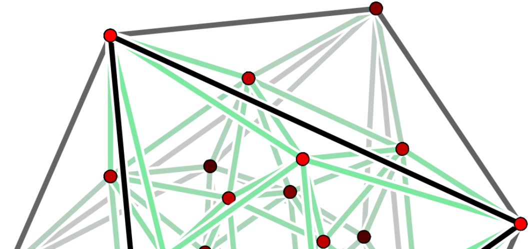
Convex Polytopes: Examples and Counterexamples, Problems and Conjectures

Günter M. Ziegler
Freie Universität Berlin

Geometry meets Combinatorics in Bielefeld, September 7, 2022



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in polytope theory and to illustrate this with some of my favourite (open and solved) polytope problems from the last 35 years.

The Value of Examples

"It is not unusual that a single example or a very few shape an entire mathematical discipline. Examples are the Petersen graph, cyclic polytopes, the Fano plane, the prisoner dilemma, the real n-dimensional projective space and the group of two by two nonsingular matrices. And it seems that overall, we are short of examples."

— Gil Kalai: *Combinatorics with a Geometric Flavor*, 2000

Call for Papers

Examples and Counterexamples

Dear G?Nter M. Ziegler,

The **volume 2** of the gold open access journal *Examples and Counterexamples* is now available on ScienceDirect. In this email, we've linked to just a few of those articles, which we hope you will enjoy reading.

- A study of Hilfer-Katugampola type pantograph equations with complex order
- The maximum cardinality of trifferent codes with lengths 5 and 6

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1. Diameters of 4-polytopes: 4D Hirsch

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References

- Dantzig, *Linear Programming and Extensions*, 1963
- Santos, *A counterexample to the Hirsch Conjecture*, 2012
- Matschke, Santos, Weibel, *The width of five-dimensional prismatoids*, 2015

2. Deformations of the 24-cell

The **Centered Realization Space** $\mathcal{R}_0(P)$ is

$$\left\{(A, V) \in \mathbb{R}^{d\times(f_0+f_{d-1})} : \mathrm{conv}(V) = \{x : Ax \leq 1\} \text{ realizes } P\right\}$$

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We know from Mnev's Universality Theorem (1986ff) that this is completely false in general.

2. Deformations of the 24-cell?

But what about the **24-cell**?

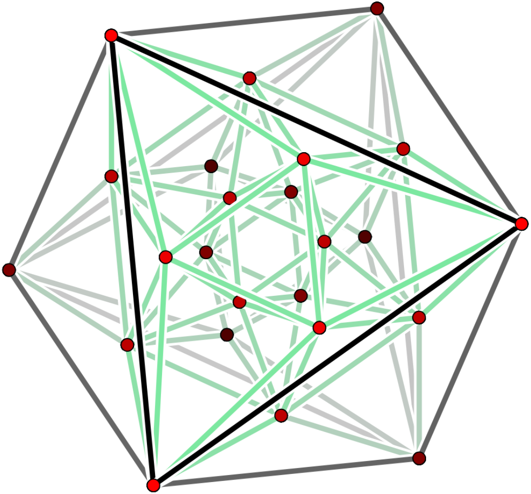

Image: javaview/M. Joswig

The 24-cell: regular, f-vector $(24, 96, 96, 24)$, 2-simple, 2-simplicial

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$$\mathrm{NG}(P_4^{24}) = 192 - 144 = 48$$

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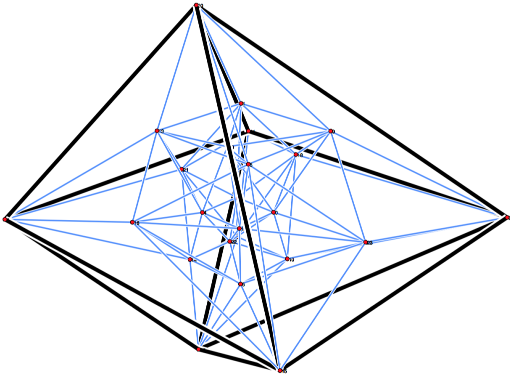

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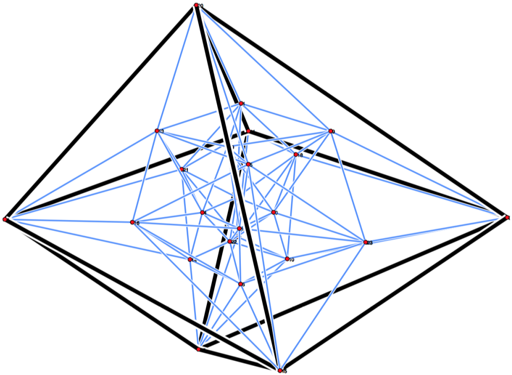

In 2005, Andreas Paffenholz described a 4-parameter deformation (in addition to the 24-dimensional family of deformations by projective transformations)

In 2021, Laith Rastanawi, Rainer Sinn & Z. showed that parts of the realization space look like a 48-dimensional manifold

2. Deformations of the 24-cell?

Open problems:

- How many deformations are there (including projective transformations)? 48?
  For every realization?
- Is the realization space pure?
  Is it a topological manifold of dimension 48?
  (It is not a *smooth* manifold!)

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References:

- Robertson, *Polytopes and Symmetry*, 1984
- Richter-Gebert, *Realization Spaces of Polytopes*, 1996
- Paffenholz, PhD thesis, FU Berlin 2005
- Rastanawi, Sinn & Z., *On the dimensions of the realization spaces of polytopes*, Mathematika 2021

3. The stellated 120-cell – irrational?

Apply the E-construction to:

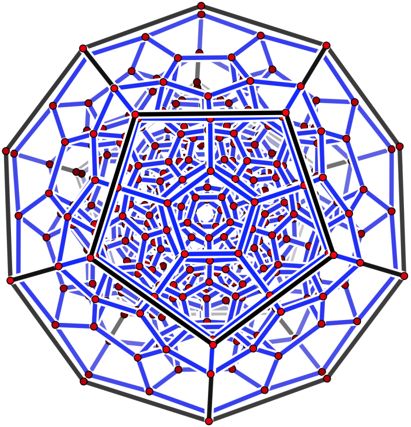

Image: javaview/M. Joswig

This yields a 2-simple 2-simplicial 4-polytope with 720 facets that are bipyramids over regular pentagons with f-vector
$f = (720, 5040, 5040, 720)$

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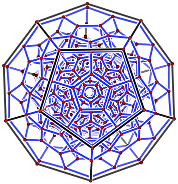
$E\Bigg($  $\Bigg)$

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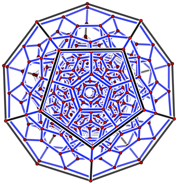
$E\Big($  $\Big)$

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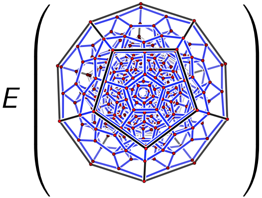

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References:

- Gévay, *Kepler hypersolids*, 1994
- Eppstein, Kuperberg & Z., *Fat 4-polytopes and fatter 3-spheres*, 2003
- Paffenholz & Z.: *The E-construction*, 2004

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Construction		$(f_0, f_1, f_2, f_3; f_{03})$	NG	facets
Δ_4	selfdual	(5,10,10,5;20)	20	5 tetrahedra
$\square * \Delta_1$	selfdual	(6,11,11,6;26)	22	4 tetrah., 2 square pyramids
$(\Delta_2 \oplus \Delta_1) * \Delta_0$		(6,14,15,7;29)	23	6 tetrah., 1 bipyramid
$(\Delta_2 \times \Delta_1) * \Delta_0$		(7,15,14,6;29)	23	2 tetrah., 3 sq. pyr., 1 prism
$\Delta_3 \oplus \Delta_1$	simplicial	(6,14,16,8;32)	24	8 tetrah.
$\Delta_3 \times \Delta_1$	simple	(8,16,14,6;32)	24	2 tetrah., 4 prisms
$\Delta_2 \oplus \Delta_2$	simplicial	(6,15,18,9;36)	24	9 tetrah.
$\Delta_2 \times \Delta_2$	simple	(9,18,15,6;36)	24	6 prisms
$(\square, v) \oplus (\square, v)$		(7,17,18,8;36)	24	4 square pyramids, 4 tetrah.
... its dual		(8,18,17,7;36)	24	2 prisms, 4 sq. pyr., 1 tetrah.
v.split$(\Delta_2 \times \Delta_1)$	selfdual	(7,17,17,7;32)	24	3 tetrah., 2 sq. pyr., 2 bipyr.

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Conjecture: *For every dimension $d \geq 4$, there are infinitely many projectively unique d-polytopes.*
In particular, Shephard's list is not complete.

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References:

- Grünbaum, *Convex Polytopes*, 1967/2003.
- McMullen, *Constructions for projectively unique polytopes*, 1976.
- Adiprasito & Z., *Many projectively unique polytopes*, 2015.

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Problem (Perles and Shephard ; Schulte 2010):

Is there a 4-polytope all whose facets are icosahedra?

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References:

- Z., Proceedings ICM 2002 Beijing.
- Z., *Projected products of polygons*, 2004.
- Kalai, *Polytope skeletons and paths*, DCG Handbook.

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References

- Ziegler, *Convex polytopes: Examples and conjectures*, DocCourse Barcelona 2010
- Pfeifle, Pilaud, Santos, *Polytopality and Cartesian products of graphs*, 2012

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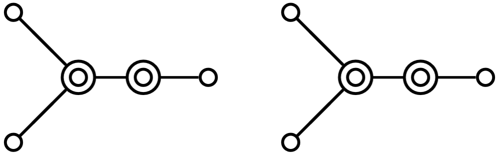
Theorem (Pfeifle): *There is a 10-dimensional Wythoff polytope without a tetrahedron or cube 3-face.*



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The Wythoff Construction:

- Take a reflection group acting on $\mathbb{R}^d$
- Choose a point on/off specified reflection hyperplanes
- Take the convex hull of its orbit.

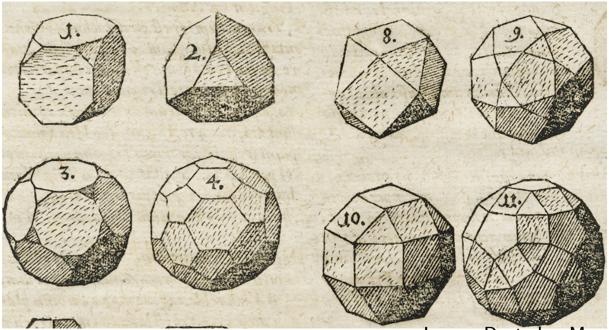

Image: Deutsches Museum München

References:

- Johannes Kepler, *Harmonices Mundi*, 1619
- Coxeter's *Regular Polytopes* for the Wythoff construction.
- Pfeifle, *Polytopes without simplices or cubes*, 2009.

9. k-simple k-simplicial polytopes for all k?

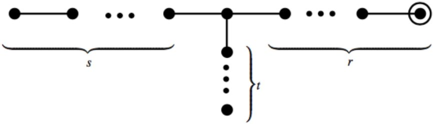

Figure 4.3: The general graph for Gosset–Elte polytopes.

- The Coxeter group is finite if

$$\frac{1}{r+1} + \frac{1}{s+1} + \frac{1}{t+1} > 1$$

- The resulting Wythoff polytope "r_{st}" has dimension $d = r + s + t + 1$
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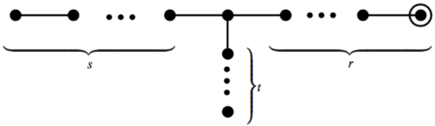

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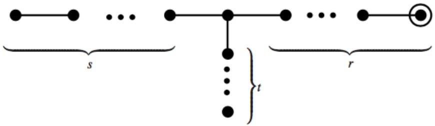

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Open problem:
Is there any 5-simple 5-simplicial polytope (other than the simplex)?

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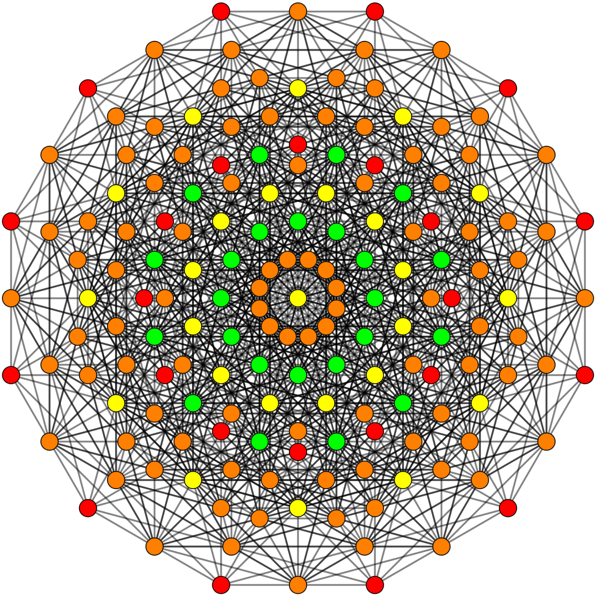

Image: Wikipedia 2_{41} polytope

10. Kalai's 3^d conjecture and its relatives

Kalai's conjectures (1989)

Open problems:

- Conjecture A, the "3^d conjecture":
  Does every c.s.-polytope have at least 3^d non-empty faces?
- Also Kalai:
  Does every c.s.-polytope have at least $2^d d!$ complete flags?
- Mahler's conjecture:
  Does every c.s. convex body have $V(B)V(B^*) \geq 4^d/d!$
- And what is the connection between these three conjectures?

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All of them are supposed to be tight at the **Hanner polytopes**:
Whatever you can construct from $I = [+1, -1]$ by taking products, direct sums or dualization — finitely many examples in each dimension.

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Hansen's (1977) construction

$$\text{Hansen}(G) \quad := \quad \text{conv}\Big(\text{Ind}(G) \times \{+1\} \,\cup\, \text{Ind}(G) \times \{-1\}\Big) \quad (1)$$

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applied to the path G_4: Hansen(G_4) is

- centrally-symmetric
- dimension $d = 5$
- f-vector $(16, 64, 98, 64, 16)$

i.e. it has $3^d + 16$ non-empty faces.

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In dimension 5:

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Is this all the counter-examples of dimension 5? (Probably not!)

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References:

- Hanner, *Intersections of translates of convex bodies*, 1956
- Hansen, *On a certain class of polytopes associated with independence systems*, 1977
- Kalai, *The number of faces of centrally-symmetric polytopes*, 1989
- Sanyal, Werner, Ziegler, *On Kalai's conjectures concerning centrally symmetric polytopes*, 2009

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