Valuative invariants for large classes of matroids

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Matroid invariants

Some examples of matroid invariants.

- The Tutte polynomial of M.
- The Whitney numbers of the second kind of M.
- The chain polynomial of the lattice of flats $\mathcal{L}(\mathrm{M})$.
- The Kazhdan–Lusztig–Stanley polynomial arising from the lattice of flats $\mathcal{L}(\mathrm{M})$.
- The Ehrhart polynomial of the base polytope $\mathscr{P}(\mathrm{M})$.
- The Hilbert–Poincaré series of the Chow ring of M
- The spectrum of the independence complex $\mathrm{IN}(\mathrm{M})$.
- The Poincaré polynomial of the intersection cohomology module of M.

Matroid polytopes and subdivisions

Definition

When we have a polytope $\mathscr{P}$, we will say that a *subdivision* of $\mathscr{P}$ is a collection of polytopes $\mathcal{S} = \{\mathscr{P}_1, \ldots, \mathscr{P}_s\}$ satisfying

- $\mathscr{P} = \bigcup_{i=1}^{s} \mathscr{P}_i$.
- Any face of a polytope $\mathscr{P}_i \in \mathcal{S}$ belongs to $\mathcal{S}$.
- For each $i \neq j$, the intersection $\mathscr{P}_i \cap \mathscr{P}_j$ is a common (possibly empty) face of both $\mathscr{P}_i$ and $\mathscr{P}_j$.

We denote by $\mathcal{S}^{\text{int}}$ the set of all the internal faces of $\mathcal{S}$.

Base polytopes

Definition

Let M be a matroid on the set E with set of bases $\mathscr{B}$. For each $A \subseteq E$ let us define a point in $\mathbb{R}^E$ by $e_A = \sum_{i \in A} e_i$, where e_i is the i-th canonical vector.
We define the *matroid (base) polytope* of M as

$$\mathscr{P}(\mathsf{M}) \doteq \textit{convex hull } \{e_B : B \in \mathscr{B}\} \subseteq \mathbb{R}^E$$

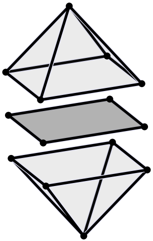
Example

If $\mathsf{M} = \mathsf{U}_{k,n}$ the polytope arising from the above construction is called the *hypersimplex* $\Delta_{k,n}$. Its vertices are all the $0/1$ points in $\mathbb{R}^n$ having exactly k ones and $n-k$ zeros.

Consider the matroid $\mathsf{U}_{2,4}$ and the matroids M_1 and M_2 on $\{1,2,3,4\}$ having sets of bases

$$\mathscr{B}(\mathsf{M}_1) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\},$$
$$\mathscr{B}(\mathsf{M}_2) = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\},$$

Take the set $\mathcal{S}$ consisting of all the faces of $\mathscr{P}(\mathsf{M}_1)$ and $\mathscr{P}(\mathsf{M}_2)$. The collection of interior faces $\mathcal{S}^{\mathsf{int}}$ consists of three polytopes.



Valuations and invariants

Throughout we consider matroids with ground sets of the form $E \subseteq \mathbb{Z}_{\geq 1}$ with $|E| < \infty$. For any such E, we denote by $\mathfrak{M}_E$ the set of all base polytopes of matroids on E, and by $\mathfrak{M}$ the union of all the $\mathfrak{M}_E$'s.

Definition

Let G some abelian group (typically, a ring). Consider a map $f : \mathfrak{M} \to \mathrm{G}$.

- If $f(\mathsf{M}) = f(\mathsf{N})$ whenever $\mathsf{M} \cong \mathsf{N}$, we say that f is an *invariant*.
- If f has the property that for every subdivision $\mathcal{S}$

$$f(\mathscr{P}) = \sum_{\mathscr{P}_i \in \mathcal{S}^{\text{int}}} (-1)^{\dim \mathscr{P} - \dim \mathscr{P}_i} f(\mathscr{P}_i).$$

  then we say that f is a *valuation*.

Large classes of matroids

We have a hierarchy of families of matroids that appear frequently in the literature.

$$\left\{\begin{matrix}\text{sparse paving}\\ \text{matroids}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{paving}\\ \text{matroids}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{elementary split}\\ \text{matroids}\end{matrix}\right\} \subseteq \{\text{split matroids}\}$$

A famous conjecture posed informally by Crapo and Rota fifty years ago, and formalized a decade ago.

Conjecture (Mayhew, Newman, Welsh and Whittle 2011)

Let $\mathrm{mat}(n)$ *be the number of matroids on* $[n]$ *and* $\mathrm{sp}(n)$ *the number of them that are sparse paving. Then:*

$$\lim_{n\to\infty} \frac{\#\,\mathrm{sp}(n)}{\#\,\mathrm{mat}(n)} = 1.$$

More than 75% of matroids on $\{1,\ldots,9\}$ are sparse paving and more than 85% are split.

Split matroids

But... What is an (elementary) split matroid?

Definition (Informal)

A connected split matroid is a matroid whose base polytope is obtained from a hypersimplex via a sequence of hyperplane splits.

Later we'll be more precise.

Goal

Provide a "good" way of computing any valuative invariant on any (elementary) split matroid.

Basic techniques for valuations

How to prove that $f : \mathfrak{M} \to \mathrm{G}$ is a valuative invariant?

- Some ad-hoc reasoning relying on specific properties of f.
- Prove that f is the linear specification of another invariant that we already know is valuative (typically, the $\mathcal{G}$-invariant, of Derksen and Fink).
- Prove that f is obtained by taking "convolutions" of other valuative invariants.

Theorem (Ardila and Sanchez)

Let G be a commutative ring and $f, g : \mathfrak{M} \to \mathrm{G}$ be two valuative invariants. Define the convolution $f \star g : \mathfrak{M} \to \mathrm{G}$ by

$$(f \star g)(\mathsf{M}) = \sum_{A \subseteq E(\mathsf{M})} f(\mathsf{M}|_A)\, g(\mathsf{M}/A).$$

If f and g are valuative then $f \star g$ is valuative too.

Many valuations

Using at least one of the above techniques yields that the following (and many more!) are valuative.

- The Ehrhart polynomial of the base polytope $\mathscr{P}(\mathrm{M})$ (ad-hoc)
- The Tutte polynomial of M (Speyer, Ardila–Rincón–Fink).
- The flag f-vector of the lattice of flats $\mathcal{L}(\mathrm{M})$ (F.–Schröter).
- The Kazhdan–Lusztig polynomial of M (Ardila–Sanchez)
- The Hilbert–Poincaré series of the Chow ring of M (F.–Schröter)
- The spectrum polynomial of the independence complex $\mathrm{IN}(\mathrm{M})$ (F.–Schröter)
- The Poincaré polynomial of the intersection cohomology module of M (Ardila–Sanchez, F.–Schröter)

How to actually compute stuff?

- Instead of subdividing your matroid M into smaller pieces, try to “add pieces to it” until you reach a hypersimplex!
- In the end you’ll have a subdivision of a hypersimplex in which one of the cells will be the matroid you started with. Hence, it’s desirable to control how “ugly” are the remaining pieces, i.e., those that you added.
- We introduce the notion of “cuspidal matroid”. These will be what we think of as a “valid piece to add”.

Theorem (F. –Schröter)

A matroid M *is elementary split if and only if adding pieces corresponding to cuspidal matroids yields a uniform matroid.*

Moreover, if M is elementary split then it already encodes all the information about the pieces we need to add!

A little bit of formalism

Definition

Let M be a matroid on E and let A be a subset.

- We say that A is a *stressed subset* if the matroids $\mathsf{M}|_A$ and M/A are both uniform matroids.
- We define the *cusp* of A to be the set

$$\operatorname{cusp}(A) = \left\{ S \in \binom{E}{\operatorname{rk}(\mathsf{M})} : |S \cap A| \geq \operatorname{rk}(A) + 1 \right\}.$$

Theorem (F.-Schröter)

Let M *be a matroid on* E *with set of bases* $\mathscr{B}$ *and* A *be a stressed subset. Then, the set* $\mathscr{B} \sqcup \operatorname{cusp}(A)$ *is the set of bases of another matroid on* E *called the relaxation* $\operatorname{Rel}(\mathsf{M}, A)$.

Cuspidal matroids

Example (The basic example)

Consider the matroid $\mathsf{M} = \mathsf{U}_{r,h} \oplus \mathsf{U}_{k-r,n-h}$ (it has rank k and size n). The ground sets of the direct summands are both stressed subsets. Let us consider the following matroid:

$$\Lambda_{r,k,h,n} = \operatorname{Rel}(\mathsf{M}, \mathsf{U}_{r,h}).$$

This is what we call a *cuspidal matroid*.

Theorem

Let $f : \mathfrak{M} \to \mathrm{G}$ *be a valuative invariant. Let* M *be a matroid of rank* k *and size* n *having a stressed subset* A *of size* h *and rank* r. *Denote* $\widetilde{\mathsf{M}} = \operatorname{Rel}(\mathsf{M}, A)$. *Then*

$$f(\widetilde{\mathsf{M}}) = f(\mathsf{M}) + f(\Lambda_{r,k,h,n}) - f(\mathsf{U}_{r,h} \oplus \mathsf{U}_{k-r,n-h}).$$

Geometry meets combinatorics

We can define/characterize purely combinatorially the class of (elementary) split matroids using the following result.

Theorem (F.–Schröter)

A matroid M *is elementary split if and only if after relaxing all of its stressed subsets with non-empty cusp one obtains a uniform matroid.*

Theorem

Let M *be a (elementary) split matroid of rank* k *and cardinality* n*, and let* f *be a valuative invariant. Then,*

$$f(\mathsf{M}) = f(\mathsf{U}_{k,n}) - \sum_{r,h} \lambda_{r,h} \left(f(\Lambda_{r,k,h,n}) - f(\mathsf{U}_{k-r,n-h} \oplus \mathsf{U}_{r,h}) \right),$$

where each $\lambda_{r,h}$ *denotes the number of stressed subsets with non-empty cusp of size* h *and rank* r*.*

A quick application

Now that we have methods to compute "fastly" any valuative invariant on any elementary split matroid, and this class is huge, then it is reasonable to search for potential counterexamples to conjectures within this class.

Conjecture (De Loera–Haws–Köppe 2007)

The Ehrhart polynomial of a matroid polytope has positive coefficients.

The above conjecture turns out to be false, and the techniques we described before help to "build" a counterexample. A result from coding theory by Graham and Sloane allows us to construct an elementary split matroid with sufficiently large λ's.

Another quick application

Elias and Williamson proved that the KL polynomials of Coxeter groups have positive coefficients. Braden–Huh–Matherne–Proudfoot-Wang proved that the same holds for matroids. However, there's another open conjecture:

Conjecture (Gedeon 2018 + Karn–Proudfoot–Nasr–Vecchi 2022)

Let M *be a matroid of rank* k *and size* n*. The Kazhdan–Lusztig polynomial of* M *is coefficient-wisely smaller than that of* $\mathsf{U}_{k,n}$*.*

By particularizing our valuation f to be the Kazhdan–Lusztig polynomials, we can prove with some effort that:

Theorem (F.–Nasr–Vecchi, F.–Schröter)

The above conjecture is true for the class of paving matroids, and for all matroids having corank 2*.*

Two further extra applications

Conjecture (Athanasiadis and Kalampogia-Evangelinou 2022)

The chain polynomial of the lattice of flats of a matroid is always a real rooted polynomial.

Particularizing our main theorem to the chain polynomial, one can prove using interlacing properties that the above conjecture holds for paving matroids.

Conjecture (Speyer 2009)

The g-polynomial of a matroid is always a polynomial with nonnegative coefficients.

Theorem (F.–Schröter)

The above conjecture is true for all sparse paving matroids.

References and recommended reading

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Thank you