FROBENIUS

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INTRODUCTION

This paper is a blown up elaboration of [Liu02, Exercise 2.20 (a)] which states that the relative Frobenius commutes with base change. This is concluded by Corollary [2.11]. The absolute and relative Frobenius are introduced in Section 1. In Section 2 we will give a conceptional approach to the process of replacing the absolute Frobenius by the relative one. We will only need one property of the Frobenius, namely that the absolute Frobenius commutes with every morphism of $\mathbb{F}_p$-schemes. The other arguments are only abstract nonsense ones and no local intepretation in terms of algebras are used.

1. THE ABSOLUTE AND RELATIVE FROBENIUS

Throughout this paper fix a prime $p$ and let all schemes be over $\mathbb{F}_p$. If $X$ is a $S$-scheme and $s : T \to S$ a morphism of $\mathbb{F}_p$-schemes, denote by $X_T := X \times_S T$ the base change under $s$.

**Definition 1.1** (Absolute Frobenius). Denote by

$$F : X \to X$$

the absolute Frobenius morphism which locally looks like

$$R \leftarrow R$$

$$x^p \leftarrow x$$

If $S$ is a scheme over $\mathbb{F}_p$ and $\pi_X : X \to S$ a $S$-scheme, we obtain a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{F} & X \\
\pi_X \downarrow & & \pi_X \downarrow \\
S & \xrightarrow{F} & S
\end{array}
$$

If $F$ is the identity on $S$, then $F : X \to X$ is a morphism of $S$-schemes. This happens for example for $S = \text{Spec}(\mathbb{F}_p)$. In the general case we have to
change \( F \) a little bit to obtain a \( S \)-morphism which leads to the following definition.

**Definition 1.2** (Relative Frobenius). Denote by \( X^{(p)} \) the fiber product \( X \times_S S \) using \( \pi_X \) and \( F \) on \( S \) and denote by \( W : X^{(p)} \to X \) and \( \pi_X(p) : X^{(p)} \to S \) the two projections where the second map gives the \( S \)-structure of \( X^{(p)} \). Then the relative Frobenius morphism

\[
F_{X/S} : X \to X^{(p)}
\]

is the unique map with the property that

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/S}} & X^{(p)} \\
\downarrow & & \downarrow \pi_X \\
\downarrow & & \downarrow \pi_X \\
S & \xrightarrow{F} & S
\end{array}
\]

commutes.

**Remark 1.3.** Note that \( F_{X/S} \) is indeed a morphism of \( S \)-schemes since it is induced by the universal property of the fiber product and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F} & X \\
\downarrow & & \downarrow \pi_X \\
S & \xrightarrow{F} & S
\end{array}
\]

If \( F \) is an isomorphism on \( S \), then \( X^{(p)} \cong X \) and the relative Frobenius coincides with the absolute one up to ismorphism (cf. Proposition 2.9 and Example 2.6).

2. **Replacements/A Reinterpretation of \( X^{(p)} \)**

**Definition 2.1.** Let \( S \) be a \( \mathbb{F}_p \)-scheme. Denote by \( \mathcal{Frob}(S) \) the Frobenius category over \( S \). The objects are morphism \( f : X \to Y \) between \( S \)-schemes which are not necessarily morphisms over \( S \) but make the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \pi_Y \\
S & \xrightarrow{F} & S
\end{array}
\]

commutative. Morphisms between objects \( f : X \to Y \) and \( g : A \to B \) are pairs \((\phi : X \to A, \psi : Y \to B)\) of morphisms over \( S \) which make the following diagram commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\phi & & \psi \\
A & \xrightarrow{g} & B
\end{array}
\]

In fact there is a 3-dimensional diagram which expresses a whole morphisms but I am too lazy to pin it down :-). The composition of morphisms is just given by componentwise application of the ordinary composition of morphisms over \( S \).
**Remark 2.2.** Note that in the case that \( F : S \rightarrow S \) is the identity, the objects of \( \mathcal{F}rob(S) \) are precisely the morphisms of \( S \)-schemes over \( S \). Hence \( \mathcal{F}rob(S) = \text{Ar}(\text{Sch}/S) \), the category of arrows in \( \text{Sch}/S \). If \( F : S \rightarrow S \) is an isomorphism (e.g. for \( S = \text{Spec}(k) \) for \( k \) a perfect field), then we will see in Proposition 2.9 that these two categories are equivalent.

**Example 2.3.** Canonical objects of \( \mathcal{F}rob(S) \) are given by the absolute Frobenius morphisms \( F : X \rightarrow X \) for \( S \)-schemes \( X \).

A morphism \( s : T \rightarrow S \) of \( \mathbb{F}_p \)-schemes induces a base change functor
\[
s^* : \mathcal{F}rob(S) \rightarrow \mathcal{F}rob(T)
\]
which is the following on objects: For an object \( f : X \rightarrow Y \) of \( \mathcal{F}rob(S) \) define the object \( s^*(f) \) of \( \mathcal{F}rob(T) \) by the unique morphism \( s^* : X_T \rightarrow Y_T \) which is induced by the commutative diagram
\[
\begin{array}{ccc}
X_T & \xrightarrow{f} & Y_T \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
T & \xrightarrow{s} & S
\end{array}
\]
The commutativity follows from the commutativity of \( f \) with the Frobenius on \( S \) and the fact that the diagram
\[
\begin{array}{ccc}
T & \xrightarrow{F} & T \\
\downarrow{s} & & \downarrow{s} \\
S & \xrightarrow{F} & S
\end{array}
\]
commutes. By definition of \( s^*(f) \) it is indeed an object of \( \mathcal{F}rob(T) \). The base change functor works on morphisms as follows: Given a morphism
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi} & & \downarrow{\psi} \\
A & \xrightarrow{\sigma} & B
\end{array}
\]
in \( \mathcal{F}rob(S) \) we get the induced morphism
\[
\begin{array}{ccc}
X_T & \xrightarrow{s^*(f)} & Y_T \\
\downarrow{\phi \times \text{id}_T} & & \downarrow{\psi \times \text{id}_T} \\
A_T & \xrightarrow{s^*(\sigma)} & B_T
\end{array}
\]
just by bringing in the given morphism in the inducing diagram of \( s^*(f) \) and \( s^*(\sigma) \) (3-dimensional again :-)).

**Example 2.4.** Given \( s : T \rightarrow S \), a \( S \)-scheme \( X \) and the absolute Frobenius \( F_X : X \rightarrow X \) we get
\[
s^*(F_X) = F_{X_T}
\]
that is, the base change of the absolute Frobenius is again the absolute Frobenius.

**Proof.** This is true since all three “squares” in the following diagram commute

\[
\begin{array}{ccc}
X_T & \xrightarrow{X} & Y \\
\downarrow \pi_X & & \downarrow \pi_Y \\
T & \xrightarrow{F} & T \\
\end{array}
\]

Our next aim is to obtain a replacement functor

\[ R_S : \mathcal{Frob}(S) \to \text{Ar}(\text{Sch}/S) \]

which replaces a morphism over the Frobenius on \( S \) by a morphism over \( S \).

For this note that we have the following lemma.

**Lemma 2.5.** The process

\[
\begin{array}{ccc}
\text{Sch}/S & \xrightarrow{(-)^{(p)}} & \text{Sch}/S \\
X & \mapsto & X^{(p)} \\
\end{array}
\]

is a functor. Furthermore for a morphism \( s : T \to S \) of \( \mathbb{F}_p \)-schemes the diagram

\[
\begin{array}{ccc}
\text{Sch}/S & \xrightarrow{(-)^{(p)}} & \text{Sch}/S \\
\downarrow s^* & & \downarrow s^* \\
\text{Sch}/T & \xrightarrow{(-)^{(p)}} & \text{Sch}/T \\
\end{array}
\]

commutes up to functor isomorphism where \( s^* \) denotes the usual base change functor.

**Proof.** The universal property of the base change via \( F : S \to S \) provides that \((-)^{(p)}\) is a functor. The commutativity with the base change \( s^* \) follows from the fact that \((X^{(p)})_T\) is the pullback of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{X} & S \\
\downarrow \pi_X & & \downarrow F \\
T & \xrightarrow{s} & S \\
\end{array}
\]

and \((X_T)^{(p)}\) is the pullback of the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{X} & S \\
\downarrow \pi_X & & \downarrow F \\
T & \xrightarrow{F} & T \\
\end{array}
\]

\[
\begin{array}{ccc}
T & \xrightarrow{s} & S \\
\end{array}
\]
But these two diagrams coincide since the diagram
\[
\begin{array}{c}
T \\ s \\
\downarrow \\ S \\
\end{array}
\xrightarrow{F}
\begin{array}{c}
T \\ s \\
\downarrow \\ S \\
\end{array}
\]
commutes. □

Now define the functor $R_S$ as follows: Given an object $f$ of $\mathcal{F}rob(S)$, that is, a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
S & \xrightarrow{\pi_Y} & S
\end{array}
\]
We obtain a unique factorization
\[
\begin{array}{ccc}
X & \xrightarrow{f/S} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
S & \xrightarrow{\pi_Y} & S
\end{array}
\]
by the pullback property of $Y^{(p)}$. Define
\[
R_S(f) := f/S
\]
on objects. For a morphism
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi} & & \downarrow{\psi} \\
A & \xrightarrow{g} & B
\end{array}
\]
of $\mathcal{F}rob(S)$, the induced diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f/S} & Y^{(p)} \\
\downarrow{\phi} & & \downarrow{\psi^{(p)}} \\
A & \xrightarrow{g/S} & B^{(p)}
\end{array}
\]
commutes and gives a morphism in $\text{Ar}(\text{Sch}/S)$, since all involved morphisms are morphisms over $S$. This defines $R_S$ on morphisms of $\mathcal{F}rob(S)$. The functoriality follows from Lemma 2.5.

**Example 2.6.** By the definition of the relative Frobenius, for a $S$-scheme $X$ and the absolute Frobenius $F : X \to X$ which is an object of $\mathcal{F}rob(S)$ we have
\[
R_S(F) = F_{X/S}
\]
that is, the replacement of the absolute Frobenius is the relative one.
Now we want to define an “inclusion” functor

\[ I_S : \text{Ar}(\text{Sch}/S) \to \mathcal{F}\text{rob}(S) \]

as follows: For an \( S \)-scheme \( X \xrightarrow{\pi} S \) denote by \( X' \) the \( S \)-scheme \( X \xrightarrow{\pi} S \xrightarrow{F} S \). Then we can define for \( f : X \to Y \) a morphism in \( \text{Sch}/S \) the object \( I_S(f) := (f : X \to Y') \) in \( \mathcal{F}\text{rob}(S) \). The action on morphisms is the obvious one since a morphism \( \phi : X \to A \) over \( S \) gives also a morphism \( \phi : X' \to A' \) over \( S \).

**Remark 2.7.** Clearly the functor \( I_S \) is faithful. But a morphism \( \phi : X' \to Y' \) over \( S \) does not give a morphism \( \phi : X \to A \) over \( S \) unless \( F : S \to S \) is a monomorphism. Then \( I_S \) is fully faithful.

**Lemma 2.8.** The functor pair

\[ I_S : \text{Ar}(\text{Sch}/S) \dashv \mathcal{F}\text{rob}(S) : R_S \]

is an adjunction.

*Proof.* Let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\phi \downarrow & & \psi \downarrow \\
A & \xrightarrow{g/S} & B^{(p)}
\end{array}
\]

be a morphism \( f \to R_S(g) \) in \( \text{Ar}(\text{Sch}/S) \). This gives

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y' \\
\phi \downarrow & & W\psi \downarrow \\
A & \xrightarrow{g} & B
\end{array}
\]

which is a morphism \( I_S(f) \to g \) in \( \mathcal{F}\text{rob}(S) \). On the other hand let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y' \\
\sigma \downarrow & & \delta \downarrow \\
A & \xrightarrow{g} & B
\end{array}
\]

be a morphism \( I_S(f) \to g \) in \( \mathcal{F}\text{rob}(S) \). Then

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\sigma \downarrow & & \delta/S \downarrow \\
A & \xrightarrow{g/S} & B^{(p)}
\end{array}
\]

gives a morphism \( f \to R_S(g) \) in \( \text{Ar}(\text{Sch}/S) \). These establishes a natural bijection

\[
\text{Hom}_{\text{Ar}(\text{Sch}/S)}(f, R_S(g)) \cong \text{Hom}_{\mathcal{F}\text{rob}(S)}(I_S(f), g)
\]

\[ \square \]

**Proposition 2.9.** If \( F : S \to S \) is an isomorphism then \( (I_S, R_S) \) is an equivalence of categories.
Proof. First of all, $W : X^{(p)} \to X$ is an isomorphism for all $S$-schemes $X$. Thus $X \to S$ is isomorphic to $(X^{(p)})'$ in $\text{Sch}/S$ for all $X \in \text{Sch}/S$ which implies that

$$I_S(R_S(f)) = I_S(f/S) = (Y \xrightarrow{f/S} (X^{(p)})') \cong (Y \xrightarrow{W \circ f/S} X) = f$$

Since $F$ is a monomorphism as an isomorphism it follows that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{id} & Y \\
\pi_Y & \downarrow & \pi_Y' \\
S & \xrightarrow{F} & S \\
\sigma & \downarrow & \sigma \\
T & \xrightarrow{R_T} & S
\end{array}
\]

has the universal property of a pullback diagram, that is, $(Y')^{(p)} \cong Y \xrightarrow{\pi_Y} S$. Hence $R_S(I_S(f)) \cong f$. $\square$

**Proposition 2.10.** For a morphism $s : T \to S$ of $\mathbb{F}_p$-schemes, the diagram

\[
\begin{array}{ccc}
F_{\text{rob}}(S) & \xrightarrow{R_S} & \text{Ar}(\text{Sch}/S) \\
\downarrow s^* & & \downarrow s^* \\
F_{\text{rob}}(T) & \xrightarrow{R_T} & \text{Ar}(\text{Sch}/T)
\end{array}
\]

commutes up to functor isomorphism.

**Proof.** Recall that we have a natural isomorphism

$$(X^{(p)})_T \xrightarrow{\phi_X} (X_T)^{(p)}$$

by Lemma 2.5. Now we have to check that this gives

$$\phi_Y \circ (s^*(R_S(f))) = \phi_Y \circ (f/S \times_S \text{id}_T) = s^*(f)/_T$$

for an object $f : X \to Y$ in $F_{\text{rob}}(S)$. By the universal property of the pullback diagram

\[
\begin{array}{ccc}
(Y_T)^{(p)} & \xrightarrow{\sigma} & Y \\
\pi & \downarrow & \pi_X \\
T & \xrightarrow{F} & T \xrightarrow{s} S
\end{array}
\]

it suffices to show that the equality above holds after the composition with $\pi \circ$ and $\sigma \circ$. The first one is clear since the application of $\pi \circ$ gives $\pi_X : X_T \to T$ on both sides. For the application of $\sigma \circ$ note that

$$\sigma \circ (s^*(f)/_T) = (X_T \to X \xrightarrow{f} Y)$$
and that the diagram

\[
\begin{array}{ccc}
X_T & \xrightarrow{f/S \times \text{id}_T} & (Y^{(p)})_T \\
\downarrow & & \downarrow
d & f/S & \rightarrow Y^{(p)} \\
\downarrow & & \downarrow
d & \rightarrow & Y \\
X & \xrightarrow{f/S} & Y^{(p)} \\
\end{array}
\]

commutes which gives the desired equality. □

**Corollary 2.11.** Together with Example 2.4 and 2.6 we obtain

\[
F_{X/S} \times \text{id}_T \cong F_{X/T}
\]

in $\text{Ar}(\text{Sch}/T)$.

**References**