Inequalities for $f^*$-vectors of lattice polytopes

Danai Deligeorgaki
KTH Royal Institute of Technology

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We started this project at the online workshop

Research Encounters in Algebraic and Combinatorial Topics
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Ehrhart theory background

- $P \subset \mathbb{R}^d$ is a $d$-dimensional lattice polytope.
- $nP := \{np : p \in P\}$ is the $n$-th dilate of $P$, $n \in \mathbb{N}$.

**Definition**

The function

$$ehr_P(n) := |nP \cap \mathbb{Z}^d|$$

is a polynomial in $n$ (Ehrhart, 1962), known as the **Ehrhart polynomial** of $P$.

Change of basis

Study $\text{ehr}_P(n)$ in different basis:
\[ \left\{ \binom{n+d}{d}, \binom{n+d-1}{d}, \ldots, \binom{n}{d} \right\} \text{ and } \left\{ \binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{d} \right\} \]

- $\text{ehr}_P(n) = \sum_{k=0}^{d} h_k^*(\binom{n+d-k}{d})$ and $\text{ehr}_P(n) = \sum_{k=0}^{d} f_k^*(\binom{n-1}{k})$.

Proposition

*For every lattice polytope $P$ of dimension $d$, for every $0 \leq k \leq d$,*

\[ h_k^* \geq 0, \quad f_k^* \geq 0. \]

*Stanley’s nonnegativity theorem*  
*Breuer (2012)*
Let $P = [0, 1]^2$ be the 2-dimensional unit cube. Then

$$\text{ehr}_P(n) = |nP \cap \mathbb{Z}^2| = |[0, n]^2 \cap \mathbb{Z}^d| = (n+1)^2,$$

and

$$\begin{cases} (n + 1)^2 = h_0^*(\frac{n+2}{2}) + h_1^*(\frac{n+1}{2}) + h_2^*(\frac{n}{2}) \\ (n + 1)^2 = f_0^*(\frac{n-1}{0}) + f_1^*(\frac{n-1}{1}) + f_2^*(\frac{n-1}{2}) \end{cases},$$

hence

$$(h_0^*, h_1^*, h_2^*) = (1, 1, 0),$$

$$(f_0^*, f_1^*, f_2^*) = (4, 5, 2).$$

The 6th dilate of $P = [0, 1]^2$. 
The $f^*$- and $h^*$-vector can also be defined through the **Ehrhart** series of $P$:

$$
\text{Ehr}_P(z) := 1 + \sum_{n \geq 1} \text{ehr}_P(n)z^n = \sum_{k=0}^{d} h^*_k z^k \over (1 - z)^{d+1}
$$

$$
= \sum_{k=-1}^{d} f^*_k \left( \frac{z}{1 - z} \right)^{k+1},
$$

where we let $f^*_{-1} := 1$. 

**Generating function**

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Inequalities for $f^*$-vectors
Let $P := \Delta$ be a unimodular $d$-dimensional simplex, i.e., lattice equivalent to $\text{conv}\{(0, \ldots, 0), (1, \ldots, 0), \ldots, (0, \ldots, 1)\} \subset \mathbb{R}^d$.

**Example**

Then, for $\Delta$ we have $[h_0^*, \ldots, h_d^*] = [1, 0, \ldots, 0]$ and

$$[f_{-1}^*, f_0^*, \ldots, f_d^*] = \begin{bmatrix} 1, \binom{d+1}{1}, \binom{d+1}{2}, \ldots, \binom{d+1}{d+1} \end{bmatrix}.$$ 

**Observation:** Notice that $[f_{-1}^*, f_0^*, \ldots, f_d^*]$ is **symmetric**. This is the only lattice polytope with symmetric $f^*$-vector!

- Does $[f_{-1}^*, f_0^*, \ldots, f_d^*]$ look familiar?

**image source:** Wikipedia
**f-vectors**

**Definition**

For a \(d\)-dimensional polytopal complex \(C\), let

\[
f(C) := \{f_{-1}, f_0, \ldots, f_d\}, \quad f_k = \#\{k\text{-dimensional faces in } C\}.
\]

For a \(d\)-dimensional polytope \(P\), let

\[
f(P) := \{f_{-1}, f_0, \ldots, f_{d-1}\}, \quad f_k = \#\{k\text{-dimensional faces in } P\}.
\]

**Connection:** If \(P\) admits a unimodular triangulation \(T\) then

\[
f^*(P) = f(T).
\]

**Example:**

\(P = [0, 1]^2, \ T = T_1 \cup T_2\)

\[
f^*(P) = (f_{-1}^*, f_0^*, f_1^*, f_2^*) = (1, 4, 5, 2)
\]

\[
f(T) = (f_{-1}, f_0, f_1, f_2) = (1, 4, 5, 2)
\]
There are several inequalities holding among the coefficients of the $h^*$-vector of a $d$-dimensional lattice polytope $P$, for example:

- $h_0^* + h_1^* + \cdots + h_{k+1}^* \geq h_d^* + h_{d-1}^* + \cdots + h_{d-k}^*$

  for $k = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor - 1$ (Hibi, 1990).

- If $h_d^* \neq 0$ then

  $$h_0^* + h_1^* + \cdots + h_k^* \leq h_d^* + h_{d-1}^* + \cdots + h_{d-k}^*$$

  for $k = 0, \ldots, d$ (Stanley, 1991),

  and $1 \leq h_1^* \leq h_k^*$ for $k = 2, \ldots, d-1$ (Hibi, 1994).

Is $h^*(P)$ unimodal for every lattice polytope $P$?

(i.e., $\exists j : h_0^* \leq \cdots \leq h_{p-1}^* \leq h_p^* \geq h_{p+1}^* \geq \cdots \geq h_d^*$) No!
Higashitani (2012) constructed an infinite family of simplices with nonunimodal $h^*$-vectors.

**Example**

The simplex of dimension 15 such that

$$\Delta_w = \text{conv}\{0, e_1, e_2, \ldots, e_{14}, w\},$$

where

$$w = (1, 1, \ldots, 1, \underbrace{131, 131, \ldots, 131, 132}_7, \underbrace{131, 132}_7),$$

has $h^*$-vector

$$h^*(\Delta_w) = (1, 0, 0, \ldots, 0, \underbrace{131, 0, 0, \ldots, 0}_7).$$

What about the unimodality of $f^*$-vectors?
Nonunimodal $f^*$-vectors

Example

The simplex of dimension 15 such that

$$\Delta_w = \text{conv}\{0, e_1, e_2, \ldots, e_{14}, w\},$$

where

$$w = (1, 1, \ldots, 1, 131, 131, \ldots, 131, 132),$$

has $f^*$-vector

$$f^*(\Delta_w) = (1, 16, 120, 560, 1820, 4368, 8008, 11440, 13001, 12488, 11676, 11704, 10990, 7896, 3788, 1064, 132).$$

In particular,

$$f_8^* \geq f_9^* \leq f_{10}^* \geq f_{11}^*. $$

So far, this is the smallest-dimensional example we have found.

▶ Hence $f^*$-vectors are not unimodal in general!

Happens in the best vector families...
Let $P$ be a $d$-dimensional polytope that is simplcial, i.e., all its faces are simplices. Bj"orner showed that $f(P)$ is not unimodal for all $P$, but ...

**Theorem (Bj"orner,1981)**

The $f$-vector of a simplicial $d$-polytope $P$ with $d \geq 3$ satisfies

$$f_{-1} < f_0 < f_1 < \cdots < f_{\lfloor d/2 \rfloor - 1} \leq f_{\lfloor d/2 \rfloor} \quad \text{and} \quad f_{\lfloor 3(d-1)/4 \rfloor - 1} > \cdots > f_{d-1}.$$ 

(Bj"orner,1994): In fact, for $p$ with $\lfloor d/2 \rfloor \leq p \leq \lfloor 3(d-1)/4 \rfloor$, there is a simplicial $d$-polytope whose $f$-vector is unimodal with a peak at $p$:

$$f_{-1} < f_0 < f_1 < \cdots < f_{p-1} < f_p > f_{p+1} > \cdots > f_{d-1}.$$
Shape of $f$-vectors

Theorem (Björner, 1986)

Moreover, $f(P) = (f_0, f_1, ..., f_k, ..., f_{\lfloor \frac{d}{2} \rfloor}, ..., f_{d-2-k}, f_{d-1-k}, ..., f_{d-1})$ satisfies

$$f_k < f_{d-2-k},$$

$$f_k \leq f_{d-1-k},$$

for $0 \leq k \leq \lfloor \frac{(d-3)}{2} \rfloor$.

Thus, this is roughly how the “shape” of the face lattice of simplicial polytopes looks like.


**Question:** Are there analogous inequalities for $f^*$-vectors?
Theorem (BDHV)

The $f^*$-vector of a $d$-dimensional lattice polytope, $d \geq 2$, satisfies

\[ f_{-1}^* < f_0^* < f_1^* < \cdots < f_{\left\lfloor \frac{d}{2} \right\rfloor - 1}^* \leq f_{\left\lfloor \frac{d}{2} \right\rfloor}^* \text{ and } f_{\left\lfloor \frac{3d-1}{4} \right\rfloor}^* > \cdots > f_d^*. \]

Moreover, for the bounds $\left\lfloor \frac{d}{2} \right\rfloor$ and $\left\lfloor \frac{3d-1}{4} \right\rfloor$ we have:

- If $P$ is the $d$-dimensional unimodular simplex $\Delta$ then
  \[ f_{-1}^* < f_0^* < \cdots < f_{\left\lfloor \frac{d}{2} \right\rfloor - 1}^* \leq f_{\left\lfloor \frac{d}{2} \right\rfloor}^* > f_{\left\lfloor \frac{d}{2} \right\rfloor + 1}^* > \cdots > f_d^*. \]

- If $P$ is the $d$-dimensional cube $[-1, 1]^d$ then
  \[ f_{-1}^* < f_0^* < \cdots < f_{\left\lfloor \frac{3d-1}{4} \right\rfloor}^* \cdots > f_d^* \]

holds (at least) for $d \leq 9$. 
Inequalities for $f^*$-vectors

Theorem (BDHV)

The $f^*$-vector of a $d$-dimensional lattice polytope, $d \geq 2$, satisfies

$$f_{-1}^* < f_0^* < f_1^* < \cdots < f_{\lfloor d/2 \rfloor - 1}^* \leq f_{\lfloor d/2 \rfloor}^* \quad \text{and} \quad f_{\lfloor 3d-1/4 \rfloor}^* > \cdots > f_d^*.$$ 

Comment: When $P$ is the unimodular simplex, the theorem holds by Björner's inequalities for $f$-vectors.

- The proof makes use of the relation between $f^*$ and $h^*$ vectors, and the inequalities given by Hibi:

$$h_d^* + h_{d-1}^* + \cdots + h_{d-k}^* \leq h_0^* + h_1^* + \cdots + h_{k+1}^*$$

for $k = 0, \ldots, \lfloor d/2 \rfloor - 1$, for a $d$-dimensional lattice polytope $P$. 

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Gorenstein polytopes

**Definition**

We say that the $d$-dimensional lattice polytope $P$ is **Gorenstein of index** $g$, $g \geq 1$, whenever the polynomial $h^*(P; z) := \sum_{k=0}^{d} h_k^* z$ has degree $d + 1 - g$ and is symmetric with respect to its degree.

**Example:** The 5-dimensional unit cube $P = [0, 1]^5$ with $h^*$-vector

$$h^*(P) = (1, 26, 66, 26, 1, 0)$$

is Gorenstein of index 2.

**Theorem (BDHV)**

*Let $P$ be a $d$-dimensional Gorenstein polytope of index $g$. Then*

$$f_k^* \gg \cdots \gg f_{\lfloor \frac{3d-1}{4} \rfloor}^* \gg \cdots \gg f_d^* \quad \text{for} \quad k = \frac{1}{2} \left( d - 1 + \left\lfloor \frac{d + 1 - g}{2} \right\rfloor \right).$$
Proposition (BDHV)

Let $P$ be a $d$-dimensional lattice polytope. Then,

$$f^*_k < f^*_{d-2-k} \quad \text{and} \quad f^*_k \leq f^*_{d-1-k}, \quad \text{for} \quad 0 \leq k \leq \frac{(d-3)}{2}.$$

Moreover, if $h^*_d \neq 0$ and $h^*(P) \neq (1, 1, \ldots, 1)$ then

$$f^*_k < f^*_{d-k}, \quad \text{for} \quad 0 < k < \frac{d}{2},$$

and $f^*_0 \leq f^*_d$.

Note: It follows that for every $d$-dimensional lattice polytope:

$$\min\{f^*_0, f^*_d\} \leq f^*_k, \quad \text{for} \quad 0 \leq k \leq d.$$

The analogous question for $f$-vectors is harder. Bárány asked it in 1997 and it was only recently answered positively:

$$\min\{f_0, f_{d-1}\} \leq f_k, \quad \text{for} \quad 0 \leq k \leq d - 1 \quad (\text{Hinman, 2022+}).$$
Examples of unimodal $f^*$-vectors

**Corollary:** It directly follows from the theorem that polytopes of dimension $2 \leq d \leq 6$ have unimodal $f^*$-vectors. In fact:

**Proposition (BDHV)**

The $f^*$-vector of a $d$-dimensional lattice polytope, where $1 \leq d \leq 10$, is unimodal.

Another family of unimodal $f^*$-vectors is the following:

**Proposition (BDHV)**

Let $P$ be a $d$-dimensional lattice polytope such that $h_k^* = 0$ for $k \geq 4$.

Then $f^*(P)$ is unimodal with a peak either at $f^*\left\lfloor \frac{d}{2} \right\rfloor$ or $f^*\left\lfloor \frac{d}{2} \right\rfloor + 1$. 
Questions

Future work:

▶ Compute $f^*$-vectors for other families of polytopes.

▶ Is $f^*(P)$ unimodal when $P$ admits a unimodular triangulation?

▶ Are there polytopes with unimodal $h^*$-vector and nonunimodal $f^*$-vector?

▶ So far we know that for a lattice $d$-dimensional polytope $P$, $f^*(P)$ is unimodal when $d \leq 10$, but unimodality fails for polytopes of order 15. Can we close this gap for $11 \leq d \leq 14$?
References

• Felix Breuer, *Ehrhart f*-coefficients of polytopal complexes are non-negative integers*, 2012.