze patterns

Theorem (C., Holm, '21): Let $\mathcal{C}$ be a specialization of $\mathcal{A}_{2,n}$ to $\mathbb{Z}_{>0}$. Then $\mathcal{C}$ appears as a subpolygon of some frieze pattern if and only if:

(1) $\forall$ triangle (a,b,c) in $\mathcal{C}$: $\gcd(a,b) = \gcd(b,c) = \gcd(a,c)$.

(2) Let $p<n$ be a prime number. Then for each $(p+1)$-subpolygon $\mathcal{D}$ of $\mathcal{C}$ the labels of edges and diagonals in $\mathcal{D}$ are either all not divisible by p or they do not all have the same p-valuation.

Subpolygons in general

Theorem (C., '22): Let R be a principal ideal domain and ψ a specialization of $A_{k,n}$ such that $\psi(t_{i_1,\dots,i_k}) \in R \setminus \{0\}$ $\forall\ 1 \le i_1 < \dots < i_k \le n$.

Let $\varepsilon \in R$ be such that $(\varepsilon) = (\psi(t_{i_1,\dots,i_k}) \mid 1 \le i_1 < \dots < i_k \le n)$.
Write $(\varepsilon) = (q_1^{d_1}) \cap \dots \cap (q_e^{d_e})$ for primes $q_1, \dots, q_e$.

Then ψ has a representation $X = (x_1, \dots, x_n) \in R^{k \times n}$ such that $|x_i| = 1$ for all $1 \le i \le n$ if and only if:

(1) $(\varepsilon) = (\psi(t_{i, i_2, \dots, i_k}) \mid 1 \le i_2 < \dots < i_k \le n)$

(2) $\forall\ \ell = 1, \dots, e,\ S \subseteq \{1, \dots, n\}$:

$$|S| = |\mathbb{P}(R/_{(q_\ell)})^k| \implies \exists\, i_j \in S\ \forall\, i_1, \dots, i_{k-2} : \nu_{q_\ell}(\psi(t_{i_1, \dots, i_{k-2}, i_j})) \neq \nu_{q_\ell}(\varepsilon).$$

Subpolygons and SL_k-frieze patterns

Theorem (C., '22): $X = (x_1, \dots, x_n) \in \mathbb{Z}^{k \times n}$ representation of a specialization of $A_{k,n}$ with $|x_i| = 1$ $\forall\, i = 1, \dots, n$.

If $|x_{i_1} \cdots x_{i_k}| > 0$ for all $1 \le i_1 < \dots < i_k \le n$, then the specialization may be extended to an SL_k-frieze pattern.

Example: $k = 3$:

$$\begin{array}{ccccccccc}
 & & & \vdots & & \vdots & & & \\
 & 0 & 0 & 1 & 2 & 1 & 3 & 1 & \\
\cdots & 1 & 0 & 0 & 1 & 1 & 4 & 2 & \cdots \\
 & 5 & 1 & 0 & 0 & 1 & 6 & 5 & \\
\cdots & 3 & 1 & 1 & 0 & 0 & 1 & 2 & \cdots \\
 & 2 & 1 & 2 & 1 & 0 & 0 & 1 & \\
\cdots & 1 & 2 & 8 & 7 & 1 & 0 & 0 & \cdots \\
 & 0 & 1 & 5 & 5 & 1 & 1 & 0 & \\
 & & & \vdots & & \vdots & & &
\end{array}$$

Specializing clusters to ± 1

Example: $k=3$, $n=9$ specialization:

$$
\begin{array}{cccccccccc}
\{1,2,4\}, & \{1,2,8\}, & \{1,4,8\}, & \{2,4,5\}, & \{2,4,8\}, & \{4,5,7\}, & \{4,5,8\}, & \{4,7,8\}, & \{4,8,9\}, & \{5,7,8\} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1
\end{array}
$$

representation:

$$
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & 1 & -1 & 2 & -1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 2 & -1 & 2 & 2
\end{pmatrix}
$$

$\longrightarrow$ $\pm$ roots of the root system of type B_3

Thank you!

1	1	1	2	1	1	0	0
0	1	1	1	2	4	1	0
0	0	1	1	1	2	1	1
1	0	0	1	1	1	2	4
1	1	0	0	1	1	1	2
2	4	1	0	0	1	1	1

(wild SL_3-frieze pattern)