Geometry & Combinatorics meet zonoids

joint work with Fulvio Gesmundo

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Geometry Meets Combinatorics in Bielefeld

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The world of convex bodies

DISCOTOPES!
→ How to recognise a zonoid?
  hard: \( \{\text{zonoids}\} \subsetneq \{\text{centrally symmetric convex bodies}\} \)
  [Schneider, Weil, ...]

→ How to recognise a semialgebraic zonoid?
  still hard [Lerario, Mathis]

→ How to recognise a discotope?

→ How to recognise a zonotope?
  easy: check if its 2-dimensional faces are centrally symmetric
  [Bolker, Schneider, ...]
Previous work

Karim A. Adiprasito and Raman Sanyal,
*Whitney numbers of arrangements via measure concentration of intrinsic volumes*,
arXiv:1606.09412

Léo Mathis and C.M.,
*Fiber Convex Bodies*,
arXiv:2105.12406, to appear in Discrete & Computational Geometry
Setting and definition

Disc $= \text{linear image of the unit ball of } \mathbb{R}^n \text{ in } \mathbb{R}^d \ (n \leq d)$.

*$\Rightarrow$* they are semialgebraic zonoids

**Definition**

Consider the discs $D_1, \ldots, D_N \subset \mathbb{R}^d$. The associated discotope is the Minkowski sum

$$D = D_1 + \ldots + D_N.$$  

Denote by $N_m$ the number of discs of dimension $m$; we say that $D$ is of type $\mathbf{N} = (N_1, \ldots, N_d)$. Notice: $N = \sum_{m=1}^{N} N_m$.

*$\Rightarrow$* they are semialgebraic zonoids

**Our goal:** characterize *generic* discotopes according to their type.
Geometry and Combinatorics meet zonoids

$\mathbf{d = 2:}$

$\mathbf{d = 3:}$
Our goal: study the exposed points \textit{algebraically}.

\( \overset{\sim}{\operatorname{Ex}}(\mathcal{D}) \) is the Zariski closure in \( \mathbb{C}^d \) of the set of exposed points.

Consider the addition map

\[
\Sigma : \partial D_1 \times \ldots \times \partial D_N \to \mathbb{R}^d \subset \mathbb{C}^d
\]

and define

\[
S = \overline{\text{im}(\Sigma)} \cap \partial \mathcal{D} \subset \mathbb{C}^d
\]

the purely nonlinear part of \( \mathcal{D} \). Then \( \operatorname{Ex}(\mathcal{D}) \subseteq S \).
Some results on $\mathcal{S}$

Let $\mathcal{D}$ be a discotope of type $\mathbb{N} = (0, N_2, \ldots, N_d)$ and consider

$$(\star) = \sum_{m=1}^{d} (m-1)N_m$$

**Theorem (Gesmundo-M. 2022)**

- if $(\star) \leq d - 1$ then $\mathcal{S}$ is an irreducible variety with
  $$\dim \mathcal{S} = (\star) \quad \text{and} \quad \deg \mathcal{S} = 2^N,$$
  - if $(\star) \geq d - 1$ then $\dim \mathcal{S} = d - 1$.

Degree and irreducibility in the case $(\star) > d - 1$?
Conjecture

$S$ is irreducible.

This would imply that actually $S = \text{Ex}(D)$.

Introduce another variety: the critical locus of the addition map $\Sigma$.

**General fact:** $\Sigma^{-1}(S) \subseteq \text{crit } \Sigma$

**Idea:** we conjecture that already the critical locus of the addition map is irreducible. This would imply that $S$ is irreducible as well.
Type $(0, N, 0, \ldots, 0)$ with $N \geq d$

Let $\mathcal{D} = D_1 + \ldots + D_N$ where $\dim D_i = 2$ for every $i$.

**Theorem (Gesmundo-M. 2022)**

The variety $\text{crit } \Sigma$ is irreducible, of dimension $d - 1$ and degree $2^N \cdot \binom{N}{d-1}$.

**Idea of the proof.**

Adapt Bertini’s Theorem to specific non-generic (but generic enough) linear cuts of a determinantal variety.

**Corollary**

*The variety $S$ is irreducible, of dimension $d - 1$ and degree $\deg S \leq 2^N \cdot \binom{N}{d-1}$.***
A case of study: the dice

Consider the dice $\mathcal{D} = D_1 + D_2 + D_3 \subset \mathbb{R}^3$, where

$$D_1 = \{(x_1, x_2, x_3): x_1 = 0; x_2^2 + x_3^2 \leq 1\},$$
$$D_2 = \{(x_1, x_2, x_3): x_2 = 0; x_1^2 + x_3^2 \leq 1\},$$
$$D_3 = \{(x_1, x_2, x_3): x_3 = 0; x_1^2 + x_2^2 \leq 1\}.$$

Its purely nonlinear part $S = \text{Ex} \mathcal{D}$ is an irreducible surface of degree $24 = 2^3 \cdot \binom{3}{2}$.

**Theorem (Gesmundo-M. 2022)**

The surface $S$ is birational to a K3 surface. Explicitly, a desingularization of $S$ is the variety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$(y_1^2 - z_1^2)(y_2^2 - z_2^2)(y_3^2 - z_3^2) - 8y_1y_2y_3z_1z_2z_3 = 0.$$
Fix a generic discotope $\mathcal{D} \subset \mathbb{R}^d$ and let $L_i = \langle D_i \rangle$, so that $L_1, \ldots, L_N$ are $N$ generic linear subspaces of $\mathbb{R}^d$. Consider a hyperplane $H$ transversal to the $L_i$.

A point $p \in \partial \mathcal{D}$ has a normal cone of dimension bigger than one if and only if

$$\dim \langle (H \cap L_1), \ldots, (H \cap L_N) \rangle < d - 1.$$ 

**Question**

When can such $H$ exist? Are there conditions on $\dim L_i$?
Thank you!

Fulvio Gesmundo and C.M.,
*The Geometry of Discotopes*,
Le Matematiche, 77(1), 143–171 (2022)