ON SPECIAL ZEROS OF p-ADIC L-FUNCTIONS OF HILBERT MODULAR FORMS

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ABSTRACT. Let E be a modular elliptic curve over a totally real number field F. We prove the weak exceptional zero conjecture which links a (higher) derivative of the p-adic L-function attached to E to certain p-adic periods attached to the corresponding Hilbert modular form at the places above p where E has split multiplicative reduction. Under some mild restrictions on p and the conductor of E we deduce the exceptional zero conjecture in the strong form (i.e. where the automorphic p-adic periods are replaced by the $\mathcal{L}$-invariants of E defined in terms of Tate periods) from a special case proved earlier by Mok. Crucial for our method is a new construction of the p-adic L-function of E in terms of local data.

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Introduction

Let E be a modular elliptic curve over a totally real number field F and let p be a prime number and such that E has either good ordinary or multiplicative reduction at all places $\mathfrak{p}$ above p. Attached to E are the (Hasse-Weil) L-function $L(E,s)$ (a function in the complex variable s) and a p-adic L-function $L_p(E,s)$ (here $s \in \mathbb{Z}_p$). Both are linked by the *interpolation property* which expresses the p-adic measure associated to $L_p(E,s)$ in terms of twisted special L-values $L(E,\chi,1)$. A special case is the formula

$$L_p(E,0) \;=\; \prod_{\mathfrak{p}|p} e(\alpha_{\mathfrak{p}},1)\,\cdot L(E,1).$$

Here $e(\alpha_{\mathfrak{p}},1)$ is certain Euler factor defined in terms of the reduction of E at $\mathfrak{p}$ (see Prop. 4.10 for its definition). It is $=0$ if and only if E has split multiplicative reduction at $\mathfrak{p}$. Let S_1 be the set of primes $\mathfrak{p}$ of F above p where E has split multiplicative reduction, let S_p be the set of all primes above p and let $S_2 = S_p - S_1$. Thus we have $L_p(E,0)=0$ if $S_1 \neq \emptyset$. In [17] it has been conjectured that

$$\operatorname{ord}_{s=0} L_p(E,s) \geq r :=\; \sharp(S_1); \tag{1}$$

$$\frac{d^r}{ds^r} L_p(E,s)|_{s=0} \;=\; r! \prod_{\mathfrak{p}\in S_1} \mathcal{L}_{\mathfrak{p}}(E) \cdot \prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}},1)\,\cdot\, L(E,1). \tag{2}$$

Here the *$\mathcal{L}$-invariant* $\mathcal{L}_{\mathfrak{p}}(E)$ is defined as $\mathcal{L}_{\mathfrak{p}}(E) = \ell_{\mathfrak{p}}(q_{E/F_{\mathfrak{p}}}))/\,\mathrm{o}_{\mathfrak{p}}(q_{E/F_{\mathfrak{p}}})$ where $q_{E/F_{\mathfrak{p}}}$ is the Tate period of $E/F_{\mathfrak{p}}$, $\ell_{\mathfrak{p}} = \log_p \circ N_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ and $\mathrm{o}_{\mathfrak{p}} = \mathrm{ord}_{\mathfrak{p}}$ is the normalized additive valuation of $F_{\mathfrak{p}}$.

In this paper we prove (1) unconditionally and (2) under the following assumptions (see Theorem 5.10): (i) $p \geq 5$ is unramified in F; (ii) E has multiplicative reduction at a prime $\mathfrak{q} \nmid p$, or r is odd, or the sign $w(E)$ of the functional equation for $L(E,s)$ (i.e. the root number of E) is $= -1$.

The statements (1) and (2) are known as *exceptional zero conjecture*. In the case $F = \mathbb{Q}$ it was formulated by Mazur, Tate and Teitelbaum [20] and proved by Greenberg and Stevens [15] and independently by Kato, Kurihara and Tsuji. In the case $r = 1$, (2) was proved by Mok [21] under the assumption (i), by extending the method of [15].

To explain our proof let π be the automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ associated to E. Thus π has trivial central character and the local factor π_v is discrete series of weight 2 at all archimedean places v. The Hasse-Weil L-function of E is then equal to the automorphic L-function $L(s-\frac{1}{2},\pi)$. Moreover $L_p(E,s)$ is solely defined in terms of π (thus we write $L_p(s,\pi)$ for $L_p(E,s)$).

In section 5.1 we shall introduce a second type of $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}(\pi)$. It is defined in terms of the cohomology of (S_p-)arithmetic groups. We show that $\mathcal{L}_{\mathfrak{p}}(\pi)$ does not change under certain quadratic twists, i.e. we have $\mathcal{L}_{\mathfrak{p}}(\pi\otimes\chi) = \mathcal{L}_{\mathfrak{p}}(\pi)$ for any quadratic character χ of the idele class group $\mathbf{I}/F^*$ of F such that the local components χ_v of χ at infinite places and at $v = \mathfrak{p}$ are trivial. We prove an analogue of (2) above (unconditionally) with the arithmetic $\mathcal{L}$-invariants $\mathcal{L}_{\mathfrak{p}}(E)$ replaced by the automorphic $\mathcal{L}$-invariants $\mathcal{L}_{\mathfrak{p}}(\pi)$, i.e. we show

$$\frac{d^r}{ds^r}L_p(s,\pi)|_{s=0} \;=\; r!\,\prod_{\mathfrak{p}\in S_1}\mathcal{L}_{\mathfrak{p}}(\pi)\,\cdot\,\prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}},1)\,\cdot\,L(\tfrac{1}{2},\pi). \tag{3}$$

In the case $F = \mathbb{Q}$ these $\mathcal{L}$-invariants have been introduced by Darmon ([9], section 3.2). He showed that they are invariant under twists and also proved (3). Also if the narrow class number of F is $= 1$ a different construction of $\mathcal{L}_{\mathfrak{p}}(\pi)$ has been given in [13].

To deduce (2) from (3) it is therefore enough to show $\mathcal{L}_{\mathfrak{p}}(\pi) = \mathcal{L}_{\mathfrak{p}}(E)$ for all $\mathfrak{p}\in S_1$. In future work [14] we plan to give an unconditional proof of it (and thus of (2)) by comparing $\mathcal{L}_{\mathfrak{p}}(\pi)$ to the (similarly defined) $\mathcal{L}$-invariant of an automorphic representation π' of a totally definite quaternion algebra – which corresponds to π under Jacquet-Langlands functoriality – and by using p-adic uniformization of Shimura curves (compare also [2] where a similar proof has been given in the case $F = \mathbb{Q}$ under certain assumptions on π).

However if p satisfies the conditions (i) above and E satisfies (ii) then we can deduce the equality $\mathcal{L}_{\mathfrak{p}}(\pi) = \mathcal{L}_{\mathfrak{p}}(E)$ for fixed $\mathfrak{p}\in S_1$ by comparing the formulas (2) and (3) in the case $r = 1$ for certain quadratic twists of E and π. More precisely, by a result of Waldspurger [29], we can choose a quadratic character χ such that the arithmetic and automorphic $\mathcal{L}$-invariants at $\mathfrak{p}$ do not change under twisting with χ, $L(\frac{1}{2},\pi\otimes\chi)$ does not vanish and $\mathfrak{p}$ is the only place above p where the twisted elliptic curve E_χ has split multiplicative reduction. Then by Mok's result and (3) we can express both $\mathcal{L}_{\mathfrak{p}}(E)$ and $\mathcal{L}_{\mathfrak{p}}(\pi)$ by the same formula.

The p-adic L-function attached to π is the Γ-transform of a certain canonical measure μ_π on the Galois group $\mathcal{G}_p$ of the maximal abelian extension of F which is unramified outside p and ∞, i.e. it is given by

$$L_p(s,\pi) \;=\; \int_{\mathcal{G}_p} \langle\gamma\rangle^s \mu_\pi(d\gamma)$$

(for the definition of $\langle\gamma\rangle^s$ see section 3.3).

Crucial for the proof of (1) and (3) is a new construction of μ_π[1]. We shall briefly explain it (for details see 4.6). Heuristically, we define μ_π as the direct image of the distribution $\mu_{\pi_p}\times W^p\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix} d^\times x$ under the reciprocity map $\mathbf{I} = F_p^*\times \mathbf{I}^p \to \mathcal{G}_p$ of class field theory. Here the first factor μ_{π_p} is the product distribution on $F_p^* = \prod_{\mathfrak{p}\in S_p} F_{\mathfrak{p}}^*$ of certain canonical distributions $\mu_{\mathfrak{p}}$ on $F_{\mathfrak{p}}^*$ attached to each local factors $\pi_{\mathfrak{p}}$, $\mathfrak{p}\in S_p$. Moreover $d^\times x$ denotes the Haar measure on the group of S_p-ideles $\mathbf{I}^p = \prod'_{v\nmid p} F_v^*$ (i.e. the group of ideles with components outside of S_p) and W^p is a certain Whittaker function of $\pi^p = \bigotimes'_{v\nmid p}\pi_v$ (it is the product of local Whittaker functions).

To put this construction on a sound foundation consider the map ϕ_π given by

$$\phi_\pi(U,x^p) \;=\; \sum_{\zeta\in F^*} \mu_{\pi_p}(\zeta U)\; W^p\begin{pmatrix} \zeta x^p & 0\\ 0 & 1\end{pmatrix}$$

where the first argument U is a compact open subset of F_p^* and the second an idele $x^p\in\mathbf{I}^p$. Then $\phi_\pi(\zeta U,\zeta x^p) = \phi_\pi(U,x)$ for all $\zeta\in F^*$. Thus if we set $\phi_U(x_p,x^p)\colon = \phi_\pi(x_pU,x^p)$ then ϕ_U can be viewed as a function on the idele class group $\mathbf{I}/F^*$ (so the map $U\mapsto\phi_U$ is a distribution on F_p^* with values in a certain space of functions on $\mathbf{I}/F^*$).

For a locally constant map $f:\mathcal{G}_p\to\mathbb{C}$ there exists a compact open subgroup $U\subset U_p = \prod_{\mathfrak{p}\in S_p}\mathcal{O}_{\mathfrak{p}}^*\subset F_p^*$ such that $f\circ\rho:\mathbf{I}/F^*\to\mathbb{C}$ factors through $\mathbf{I}/F^*(U\times U^p)$ (here $\rho:\mathbf{I}/F^*\to\mathcal{G}_p$ denotes the reciprocity map). Then $\int_{\mathcal{G}_p} f(\gamma)\,\mu_\pi(d\gamma)$ is given by

$$\int_{\mathcal{G}_p} f(\gamma)\,\mu_\pi(d\gamma) = [U_p:U]\int_{\mathbf{I}/F^*} f(\rho(x))\phi_U(x)\,d^\times x.$$

By using properties of the cohomology groups of arithmetic subgroups of $\mathrm{GL}_2(F)$ we show that μ_π is *bounded* (i.e. it is a *p-adic measure* in the sense of section 1.2 below) and so any continuous map $\mathcal{G}_p\to\mathbb{C}_p$ can be integrated against it.

One way to describe the local distribution $\mu_{\mathfrak{p}}$ for $\mathfrak{p}\in S_p$ is that it is the image of a certain Whittaker functional of $\pi_{\mathfrak{p}}$ under a canonical map – denoted by δ – from the dual of $\pi_{\mathfrak{p}}$ to the space of distributions on $F_{\mathfrak{p}}^*$. We will give the definition of δ in the case $\mathfrak{p}\in S_1$, or equivalently, when $\pi_{\mathfrak{p}}$ is

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[1]In principle our construction is related to Manin's [19]. However in our set-up the measure μ_π is build in a simple manner from local distributions μ_{π_v} at each place v of F

the Steinberg representation St (i.e. $\pi_{\mathfrak{p}}$ is isomorphic to the space of locally constant functions $\mathbb{P}^1(F_{\mathfrak{p}}) \to \mathbb{C}$ modulo constants). For $c \in \mathrm{Hom}(\mathrm{St}, \mathbb{C})$ we define $\delta(c)$ by $\int_{F_{\mathfrak{p}}} f(x)\delta(c)(dx) = c(\tilde{f})$. Here for a locally constant map with compact support $f : F_{\mathfrak{p}} \to \mathbb{C}$ we define $\tilde{f} : \mathbb{P}^1(F_{\mathfrak{p}}) \to \mathbb{C}$ by $\tilde{f}(\infty) = 0$ and $\tilde{f}([x:1]) = f(x)$. Thus in the case $\pi_{\mathfrak{p}} = \mathrm{St}$, the target of δ is the space of distributions on $F_{\mathfrak{p}}$.

In particular the local contribution $\mu_{\mathfrak{p}}$ of μ_π at $\mathfrak{p} \in S_1$ is actually a distribution on $F_{\mathfrak{p}}$ (and not only on $F_{\mathfrak{p}}^*$). Therefore, allowed as first argument in $\phi_\pi(U, x^p)$ are not only compact open subsets U of F_p^* but also of the larger space $\prod_{\mathfrak{p}\in S_1} F_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in S_2} F_{\mathfrak{p}}^*$. This fact is crucial for our proof that the vanishing order $L_p(s,\pi)$ at $s = 0$ is $\geq r$. The map δ and distributions $\mu_{\mathfrak{p}}$ will be introduced in sections 2.5 and 2.6 respectively.

Chapter 3 is the technical heart of this paper. It provides an axiomatic approach to study trivial zeros of p-adic L-function which can be applied in other situations as well (e.g. to the case of p-adic L-functions of totally real number fields [26], [11]). We consider an arbitrary two-variable function $\phi : (U, x^p) \mapsto \phi(U, x^p)$ ($U \subset \prod_{\mathfrak{p}\in S_1} F_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in S_2} F_{\mathfrak{p}}^*$ compact open and $x^p \in \mathbf{I}^p$) satisfying certain axioms and attach a p-adic distribution μ on $\mathcal{G}_p$ as above. By "integrating away" the infinite places we obtain a certain cohomology class $\kappa \in H^d(F_+^*, \mathscr{D})$ associated to ϕ (where $d = [F:\mathbb{Q}] - 1$, F_+^* denotes the group of totally positive elements of F and $\mathscr{D}$ is a certain space of distributions on the adelic space $\prod_{\mathfrak{p}\in S_1} F_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in S_2} F_{\mathfrak{p}}^* \times \prod'_{v\nmid p\infty} F_v^*$) and the distribution μ can be defined solely in terms of κ. The space $\mathscr{D}$ contains a canonical subspace $\mathscr{D}^b$ (consisting – in a certain sense – of p-adic measures) and μ is a p-adic measure provided that κ lies in the image of $H^d(F_+^*, \mathscr{D}^b) \to H^d(F_+^*, \mathscr{D})$ (see section 3.4).

In this case we define $L_p(s,\phi)$ as the Γ-transform of μ and show that $L_p(s,\phi)$ has a zero of order $\geq r$ at $s = 0$. Furthermore we give a description of the r-th derivative $\frac{d^r}{ds^r} L_p(s,\phi)|_{s=0}$ as a certain cap-product. More precisely, we associate to any continuous homomorphism $\ell : F_{\mathfrak{p}}^* \to \mathbb{C}_p$ a cohomology class $c_\ell \in H^1(F_+^*, C_c(F_{\mathfrak{p}}, \mathbb{C}_p))$ (for its definition and the notation see 3.4). If $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ we will show

$$\frac{d^r}{ds^r} L_p(s,\phi)|_{s=0} \;=\; (-1)^{\binom{r}{2}}\, r!\, (\kappa \cup c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta. \tag{4}$$

Here ϑ is essentially the fundamental class of the quotient M/F_+^* where M is a certain $d + r$-dimensional manifold on which F_+^* acts freely (see section 3.2). If $U_0 = \prod_{\mathfrak{p}\in S_1} \mathcal{O}_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in S_2} \mathcal{O}_{\mathfrak{p}}^*$ and $\phi_0(x) := \phi(x_p U_0, x^p)$ for $x = (x_p, x^p) \in F_p^* \times \mathbf{I}^p = \mathbf{I}$, we will also prove

$$\int_{\mathbf{I}/F^*} \phi_0(x)\, d^\times x \;=\; (-1)^{\binom{r}{2}}\, r!\, (\kappa \cup c_{\mathrm{o}_{\mathfrak{p}_1}} \cup \ldots \cup c_{\mathrm{o}_{\mathfrak{p}_r}}) \cap \vartheta. \tag{5}$$

In chapter 4 we will verify that the theory developed in the previous chapter can be applied in the case $\phi = \phi_\pi$. The difficult part is to show that the cohomology class κ_π attached to ϕ_π comes from a class in $H^d(F_+^*, \mathscr{D}^b)$. This is achieved by showing that it lies in the image of a specific cohomology class

$\widehat{\kappa}_\pi \in H^d(\mathrm{PGL}_2(F), \mathscr{A})$ under a canonical map $\Delta_* : H^d(\mathrm{PGL}_2(F), \mathscr{A}) \to H^d(F_+^*, \mathscr{D})$ (for the definition of the coefficients $\mathscr{A}$ and the map Δ_* we refer to section 4.4 and 4.5). The fact that any arithmetic subgroup of $\mathrm{PGL}_2(F)$ has the finiteness property (VFL) (introduced by Serre in [24]) implies that Δ_* factors through $H^d(F_+^*, \mathscr{D}^b)$.

In the last chapter 5 we will introduce the automorphic $\mathcal{L}$-invariant $\mathcal{L}_\mathfrak{p}(\pi)$ and deduce (3) from (4) and (5). The cohomology group $H^d(\mathrm{PGL}_2(F), \mathscr{A})$ carries an action of a Hecke algebra and $\widehat{\kappa}_\pi$ lies in the π-isotypic component $H^d(\mathrm{PGL}_2(F), \mathscr{A})_\pi$. Using the fact that the classes c_ℓ "come" from certain PGL_2 cohomology classes as well (they will be introduced in section 2.7) and the fact that $H^d(\mathrm{PGL}_2(F), \mathscr{A})_\pi$ is one-dimensional (a results due to Harder [16]) we show that the cup products $\kappa \cup c_{\ell_\mathfrak{p}}$ and $\kappa \cup c_\mathfrak{p}$ differ by a factor $\mathcal{L}_\mathfrak{p}(\pi)$ which is defined in terms of the cohomology of $\mathrm{PGL}_2(F)$.

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Notation. The following notations are valid throughout this paper. A list with further notations will be given at the beginning of chapters 2 and 3.

Unless otherwise stated all rings are commutative with unit.

We fix a prime number p and embeddings

$$\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \qquad \iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p.$$

We let ord_p denote the valuation on $\mathbb{C}_p$ and $\overline{\mathbb{Q}}$ (via ι_p) normalized so that $\mathrm{ord}_p(p) = 1$. The valuation ring of $\overline{\mathbb{Q}}$ with respect to ord_p will be denoted by $\overline{\mathcal{O}}$.

If X and Y are topological spaces then $C(X, Y)$ denotes the set of continuous maps $X \to Y$. If we consider Y with the discrete topology then we shall also write $C^0(X, Y)$ instead of $C(X, Y)$. If $Y = R$ is a topological ring then $C_c(X, R)$ is the submodule of $C(X, R)$ of continuous maps with compact support. If we consider R with the discrete topology then we shall also write $C_c^0(X, R)$ instead of $C_c(X, R)$.

Put $G := \mathrm{PGL}_2$, and let B be the subgroup of upper triangular matrices (modulo the center Z of GL_2), $T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} / Z$ be the maximal torus of G in B. We write elements of G often simply as matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (and neglect the fact that we consider them only modulo the center of GL_2). We identift $\mathbb{G}_m$ with T via the isomorphism $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. If R is a ring the determinant induces a homomorphism $\det : G(R) \to R^*/(R^*)^2$.

1. Generalities on distributions and measures

1.1. **Distributions and measures.** Let $\mathcal{X}$ be a totally disconnected σ-locally compact topological space (in practice $\mathcal{X}$ will be a profinite set like an infinite Galois group or a certain space of adeles). For a topological Hausdorff ring R we denote by $C_\diamond(\mathcal{X}, R)$ the subring of $C(\mathcal{X}, R)$ consisting of maps $f : \mathcal{X} \to R$ with $f(x) \to 0$ as $x \to \infty$ (equivalently by setting $f(\infty) = 0$ the map f extends continuously to the one-point compactification of $\mathcal{X}$). We have $C_c^0(\mathcal{X}, R) \subseteq C_c(\mathcal{X}, R) \subseteq C_\diamond(\mathcal{X}, R) \subseteq C(\mathcal{X}, R)$. Note that if $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ where $\mathcal{X}_1$ and $\mathcal{X}_2$ are σ-locally compact and if $f_1 \in C_\diamond(\mathcal{X}_1, R)$, $f_2 \in C_\diamond(\mathcal{X}_1, R)$ then the map $(f_1 \otimes f_2)(g_1, g_2) := f_1(g_1) \cdot f_2(g_2)$ lies in $C_\diamond(\mathcal{X}, R)$.

Let M be an R-module. Recall that an M-valued *distribution* on $\mathcal{X}$ is a homomorphism $\mu : C_c^0(\mathcal{X}, \mathbb{Z}) \to M$. It extends to an R-linear map

$$C_c^0(\mathcal{X}, R) \longrightarrow M, \;\; f \mapsto \int_{\mathcal{X}} f \, d\mu. \tag{6}$$

We shall denote the R-module of M-valued distributions on $\mathcal{X}$ by $\mathrm{Dist}(\mathcal{X}, M)$. If $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, $\mu \in \mathrm{Dist}(\mathcal{X}, M)$ and $f_1 \in C_c^0(\mathcal{X}_1, R)$ then $f_2 \mapsto \int f_1 \otimes f_2 \, d\mu$ is an M-valued distribution on $\mathcal{X}_2$ which will be denoted by $\int_{\mathcal{X}_1} f_1 \, d\mu$ i.e. we have a pairing

$$\mathrm{Dist}(\mathcal{X}, M) \times C_c^0(\mathcal{X}_1, R) \longrightarrow \mathrm{Dist}(\mathcal{X}_2, M), \; (\mu, f_1) \mapsto \int_{\mathcal{X}_1} f_1 \, d\mu. \tag{7}$$

Next we introduce the notion of a measure on $\mathcal{X}$ with values in a p-adic Banach space. Assume that $R = K$ is a p-adic field. By that we mean that K is a field of characteristic 0 which is equipped with a p-adic absolute value, i.e. a nonarchimedian absolute value $| \; | : K \to \mathbb{R}$ whose restriction to $\mathbb{Q}$ is the usual p-adic value and such K is complete with respect to $| \; |$. We denote a p-adic value often as $| \; |_p$ and the corresponding valuation ring by $\mathcal{O}_K$.

A norm on a K-vector space V is a function $\| \; \| : V \to \mathbb{R}$ such that (i) $\|av\| = |a|_p \|v\|$, (ii) $\|v + w\| \leq \max(\|v\|, \|w\|)$ and (iii) $\|v\| \geq 0$ with equality iff $v = 0$ for all $a \in K$, $v, w \in V$. Two norms $\| \; \|_1$, $\| \; \|_2$ are *equivalent* if there exists $C_1, C_2 \in \mathbb{R}_+$ with $C_1 \|v\|_2 \leq \|v\|_1 \leq C_2 \|v\|_2$ for all $v \in V$. A normed K-vector space $(V, \| \; \|)$ is a (K-) *Banach space* if V is complete with respect to $\| \; \|$. Recall that any finite-dimensional K-vector space admits a norm, any two norms are equivalent and it is complete. The K-vector space $C_\diamond(\mathcal{X}, K)$ with the supremum norm $\|f\|_\infty = \sup_{\gamma \in \mathcal{X}} |f(\gamma)|_p$ is a K-Banach space.

Let V be a K-vector space. Recall that an $\mathcal{O}_K$-submodule $L \subseteq V$ is a *lattice* if $\bigcup_{a \in K^*} aL = V$ and $\bigcap_{a \in K^*} aL = \{0\}$. For a given lattice $L \subseteq V$ the function $p_L(v) := \inf_{v \in aL} |a|_p$ is a norm on V. If $\| \; \|$ is another norm then p_L is equivalent to $\| \; \|$ if and only if L is open and bounded in $(V, \| \; \|)$. A lattice $L \subseteq V$ is *complete* if V is complete with respect to p_L. Finally a torsion free $\mathcal{O}_K$-module L is said to be *complete* if L is a complete lattice in $L \otimes_{\mathcal{O}_K} K$. For example the $\mathcal{O}_K$-dual of a free module is a complete torsion free $\mathcal{O}_K$-module.

Let $(V, \| \ \|)$ be a Banach space. An element $\mu \in \mathrm{Dist}(\mathcal{X}, V)$ is a *measure (or bounded distribution)* if μ is continuous with respect to the supremum norm, i.e. if there exists $C \in \mathbb{R}$, $C > 0$ such that $\| \int_{\mathcal{X}} f \, d\mu \| \leq C \|f\|_\infty$ for all $f \in C_c^0(\mathcal{X}, K)$. We will denote the space of V-valued measures on $\mathcal{X}$ by $\mathrm{Dist}^{\mathrm{b}}(\mathcal{X}, V)$. If $L \subseteq V$ is an open and bounded lattice then $\mathrm{Dist}^{\mathrm{b}}(\mathcal{X}, V)$ is the image of the canonical inclusion $\mathrm{Dist}(\mathcal{X}, L) \otimes_{\mathcal{O}_K} K \to \mathrm{Dist}(\mathcal{X}, V)$. An element $\mu \in \mathrm{Dist}^{\mathrm{b}}(\mathcal{X}, V)$ can be integrated not only against locally constant functions but against any $f \in C_\diamond(\mathcal{X}, K)$. In fact since $C_c^0(\mathcal{X}, K)$ is dense in the Banach space $(C_\diamond(\mathcal{X}, K), \| \ \|_\infty)$ the functional (6) extends uniquely to a continuous functional

$$C_\diamond(\mathcal{X}, K) \longrightarrow V, \ f \mapsto \int f \, d\mu. \tag{8}$$

If $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ then we obtain as a refinement of the bilinear map (7) a pairing

$$\mathrm{Dist}^{\mathrm{b}}(\mathcal{X}, V) \times C_\diamond(\mathcal{X}_1, K) \longrightarrow \mathrm{Dist}^{\mathrm{b}}(\mathcal{X}_2, V), \ (\mu, f_1) \mapsto \int_{\mathcal{X}_1} f_1 \, d\mu. \tag{9}$$

1.2. **p-adic measures.** Given $\mu \in \mathrm{Dist}(\mathcal{X}, \mathbb{C})$ we want to clarify what do we mean by saying that μ is a p-adic measure. For simplicity assume that $\mathcal{X}$ is compact. The distribution μ extends to a $\mathbb{C}_p$-linear map

$$C^0(\mathcal{X}, \mathbb{C}_p) \longrightarrow \mathbb{C}_p \otimes_{\overline{\mathbb{Q}}} \mathbb{C}, \ f \mapsto \int f \, d\mu \tag{10}$$

and we denote its image by V_μ so that we can view μ as an element of $\mathrm{Dist}(\mathcal{X}, V_\mu)$. It is called a *$p$-adic measure* if V_μ is a finitely generated $\mathbb{C}_p$-vector space and if $\mu \in \mathrm{Dist}^b(\mathcal{X}, V_\mu)$. Equivalently, the image of μ (considered as a map $C^0(\mathcal{X}, \mathbb{Z}) \to \mathbb{C}$) is contained in a finitely generated $\overline{\mathcal{O}}$-module. So if $\mu \in \mathrm{Dist}(\mathcal{X}, \mathbb{C})$ is a p-adic measure (10) extends to continuous functional $C(\mathcal{X}, \mathbb{C}_p) \longrightarrow V_\mu$, $f \mapsto \int f \, d\mu$.

2. Local distributions attached to ordinary representations

2.1. **Gauss sums.** Throughout this chapter F denotes a finite extension of $\mathbb{Q}_p$, $\mathcal{O} = \mathcal{O}_F$ its ring of integers and $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$. We denote by U the group of units of $\mathcal{O}$ and put $U^{(n)} = \{x \in U | \ x \equiv 1 \mod \mathfrak{p}^n\}$. Let q denote the number of elements of $\mathcal{O}/\mathfrak{p}$. We fix an (additive) character $\psi : F \to \overline{\mathbb{Q}}^*$ such that $\mathrm{Ker}(\psi) = \mathcal{O}$ and a generator ϖ of $\mathfrak{p}$. We denote by $|x|$ the modulus of $x \in F^*$ (i.e. $|\varpi| = q^{-1}$) and by $\mathrm{ord} = \mathrm{ord}_F$ the additive valuation (normalized by $\mathrm{ord}(\varpi) = 1$). The normalized Haar measure on F will be denoted by dx (normalized by $\int_{\mathcal{O}} dx = 1$). We put $d^\times x = (1 - \frac{1}{q})^{-1} \frac{dx}{|x|}$ so that $\int_U d^\times x = 1$.

Lemma 2.1. *Let $X \subseteq \{x \in F^* \mid \mathrm{ord}(x) \leq -2\}$ be a compact open subset such for all $a \in X$ there exists $n \in \mathbb{Z}$, $1 \leq n \leq -\mathrm{ord}(a) - 1$ such that $aU^{(n)} \subseteq X$. Then,*

$$\int_X \psi(x) d^\times x = 0.$$

Proof. It is enough to consider the case $X = aU^{(n)}$ with $1 \le n \le -\operatorname{ord}(a) - 1$. Choose $b \in F^*$ with $\operatorname{ord}(b) + \operatorname{ord}(a) = -1$. Hence $\psi(ab) \neq 1$ and $\operatorname{ord}(b) \ge n$ and therefore

$$\begin{aligned}\int_X \psi(x) d^\times x &= \int_{U^{(n)}} \psi(ax) d^\times x = \int_{U^{(n)}} \psi(a(1+b)x) d^\times x \\ &= \int_{U^{(n)}} \psi(ax)\psi(abx) d^\times x.\end{aligned}$$

Since $\operatorname{ord}(abx - ab) = -1 + \operatorname{ord}(x-1) \ge n-1 \ge 0$, we have $\psi(abx) = \psi(ab)$ for all $x \in U^{(n)}$. It follows

$$\int_X \psi(x) d^\times x = \psi(ab) \int_{U^{(n)}} \psi(ax) d^\times x = \psi(ab) \int_X \psi(x) d^\times x,$$

hence $\int_X \psi(x) d^\times x = 0$. $\square$

Recall that the *conductor* $\mathfrak{c}(\chi)$ of a quasicharacter $\chi : F^* \to \mathbb{C}^*$ is the largest ideal $\mathfrak{p}^n$ of $\mathcal{O}$ such that $U^{(n)} \subseteq \operatorname{Ker}(\chi)$.

Lemma 2.2. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasicharacter of conductor $\mathfrak{p}^n, n \ge 1$ and let $a \in F^*$ with $\operatorname{ord}(a) \neq -n$. Then we have*

$$\int_U \psi(ax)\chi(x) d^\times x = 0.$$

Proof. 1. case $\operatorname{ord}(a) > -n$: Choose $b \in F^*$ with $\max(-\operatorname{ord}(a), 0) \le \operatorname{ord}(b) < n$, $1 + b \in U$ and $\chi(1+b) \neq 1$. Then,

$$\begin{aligned}\int_U \psi(ax)\chi(x) d^\times x &= \int_U \psi(ax(1+b))\chi(x(1+b)) d^\times x = \\ &= \chi(1+b) \int_U \psi(ax)\psi(abx)\chi(x) d^\times x \\ &= \chi(1+b) \int_U \psi(ax)\chi(x) d^\times x\end{aligned}$$

hence $\int_U \psi(ax)\chi(x) d^\times x = 0$.

2. case $\operatorname{ord}(a) < -n$: By 2.1 above we have

$$\int_U \psi(ax)\chi(x) d^\times x = \sum_{bU^{(n)} \in U/U^{(n)}} \chi(b) \int_{abU^{(n)}} \psi(x) d^\times x = 0.$$

$\square$

We recall the definition of the Gauss sum of a quasicharacter (with respect to the fixed choice of ψ).

Definition 2.3. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasicharacter with conductor $\mathfrak{p}^n$, $n \ge 0$ and $a \in F^*$ with $\operatorname{ord}(a) = -n$. We define the Gauss sum of χ by*

$$\tau(\chi) = \tau(\chi, \psi) = [U : U^{(n)}] \int_{aU} \psi(x)\chi(x) d^\times x.$$

For a quasicharacter $\chi : F^* \to \mathbb{C}^*$ we define

$$\int_{F^*} \chi(x)\psi(x)dx := \lim_{n\to+\infty} \int_{x\in F^*, -n\leq \mathrm{ord}(x)\leq n} \chi(x)\psi(x)dx. \tag{11}$$

Lemma 2.4. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasicharacter with conductor $\mathfrak{p}^f$. Assume that $|\chi(\varpi)| < q$. Then the integral (11) converges and we have*

$$\int_{F^*} \chi(x)\psi(x)dx = \begin{cases} \frac{1-\chi(\varpi)^{-1}}{1-\chi(\varpi)q^{-1}} & \text{if } f = 0; \\ \tau(\chi) & \text{if } f > 0. \end{cases}$$

Proof. Firstly, we remark

$$\int_U \psi(ax)d^\times x = \begin{cases} 1 & \text{if } \mathrm{ord}(a) \geq 0; \\ -\frac{1}{q-1} & \text{if } \mathrm{ord}(a) = -1; \\ 0 & \text{if } \mathrm{ord}(a) \leq -2; \end{cases} \tag{12}$$

for all $a \in F^*$. Since $(1 - 1/q)d^\times x = \frac{dx}{|x|}$, we obtain

$$\int_{F^*} \chi(x)\psi(x)dx = \sum_{n=-\infty}^{\infty} (1 - 1/q)q^{-n} \int_{\varpi^n U} \chi(x)\psi(x)d^\times x.$$

If $f > 0$ then by Lemma 2.2 we have

$$\int_{F^*} \chi(x)\psi(x)dx = (1 - 1/q)q^f \int_{\varpi^{-f} U} \chi(x)\psi(x)d^\times x = \tau(\chi).$$

On the other hand if $f = 0$ then by (12) we get

$$\begin{aligned} \int_{F^*} \chi(x)\psi(x)dx &= (1 - 1/q) \left(-\frac{q}{(q-1)\chi(\varpi)} + \sum_{n=0}^{\infty} (\chi(\varpi)q^{-1})^n \right) \\ &= \frac{1 - \chi(\varpi)^{-1}}{1 - \chi(\varpi)q^{-1}}. \end{aligned}$$

□

2.2. **Ordinary representations of** $\mathrm{PGL}_2(F)$**.** We introduce more notation. Let $K = G(\mathcal{O})$. For an ideal $\mathfrak{c} \subset \mathcal{O}$ let $K_0(\mathfrak{c}) \subseteq K$ denote the subgroup of matrices A (modulo Z) which are upper triangular modulo $\mathfrak{c}$. Let $\pi : G(F) \to \mathrm{GL}(V)$ be an irreducible admissible infinite-dimensional representation (where V is a $\mathbb{C}$-vector space). Recall [8] that there exists a largest ideal $\mathfrak{c}(\pi)$ – the *conductor* of π – such that $V^{K_0(\mathfrak{c})} = \{v \in V \mid \pi(k)v = v\ \forall\, k \in K_0(\mathfrak{c})\} \neq 0$. In this case $V^{K_0(\mathfrak{c})}$ is one-dimensional.

The representation π is called *tamely ramified* if the conductor divides $\mathfrak{p}$. This holds if and only if $\pi = \pi(\chi^{-1}, \chi)$ for an unramified quasicharacter $\chi : F^* \to \mathbb{C}^*$ (see e.g. [5], Ch. IV). More precisely if the conductor is $\mathcal{O}_F$, then π is spherical hence a principal series representation $\pi(\chi^{-1}, \chi)$ where $\chi : F^* \to \mathbb{C}^*$ is an unramified quasicharacter with $\chi^2 \neq |\cdot|$. If $\mathfrak{c}(\pi) = \mathfrak{p}$, then π is a special representation $\pi(\chi^{-1}, \chi)$ where χ is unramified with $\chi^2 = |\cdot|$.

Definition 2.5. *Assume that $\pi = \pi(\chi^{-1}, \chi)$ is tamely ramified. Then π is called ordinary if either $\chi^2 = |\cdot|$ or if π is spherical and tempered and if $\chi(\varpi)q^{1/2}$ is a p-adic unit (i.e. it lies in $\overline{\mathcal{O}}^*$).*

Thus if $\pi = \pi(\chi^{-1}, \chi)$ is tamely ramified and if we put $\alpha := \chi(\varpi)q^{1/2} \in \mathbb{C}$ then π is ordinary if either $\alpha = \pm 1$ or if $\alpha \in \overline{\mathcal{O}}^*$ and $|\alpha| = q^{1/2}$. Note that α determines π uniquely, i.e. there exists a one-to-one correspondence between the set (of isomorphism classes) of ordinary representations of $G(F)$ and the set $\{\alpha \in \overline{\mathcal{O}}^* |\ \alpha = \pm 1 \text{ or } |\alpha| = q^{1/2}\}$. We will call an element of the latter set an *ordinary parameter*. We will denote the class corresponding to α by π_α and define $\chi_\alpha(x) := \alpha^{\mathrm{ord}(x)}$ (thus $\pi_\alpha = \pi(\chi_\alpha^{-1}|\cdot|^{-1/2}, \chi_\alpha|\cdot|^{1/2})$). If $\alpha = \pm 1$ (resp. $\alpha \neq \pm 1$) then π_α is special (resp. spherical). If $\alpha = 1$ then $\pi_\alpha = \mathrm{St}$ is the Steinberg representation.

2.3. **Bruhat-Tits tree.** In the next section we recall the well-known construction of models of the spherical principal series representations and special representations of $G(F)$ in terms of the Bruhat-Tits tree $\mathcal{T}$ of $G(F)$ (see e.g. [1]). In fact we will work over an arbitrary ring R rather than $\mathbb{C}$. Here we recall a few facts regarding the tree $\mathcal{T}$ (see e.g. [25]). The set of vertices $\mathcal{V} = \mathcal{V}(\mathcal{T})$ of $\mathcal{T}$ is the set of homothety classes of lattices in F^2. For any two vertices v_1, v_2 of $\mathcal{T}$ denote $d(v_1, v_2)$ the distance between v_1 and v_2. A vertex v is even or odd if its distance to the standard vertex $v_0 = [\mathcal{O}^2]$ is even or odd. The set of even (resp. odd) vertices will be denoted by $\mathcal{V}_{\mathrm{even}}$ (resp. $\mathcal{V}_{\mathrm{odd}}$). The group $G(F)$ operates on $\mathcal{T}$ by $g[L] = [\widetilde{g}L]$ (where $\widetilde{g} \in \mathrm{GL}_2(F)$ is a lift of $g \in G(F)$). Let $\vec{\mathcal{E}} = \vec{\mathcal{E}}(\mathcal{T})$ (resp. $\mathcal{E} = \mathcal{E}(\mathcal{T})$) denote the set of oriented (resp. unoriented) edges of $\mathcal{T}$. For $e \in \vec{\mathcal{E}}$ let $o(e)$ (resp. $t(e)$) denote the origin (resp. target) of e and let $\bar{e}$ be the same edge as e but with opposite orientation as e. Given $e \in \vec{\mathcal{E}}$ the set of ends $U(e)$ of e is an open compact subset of $\mathbb{P}^1(F)$. We recall its definition. For $(x, y) \in F^2$ let $\ell_{(x,y)}$ denote the linear form $F^2 \to F, (x', y') \mapsto \det \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}$. Given $P = [x : y] \in \mathbb{P}^1(F)$ and representatives L_1 and L_2 of $o(e)$ and $t(e)$ respectively with $\varpi L_1 \subsetneq L_2 \subsetneq L_1$ we have $P \in U(e)$ iff $\ell_{(x,y)}(L_2) \subsetneq \ell_{(x,y)}(L_1)$. For $e \in \vec{\mathcal{E}}(\mathcal{T})$ and $g \in G(F)$ we get $U(\bar{e}) = \mathbb{P}^1(F) - U(e)$ and $gU(e) = U(ge)$ for all. For $n \in \mathbb{Z}$ put $v_n = [\mathcal{O} \oplus \mathfrak{p}^n]$. The set $\{v_n \mid n \in \mathbb{Z}\}$ determines the standard apartment A of $\mathcal{T}$. The edge of A with origin v_{n+1} and target v_n will be denoted by e_n. One easily checks that $U(e_n) = \mathfrak{p}^{-n} \subset \mathbb{P}^1(F)$, so $\bigcap_n U(e_n) = \{0\}$ and $\bigcap_n U(\bar{e}_n) = \{\infty\}$ so the sequence $\{e_n\}_{n \in \mathbb{Z}}$ is the geodesic from ∞ to 0.

Following [1] we define the height $h(v) \in \mathbb{Z}$ of $v \in \mathcal{V}$ as follows. The geodesic ray from v to ∞ has a non-empty intersection with A. If v_n is any point in the intersection we define $h(v) = n - d(v, v_n)$. It is independent of the choice of v_n and satisfies $h(v_n) = n$. We need the following simple

Lemma 2.6. *(a) For all $e \in \mathcal{E}(\mathcal{T})$ we have*

$$h(t(e)) = \begin{cases} h(o(e)) + 1 & \text{if } \infty \in U(e), \\ h(o(e)) - 1 & \text{otherwise.} \end{cases}$$

(b) For all $a, b \in F, a \neq 0$ and $v \in \mathcal{V}(\mathcal{T})$ we have

$$h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v\right) = -\operatorname{ord}(a) + h(v).$$

Proof. (a) follows immediately from the definition. For (b) it suffices to consider the case $v = v_0$ since the group $B(F)$ acts transitively on $\mathcal{V}(\mathcal{T})$. Put $e = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} e_0$. Since $\infty \notin b + a\mathcal{O} = U(e)$ we obtain

$$h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v_0\right) = h(t(e)) = h(o(e)) - 1 = h\left(\begin{pmatrix} a\varpi^{-1} & b \\ 0 & 1 \end{pmatrix} v_0\right) - 1.$$

If $b \neq 0$ we have for $n \in \mathbb{N}$ with $n \geq \operatorname{ord}(a) - \operatorname{ord}(b)$ and $m = \operatorname{ord}(a)$

$$\begin{aligned} h\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} v_0\right) &= h\left(\begin{pmatrix} a\varpi^{-n} & b \\ 0 & 1 \end{pmatrix} v_0\right) - n = h\left(\begin{pmatrix} a\varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix} v_0\right) - n \\ &= h\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} v_0\right) = h(v_{-m}) = -m. \end{aligned}$$

$\square$

2.4. **Representations of $\mathrm{PGL}_2(F)$ attached to the Bruhat-Tits tree.** Let R be an arbitrary ring. For an R-module M let $C(\mathcal{V}, M)$ denote the R-module of maps $\phi : \mathcal{V}(\mathcal{T}) \to M$ and $C(\vec{\mathcal{E}}, M)$ the R-module of maps $c : \vec{\mathcal{E}}(\mathcal{T}) \to M$. Moreover we denote by $C^{\pm}(\mathcal{E}, M) \subseteq C(\vec{\mathcal{E}}, M)$ the submodule of $c \in C(\vec{\mathcal{E}}, M)$ with $c(\bar{e}) = \mp c(e)$ for all $e \in \vec{\mathcal{E}}(\mathcal{T})$. Both $C(\mathcal{V}, M)$ and $C(\vec{\mathcal{E}}, M)$ are left $G(F)$-modules via $(g\phi)(v) := \phi(g^{-1}v)$ and $(gc)(e) = c(g^{-1}e)$ and $C^{+}(\mathcal{E}, M), C^{-}(\mathcal{E}, M) \subseteq C(\vec{\mathcal{E}}, M)$ are $G(F)$-stable submodules. We let $C_c(\mathcal{V}, M) \subseteq C(\mathcal{V}, M)$ (resp. $C_c(\vec{\mathcal{E}}, M) \subseteq C(\vec{\mathcal{E}}, M)$, $C_c^{\pm}(\mathcal{E}, M) \subseteq C^{\pm}(\mathcal{E}, M)$) be the submodule of $\phi \in C(\mathcal{V}, M)$ with $\phi(v) = 0$ for almost all v (resp. $c \in C^{\pm}(\mathcal{E}, M)$ with $c(e) = 0$ for almost all e). We define pairings

(13) $\langle\ ,\ \rangle : C_c(\mathcal{V}, R) \times C(\mathcal{V}, M) \to M,\ \ \langle \phi_1, \phi_2 \rangle := \sum_{v \in \mathcal{V}} \phi_1(v)\phi_2(v)$

(14) $\langle\ ,\ \rangle : C_c^{\pm}(\mathcal{E}, R) \times C^{\pm}(\mathcal{E}, M) \to M,\ \ \langle c_1, c_2 \rangle := \sum_{e \in \mathcal{E}} c_1(e)c_2(e)$

(note that in the second pairing the summand $c_1(e)c_2(e)$ does not depend on the choice of orientation of e). Define maps

$$\delta : C(\vec{\mathcal{E}}, M) \longrightarrow C(\mathcal{V}, M),\ \ \delta(c)(v) := \textstyle\sum_{t(e)=v} c(e),$$

$$\delta_{\pm}^{*} : C(\mathcal{V}, M) \longrightarrow C^{\pm}(\mathcal{E}, M),\ \ \delta_{\pm}^{*}(\phi)(e) := \phi(t(e)) \mp \phi(o(e)).$$

They are adjoint with respect to (13), (14), i.e. we have $\langle \delta(c), \phi \rangle = \langle c, \delta_{\pm}^{*}(\phi) \rangle$ for all $c \in C_c^{\pm}(\mathcal{E}, R), \phi \in C(\mathcal{V}, M)$ (by abuse of notation we denote the restriction of δ to any submodule of $C(\vec{\mathcal{E}}, M)$ also by δ and similarly for $\delta_{\pm}^{*}$).

For the function $\tau_{\pm} \in C(\mathcal{V}, R)$ defined by $\tau_{\pm}(v) = (\pm 1)^{d(v_0, v)} = (\pm 1)^{h(v)}$ for $v \in \mathcal{V}$ we consider the map

$$\langle \cdot, \tau_{\pm} \rangle : C_c(\mathcal{V}, R) \longrightarrow R,\ \phi \mapsto \langle \phi, \tau_{\pm} \rangle = \sum_{v \in \mathcal{V}_{\mathrm{even}}} \phi(v) \pm \sum_{v \in \mathcal{V}_{\mathrm{odd}}} \phi(v).$$

One readily verifies that the sequence of R-modules

$$0 \longrightarrow C_c^{\pm}(\mathcal{E}, R) \xrightarrow{\delta} C_c(\mathcal{V}, R) \xrightarrow{\langle\,\cdot\,,\tau_{\pm}\rangle} R \longrightarrow 0$$

is exact. Dually we have an exact sequence

$$0 \longrightarrow M \xrightarrow{m\mapsto\tau_{\pm}m} C(\mathcal{V}, M) \xrightarrow{\delta_{\pm}^*} C^{\pm}(\mathcal{E}, M) \longrightarrow 0.$$

In particular the restriction $\delta_{\pm}^* = (\delta_{\pm}^*)|_{C_c(\mathcal{V},R)} : C_c(\mathcal{V}, R) \to C_c^{\pm}(\mathcal{E}, R)$ is injective.

The kernel of $\delta : C^+(\mathcal{E}, M) \to C(\mathcal{V}, M)$ is the set of harmonic cocycles $C_{\mathrm{har}}(\mathcal{T}, M)$, i.e. the set of maps $c : \vec{\mathcal{E}}(\mathcal{T}) \to M$ such that $c(\bar{e}) = -c(e)$ for all $e \in \vec{\mathcal{E}}(\mathcal{T})$ and $\sum_{o(e)=v} c(e) = 0$ for all $v \in \mathcal{V}(\mathcal{T})$. We recall the relation between harmonic cocycles and boundary distributions on $\mathbb{P}^1(F)$. The map

$$\mathrm{Coker}(\delta_+^* : C_c(\mathcal{V}, R) \to C_c^+(\mathcal{E}, R)) \longrightarrow C^0(\mathbb{P}^1(F), R)/R$$

given by $c \mapsto \sum_{e\in\mathcal{E}(\mathcal{T})} c(e)1_{U(e)}$ is an isomorphism (note that $c(e)1_{U(e)} \equiv c(\bar{e})1_{U(\bar{e})} \mod R$ – i.e. modulo constant functions). Thus (14) induces a pairing

$$C^0(\mathbb{P}^1(F), R)/R \times C_{\mathrm{har}}(\mathcal{T}, M) \to M, (f, c) \mapsto \int_{\mathbb{P}^1(F)} f(P)\mu_c(dP) \tag{15}$$

i.e. a map $C_{\mathrm{har}}(\mathcal{T}, M) \to \mathrm{Dist}(\mathbb{P}^1(F), M), c \mapsto \mu_c$ so that μ_c has total mass $= 0$. For $f = 1_{U(e)}$, $e \in \vec{\mathcal{E}}(\mathcal{T})$ we have $\int_{\mathbb{P}^1(F)} 1_{U(e)}(P)\mu_c(dP) = c(e)$.

The Hecke operator $T : C(\mathcal{V}, M) \to C(\mathcal{V}, M)$ is defined by $(T\phi)(v) = \sum_{o(e)=v} \phi(t(e))$. By ([1], Thm. 10) the $R[T]$-module $C_c(\mathcal{V}, R)$ is free. Thus for $a \in R$ the map $T - a : C_c(\mathcal{V}, R) \to C_c(\mathcal{V}, R)$ is injective. If $a \neq \pm(q + 1)$ we define $\mathscr{B}_a(F, R)$ to be the cokernel so that there exists a short exact sequence

$$0 \longrightarrow C_c(\mathcal{V}, R) \xrightarrow{T-a\,\mathrm{id}} C_c(\mathcal{V}, R) \longrightarrow \mathscr{B}_a(F, R) \longrightarrow 0 \tag{16}$$

of $R[G(F)]$-modules. Note that $\mathscr{B}_a(F, R)$ is free as an R-module.

For $a = \pm(q+1)$ we have $\langle\,\cdot\,, \tau_{\pm}\rangle \circ (T-a) = 0$ since $\langle T\phi, \tau_{\pm}\rangle = \langle \phi, T\tau_{\pm}\rangle = a\langle \phi, \tau_{\pm}\rangle$ for all $\phi \in C_c(\mathcal{V}, R)$. We put $\mathscr{B}_a(F, R) = \mathrm{Ker}(\langle\,\cdot\,, \tau_{\pm}\rangle)/\,\mathrm{Im}(T - a)$, so that the sequence

$$0 \to \mathscr{B}_a(F, R) \to \mathrm{Coker}(T - a : C_c(\mathcal{V}, R) \to C_c(\mathcal{V}, R)) \xrightarrow{\langle\,\cdot\,,\tau_{\pm}\rangle} R \to 0 \tag{17}$$

is exact. It is easy to see that $\mathscr{B}_a(F, R)$ is again an $R[G(F)]$-module which is free as an R-module. Since $\delta \circ \delta_{\pm}^* = (q + 1)\,\mathrm{id} \pm T$ we see that δ induces a map $C_c^{\pm}(\mathcal{E}, R) \to \mathscr{B}_{\pm(q+1)}(F, R)$ such that

$$0 \longrightarrow C_c(\mathcal{V}, R) \xrightarrow{\delta_{\pm}^*} C_c^{\pm}(\mathcal{E}, R) \longrightarrow \mathscr{B}_{\pm(q+1)}(F, R) \longrightarrow 0 \tag{18}$$

is an exact sequence of $R[G(F)]$-modules.

Dually, for $a \in R$ we define $\mathscr{B}^a(F, M)$ as follows. If $a \neq \pm(q + 1)$ we let $\mathscr{B}^a(F, M)$ be the kernel of $T-a : C(\mathcal{V}, M) \to C(\mathcal{V}, M)$. If $a = \pm(q+1)$ then $T - a$ maps the submodule $\tau_{\pm}M = \{\tau_{\pm} \cdot m \mid m \in M\}$ to zero so it induces an endomorphism of the quotient $C(\mathcal{V}, M)/\tau_{\pm}M$ and we define $\mathscr{B}^a(F, M)$ to be its kernel.

Since $T : C_c(\mathcal{V}, R) \to C_c(\mathcal{V}, R)$ and $T : C(\mathcal{V}, M) \to C(\mathcal{V}, M)$ are adjoint with respect to (13) we obtain a pairing

$$\langle\ ,\ \rangle : \mathscr{B}_a(F, R) \times \mathscr{B}^a(F, M) \to M \tag{19}$$

which induces an isomorphism $\mathscr{B}^a(F, M) \to \mathrm{Hom}_R(\mathscr{B}_a(F, R), M)$.

For $\epsilon = \pm 1$ let as before $\chi_\epsilon(x) = \epsilon^{\mathrm{ord}(x)}$. We can view χ_ϵ as a character of $F^*/(F^*)^2$ and so $\chi_\epsilon(\det(g))$ is defined for $g \in G(F)$. Since $T(\tau_\pm \cdot \phi) = \pm T(\phi) \cdot \tau_\pm$ the isomorphism $C_c(\mathcal{V}, R) \to C_c(\mathcal{V}, R), \phi \mapsto \phi \cdot \tau_\pm$ induces an isomorphism

$$\mathfrak{Tw}_\epsilon : \mathscr{B}_a(F, R) \longrightarrow \mathscr{B}_{\epsilon a}(F, R) \tag{20}$$

which satisfies $\chi_\epsilon(\det(g))\mathfrak{Tw}_\epsilon(g\phi) = g\mathfrak{Tw}_\epsilon(\phi)$ for all $\phi \in \mathscr{B}_a(F, R)$ and $g \in G(F)$. If $\epsilon = +1$ then $\mathfrak{Tw}_\epsilon$ is of course the identity. In general we have $\mathfrak{Tw}_\epsilon \circ \mathfrak{Tw}_\epsilon = \mathrm{id}$. The operators (20) will be used in section 5.1 in order to show that certain $\mathcal{L}$-invariants do not change under quadratic twists.

We want to reinterpret the sequences (16) and (18) in terms of the induced representation $\mathrm{Ind}_K^{G(F)} R$ and $\mathrm{Ind}_{K_0(\mathfrak{p})}^{G(F)} R$ (here we consider compact induction). Since $G(F)$ acts transitively on $\mathcal{V}(\mathcal{T})$ and $\vec{\mathcal{E}}(\mathcal{T})$ and the stabilizer of v_0 and e_0 is K and $K_0(\mathfrak{p})$ respectively we have $C_c(\mathcal{V}, R) \cong \mathrm{Ind}_K^{G(F)} R$ and $C_c(\vec{\mathcal{E}}, R) \cong \mathrm{Ind}_{K_0(\mathfrak{p})}^{G(F)} R$. The element $W = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \in G(F)$ normalizes $K_0(\mathfrak{p})$ hence induces an involution $W : \mathrm{Ind}_{K_0(\mathfrak{p})}^{G(F)} R \to \mathrm{Ind}_{K_0(\mathfrak{p})}^{G(F)} R, W(\phi)(g) := \phi(Wg)$ and $C_c^\pm(\mathcal{E}, R)$ is mapped onto $(\mathrm{Ind}_{K_0(\mathfrak{p})}^{G(F)} R)^{W=\mp 1}$ under the above isomorphism. Hence we have exact sequences of $R[G(F)]$-modules

$$0 \longrightarrow \mathrm{Ind}_K^{G(F)} R \xrightarrow{T - a\,\mathrm{id}} \mathrm{Ind}_K^{G(F)} R \longrightarrow \mathscr{B}_a(F, R) \longrightarrow 0 \tag{21}$$

for $a \in R$, $a \neq \pm(q+1)$ and

$$0 \longrightarrow \mathrm{Ind}_K^{G(F)} R \longrightarrow (\mathrm{Ind}_{K_0(\mathfrak{p})}^{G(F)} R)^{W=\mp 1} \longrightarrow \mathscr{B}_{\pm(q+1)}(F, R) \longrightarrow 0. \tag{22}$$

If $R = \mathbb{C}$ and $a = \alpha + q/\alpha$ for some $\alpha \in \mathbb{C}^*$, $\alpha \neq \pm 1$ (resp. $\alpha = \pm 1$) then it is well-known that $\mathscr{B}_a(F, \mathbb{C})$ is a model of the principal series representation (resp. special representation) $\pi(\chi_\alpha^{-1}|\cdot|^{-1/2}, \chi_\alpha|\cdot|^{1/2})$ (see e.g. [25], or [1]). In particular $\mathscr{B}_a(F, \mathbb{C})$ admits a (up to scalar) unique Whittaker functional, i.e. a nontrivial linear map $\lambda : \mathscr{B}_a(F, \mathbb{C}) \to \mathbb{C}$ such that

$$\lambda\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \phi\right) \ = \ \psi(x)\lambda(\phi)$$

for all $x \in F$ and $\phi \in \mathscr{B}_a(F, \mathbb{C})$. This fact will be used in section 2.6.

2.5. Distributions attached to elements of $\mathscr{B}^a(F, M)$.

Given $\rho \in C(\mathcal{V}, R)$ define R-linear maps

$\widetilde{\delta}_\rho : C(\mathcal{E}, M) \longrightarrow C(\mathcal{V}, M)$, $\widetilde{\delta}_\rho(c)(v) := \sum_{t(e)=v} \rho(o(e))c(e)$,

$\widetilde{\delta}^\rho : C(\mathcal{V}, M) \longrightarrow C^+(\mathcal{E}, M)$, $\widetilde{\delta}^\rho(\phi)(e) := \rho(o(e))\phi(t(e)) - \rho(t(e))\phi(o(e))$.

They are adjoint with respect to (13), (14), i.e. we have $\langle\widetilde{\delta}_\rho(c),\phi\rangle=\langle c,\widetilde{\delta}^\rho(\phi)\rangle$ for all $c\in C_c^+(\mathcal{E},R),\phi\in C(\mathcal{V},M)$. For the constant function $\rho\equiv 1$ we have $\widetilde{\delta}^1=\delta_+^*,\widetilde{\delta}_1=\delta$. Note that for $\rho_1,\rho_2\in C(\mathcal{V},R)$ and $\phi\in C(\mathcal{V},M)$ we have

$$(\widetilde{\delta}_{\rho_1}\circ\widetilde{\delta}^{\rho_2})(\phi)=T(\rho_1\cdot\rho_2)\cdot\phi-\rho_2\cdot T(\rho_1\cdot\phi). \tag{23}$$

Hence for $a\in R$ and $\rho\in\mathscr{B}^a(F,R)$ the maps $\widetilde{\delta}_\rho$ and $\widetilde{\delta}^\rho$ induce R-linear maps

$$\widetilde{\delta}_\rho:\ C^0(\mathbb{P}^1(F),R)/R\cong\mathrm{Coker}(\delta_+^*:C_c(\mathcal{V},R)\to C_c^+(\mathcal{E},R))\longrightarrow\mathscr{B}_a(F,R),$$
$$\widetilde{\delta}^\rho:\ \mathscr{B}^a(F,M)\longrightarrow\mathrm{Ker}(\delta:C^+(\mathcal{E},M)\to C(\mathcal{V},M))=C_{\mathrm{har}}(\mathcal{T},M).$$

In fact by applying (23) to $\rho_1=1$ and $\rho_2=\rho$ (resp. $\rho_1=\rho$ and $\rho_2=1$) we see that $\widetilde{\delta}^\rho$ maps $\mathrm{Ker}(T-a)$ into $\mathrm{Ker}(\delta)$ (resp. $\widetilde{\delta}_\rho$ induces a map $\mathrm{Coker}(\delta)\to\mathrm{Coker}(T-a)$).

Let $\alpha\in R^*$ and put $a=\alpha+q/\alpha$ and $\chi_\alpha:F^*\to R^*,x\mapsto\chi_\alpha(x)=\alpha^{\mathrm{ord}(x)}$. In the following we assume that $q\pm\alpha$ is not a zero-divisor in R so that $\alpha=\pm1$ if and only if $a=\pm(q+1)$. One easily checks that the function $\rho(v):=\alpha^{h(v)}$ lies in $\mathscr{B}^a(F,R)$. Put $\widetilde{\delta}_\alpha:=\widetilde{\delta}_\rho$ and $\widetilde{\delta}^\alpha:=\widetilde{\delta}^\rho$, so

$$\widetilde{\delta}_\alpha:C^0(\mathbb{P}^1(F),R)/R\longrightarrow\mathscr{B}_a(F,R),\quad\widetilde{\delta}^\alpha:\mathscr{B}^a(F,M)\longrightarrow C_{\mathrm{har}}(\mathcal{T},M)$$

are adjoint with respect to (15) and (19).

Lemma 2.7. *(a) We have* $\widetilde{\delta}_\alpha(gf)=\chi_\alpha(a)^{-1}\,g\,\widetilde{\delta}_\alpha(f)$ *for all* $g=\begin{pmatrix}a&b\\0&1\end{pmatrix}\in B(F)$ *and* $f\in C^0(\mathbb{P}^1(F),R)/R$.

(b) If $\alpha=\pm1$ *then* $\widetilde{\delta}_\alpha:C^0(\mathbb{P}^1(F),R)/R\longrightarrow\mathscr{B}_a(F,R)$ *is an isomorphism.*

(c) If $\alpha\neq\pm1$ *then*

$$0\longrightarrow C^0(\mathbb{P}^1(F),R)/R\xrightarrow{\ \widetilde{\delta}_\alpha\ }\mathscr{B}_a(F,R)\xrightarrow{\ \phi\mapsto\langle\phi,\rho\rangle\ }R\longrightarrow 0$$

is exact.

Proof. (a) follows immediately from $\rho\left(\begin{pmatrix}a&b\\0&1\end{pmatrix}v\right)=\chi_\alpha(a)^{-1}\rho(v)$ by Lemma 2.6 and the simple proof of (b) will be left to the reader.

For (c) consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_c(\mathcal{V},R) \arrow[r, "\delta_+^*"] \arrow[d, "(24)"] & C_c^+(\mathcal{E},R) \arrow[r] \arrow[d, "\widetilde{\delta}_\alpha"] & C^0(\mathbb{P}^1(F),R)/R \arrow[r] \arrow[d, "\widetilde{\delta}_\alpha"] & 0 \\
0 \arrow[r] & C_c(\mathcal{V},R) \arrow[r, "a\,\mathrm{id}-T"] & C_c(\mathcal{V},R) \arrow[r] & \mathscr{B}_a(F,R) \arrow[r] & 0
\end{tikzcd}$$

where the first vertical map is the isomorphism

$$C_c(\mathcal{V},R)\longrightarrow C_c(\mathcal{V},R),\ \phi\mapsto(v\mapsto\phi(v)\rho(v)). \tag{24}$$

So it remains to prove that the upper row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & C_c^+(\mathcal{E},R) \arrow[r, "\widetilde{\delta}_\alpha"] \arrow[d, "(25)"] & C_c(\mathcal{V},R) \arrow[r, "\phi\mapsto\langle\phi,\rho\rangle"] \arrow[d, "(24)"] & R \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & C_c^+(\mathcal{E},R) \arrow[r, "\delta"] & C_c(\mathcal{V},R) \arrow[r, "\phi\mapsto\langle\phi,\tau_+\rangle"] & R \arrow[r] & 0
\end{tikzcd}$$

is exact where (25) is the isomorphism

$$C_c^+(\mathcal{E},R) \longrightarrow C_c^+(\mathcal{E},R),\ c \mapsto (e \mapsto c(e)\phi(t(e))\phi(o(e))). \tag{25}$$

However the lower row is exact. □

We define R-linear maps

$$\begin{array}{rcll} \delta_\alpha : C_c(F^*,R) & \longrightarrow & \mathscr{B}_a(F,R) & \text{if } \alpha \neq 1, \\ \delta_\alpha : C_c(F,R) & \longrightarrow & \mathscr{B}_a(F,R) & \text{if } \alpha = 1. \end{array} \tag{26}$$

as follows. If $\alpha \neq 1$ and $f \in C_c(F^*,R)$ we define $\delta_\alpha(f)$ by extending $\chi_\alpha(x)f(x)$ by zero to $\mathbb{P}^1(F)$ and then applying $\widetilde{\delta}_\alpha$. If $\alpha = 1$ and $f \in C_c(F,R)$ we extend f by zero to $\mathbb{P}^1(F)$ and then apply $\widetilde{\delta}_\alpha$. We let F^* act on $C_c(F^*,R)$ and $C_c(F,R)$ by $(a \cdot f)(x) = f(a^{-1}x)$. It induces a $T(F)$-action via the isomorphism $T \cong \mathbb{G}_m$. If M is a F^*-module we also define a F^* operation on $\mathrm{Dist}(F,M)$ and $\mathrm{Dist}(F^*,M)$ by

$$\int f(x)(a\mu)(dx) = a\left(\int (a^{-1}f)(x)\mu(dx)\right).$$

for all $f \in C_c(F^*,R)$ resp. in $f \in C_c(F,R)$ and $a \in F^*$. The following result is an immediate consequence of Lemma 2.7 (a).

Lemma 2.8. *The map δ_α is $T(F)$-equivariant.*

Let H be a subgroup of $G(F)$ and M a $R[H]$-module (in the applications in chapter 4 both H and M will be of "global nature"). We define a H-action on $\mathscr{B}^a(F,M)$ by requiring that $\langle \phi, h\cdot\lambda\rangle = h \cdot \langle h^{-1}\phi, \lambda\rangle$ for all $h \in H$, $\phi \in \mathscr{B}_a(F,R)$ and $c \in \mathscr{B}^a(F,M)$. By passing to duals we get $(T(F)\cap H)$-equivariant homomorphisms

$$\delta^\alpha : \mathscr{B}^a(F,M) \longrightarrow \begin{cases} \mathrm{Dist}(F^*,M) & \text{if } \alpha \neq 1, \\ \mathrm{Dist}(F,M) & \text{if } \alpha = 1 \end{cases}$$

characterized by

$$\langle \delta_\alpha(f), \lambda\rangle = \begin{cases} \int_{F^*} f(x)\,\delta^\alpha(\lambda)(dx) & \text{if } \alpha \neq 1, \\ \int_F f(x)\,\delta^\alpha(\lambda)(dx) & \text{if } \alpha = 1. \end{cases} \tag{27}$$

2.6. **Local distributions.** In this section we assume $R = \mathbb{C}$. Let $\alpha \in \overline{\mathcal{O}}^*$ be an ordinary parameter, i.e. $\alpha = \pm 1$ or $|\alpha| = q^{1/2}$. Define $\mu_\alpha := \psi(x)\chi_\alpha(x)dx \in \mathrm{Dist}(F^*,\mathbb{C})$ (resp. $\in \mathrm{Dist}(F,\mathbb{C})$ if $\alpha = 1$). We call it the *local distribution associated to* π_α (the justification for this terminology will become apparent in section 4.6). μ_α is the image of a Whittaker functional under (27) (see Prop. 2.10 below).

Proposition 2.9. *Let $\chi : F^* \to \mathbb{C}^*$ be a quasicharacter with conductor $\mathfrak{p}^f$. Assume that $|\chi(\varpi)| < q^{1/2}$. Then the integral $\int_{F^*}\chi(x)\mu_\alpha(dx)$ converges and we have*

$$\int_{F^*} \chi(x)\mu_\alpha(dx) = \tau(\chi)e(\alpha,\chi)L(\tfrac{1}{2},\pi_\alpha\otimes\chi)$$

where

$$e(\alpha,\chi) \;=\; \begin{cases} (1-\alpha\chi(\varpi)^{-1}) & \textit{if } f=0, \alpha=\pm 1;\\ \left(1-\frac{\chi(\varpi)}{\alpha}\right)\left(1-\frac{1}{\alpha\chi(\varpi)}\right) & \textit{if } f=0, \alpha\neq\pm 1;\\ \alpha^{-f} & \textit{if } f>0. \end{cases}$$

Proof. Recall that for the local L-factors we have $L(s,\pi_\alpha\otimes\chi)=1$ if $f>0$ and

$$L(s,\pi_\alpha\otimes\chi) = (1-\chi(\varpi)\alpha q^{-(s+1/2)})^{-1}$$

if $\alpha=\pm 1, f=0$ and

$$\begin{aligned} L(s,\pi_\alpha\otimes\chi) &= L(s,\chi\chi_\alpha^{-1}|\cdot|^{-1/2})L(s,\chi\chi_\alpha|\cdot|^{1/2})\\ &= (1-\chi(\varpi)\alpha^{-1}q^{-(s-1/2)})^{-1}(1-\chi(\varpi)\alpha q^{-(s+1/2)})^{-1}\end{aligned}$$

if $\alpha\neq\pm 1, f=0$. Thus the assertion follows from Lemma 2.4. □

Proposition 2.10. *(a) There exists a unique Whittaker functional $\lambda=\lambda_a$ for $\mathscr{B}_a(F,\mathbb{C})$ such that $\delta^\alpha(\lambda_a)=\mu_\alpha$.*

(b) Let $\mathcal{W}_\alpha=\mathcal{W}(\pi_\alpha)$ denote the Whittaker model of π_α. If $\alpha\neq 1$ (resp. $\alpha=1$) then for any $f\in C_c(F^*,\mathbb{C})$ (resp. $f\in C_c(F,\mathbb{C})$) there exists $W=W_f\in\mathcal{W}_\alpha$ such that

$$\int_{F^*}(af)(x)\,\mu_\alpha(dx) = W\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}$$

for all $a\in F^*$ (resp. $\int_F (af)(x)\,\mu_\alpha(dx) = W\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}$).

(c) Let H be an open subgroup of U and put $W_H=W_{1_H}$. Then, for any $f\in C_c^0(F^*,\mathbb{C})^H$ we have

$$\int_{F^*} f(x)\mu_\alpha(dx) = [U:H]\int_{F^*} f(x)\,W_H\begin{pmatrix} x & 0\\ 0 & 1\end{pmatrix}d^\times x.$$

Proof. (a) We let the (additive) group F act on the Schwartz space $C_c(F,\mathbb{C})$ as usual by $(x\cdot f)(y)\colon = f(y-x)$. Thus the functional

$$\Lambda: C_c(F,\mathbb{C}) \longrightarrow \mathbb{C},\; f\mapsto \int_F f(x)\,\psi(x)\,dx$$

satisfies $\Lambda(xf)=\psi(x)\Lambda(f)$ for all $x\in F$ and $f\in C_c(F,\mathbb{C})$. Also we let F act on $C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C}$ as $x\phi\colon = \begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\phi$ so that

$$C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C} \longrightarrow C_c(F,\mathbb{C}),\; \phi\mapsto f(x)\colon = \phi([x:1])-\phi(\infty) \tag{28}$$

is an F-equivariant isomorphism. Thus the composite

$$\mathrm{St}(F,\mathbb{C}) = C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C} \xrightarrow{(28)} C_c(F,\mathbb{C}) \xrightarrow{\Lambda} \mathbb{C} \tag{29}$$

is a Whittaker functional of the Steinberg representation. It follows from Lemma 2.7 (a), (b) that for $\alpha=\pm 1$ the composition

$$\lambda: \mathscr{B}_a(F,\mathbb{C}) \xrightarrow{\delta_\alpha^{-1}} C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C} \xrightarrow{(28)} C_c(F,\mathbb{C}) \xrightarrow{\Lambda} \mathbb{C}$$

is a Whittaker functional.

Assume now $\alpha \neq \pm 1$ and let $\lambda : \mathscr{B}_a(F,\mathbb{C}) \to \mathbb{C}$ be a Whittaker functional. Since $\langle u\phi, \rho\rangle = \langle \phi, u\rho\rangle = \langle \phi, \rho\rangle$ for all $\phi \in \mathscr{B}_a(F,\mathbb{C})$ and $u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ the map $\langle \cdot, \rho\rangle : \mathscr{B}_a(F,\mathbb{C}) \longrightarrow \mathbb{C}$, $\phi \mapsto \langle \phi, \rho\rangle$ is not a Whittaker functional. Therefore by Lemma 2.7 (a), (c) the map $\lambda \circ \widetilde{\delta}_\alpha : \mathrm{St}(F,\mathbb{C}) = C^0(\mathbb{P}^1(F),\mathbb{C})/\mathbb{C} \to \mathscr{B}_a(F,\mathbb{C}) \to \mathbb{C}$ is a Whittaker functional of $\mathrm{St}(F,\mathbb{C})$ so – after replacing λ by a scalar multiple – we may assume that $\lambda \circ \widetilde{\delta}_\alpha$ is equal to the Whittaker functional (29). Then $\delta^\alpha(\lambda)(f) = (\lambda \circ \widetilde{\delta}_\alpha)(\chi_\alpha \cdot f) = \Lambda(\chi_\alpha \cdot f) = \int_{F^*} f(x)\,\chi_\alpha(x)\,\psi(x)\,dx = \mu_\alpha(f)$ for all $f \in C_c(F^*,\mathbb{C})$.

(b) By (a) the function $W(g) := \lambda(g \cdot \delta_\alpha(f))$ lies in $\mathcal{W}_\alpha$ and we have

$$\int_{F^*} (af)(x)\,\mu_\alpha(dx) = \lambda(\delta_\alpha(af)) = \lambda\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\delta_\alpha(f)\right) = W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) It is enough to consider the case $f = 1_{aH}$ for $a \in F^*$. Then

$$\begin{aligned} \int_{F^*} f(x)\mu_\alpha(dx) = \int_{F^*} (a1_H)(x)\mu_\alpha(dx) \ &= \ W_H\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \\ = m\int_{F^*} 1_H(x)\,W_H\begin{pmatrix} ax & 0 \\ 0 & 1 \end{pmatrix} d^\times x &= m\int_{F^*} f(x)\,W_H\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} d^\times x \end{aligned}$$

with $m = [U : H]$. □

2.7. **Extensions of the Steinberg representation.** In this section we assume that R is a topological Hausdorff ring. We consider certain extensions of the $R[G(F)]$-module $\mathrm{St}(F,R) = C(\mathbb{P}^1(F),R)/R$ associated to a continuous homomorphism ℓ from F^* to the additive group of R (for a related construction see [3], 2.1). Let

$$\pi : G(F) \longrightarrow \mathbb{P}^1(F),\ \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g\infty = [a : c]$$

be the canonical $G(F)$-equivariant projection. Note that

$$\delta : C_\diamond(F,R) \longrightarrow \mathrm{St}(R),\ f \mapsto \delta(f)(P) = \begin{cases} f(x) & \text{if } P = [x:1], \\ 0 & \text{if } P = \infty \end{cases}$$

is an isomorphism of $R[T(F)]$-modules (its inverse δ^{-1} is given by $\delta^{-1}(\phi)(x) = \phi([x:1]) - \phi(\infty))$. Define $\widetilde{\mathscr{E}}(\ell)$ as the R-module of pairs $(f,y) \in C(G(F),R) \times R$ with

$$\phi\left(g \cdot \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix}\right) = \phi(g) + \ell(t_1/t_2)y$$

for all $t_1, t_2 \in F^*$, $u \in F$ and $g \in G(F)$. We denote by $\widetilde{\mathscr{E}}(\ell)_0 \cong R$ the submodule consisting of pairs $(\phi, 0)$ with $\phi : G(F) \to R$ constant and put $\mathscr{E}(\ell) = \widetilde{\mathscr{E}}(\ell)/\widetilde{\mathscr{E}}(\ell)_0$. The left $G(F)$-action on $\widetilde{\mathscr{E}}(\ell)$ given by $g \cdot (\phi(h), y) = (\phi(g^{-1}h), y)$ induces a $G(F)$-action on $\mathscr{E}(\ell)$.

Lemma 2.11. *(a) Let $\epsilon : \mathscr{E}(\ell) \to R$ be given by $\epsilon(\phi, y) = y$. Then the sequence of $R[G(F)]$-modules*

$$0 \longrightarrow \mathrm{St}(F,R) \xrightarrow{\phi\mapsto(\phi\circ\pi,0)} \mathscr{E}(\ell) \xrightarrow{\ \epsilon\ } R \longrightarrow 0 \tag{30}$$

is exact.

(b) Let $\delta^* : H^1(G(F), \mathrm{St}(F,R)) \to H^1(F^*, C_\diamond(F,R))$ be the homomorphism induced by the maps $\mathbb{G}_m \cong T \subseteq G, x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ and δ^{-1}, let $[\mathscr{E}(\ell)]$ denote the cohomology class of the extension (30) *and let $c_\ell \in H^1(F^*, C_\diamond(F,R))$ be the class of the cocycle*

$$z_\ell(a) := (1-a)(\ell \cdot 1_{\mathcal{O}}) := \begin{cases} \ell(a)1_{a\mathcal{O}} + \ell \cdot 1_{\mathcal{O}-a\mathcal{O}} & \text{if } \mathrm{ord}(a) \geq 0; \\ \ell(a)1_{a\mathcal{O}} - \ell \cdot 1_{a\mathcal{O}-\mathcal{O}} & \text{if } \mathrm{ord}(a) < 0. \end{cases} \tag{31}$$

Then, $\delta^*([\mathscr{E}(\ell)]) = 2c_\ell$.

(c) If $\ell = \mathrm{ord}_F : F \to \mathbb{Z} \to R$ and the topology on R is discrete then (30) *is isomorphic to* (17).

Proof. (a) It suffices to show that $\mathscr{E}(\ell) \to R, (\phi, y) \mapsto y$ is surjective. Define

$$\phi_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \ell\left(\frac{a^2}{ad-bc}\right) & \text{if } \mathrm{ord}(a) < \mathrm{ord}(c), \\ \ell\left(\frac{c^2}{ad-bc}\right) & \mathrm{ord}(a) \geq \mathrm{ord}(c) \end{cases}$$

so $\phi_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \ell\left(\frac{a^2}{ad-bc}\right) + 2\ell(c/a)1_{c\mathcal{O}}(a)$ if $a, c \neq 0$. One easily checks that $(\phi_0, 1) \in \mathscr{E}(\ell)$.

(b) Note that $\pi\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix} = [x:1]$ and $\pi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \infty$. Thus for $a \in F^*$ let $\phi \in \mathrm{St}(R)$ be given by $\phi \circ \pi = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \phi_0 - \phi_0$. Then for $x \in F$ we have

$$\begin{aligned} \delta^{-1}(\phi)(x) &= \phi_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}\right) - \phi_0\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}^{-1}\right) \\ &\quad -\phi_0\left(\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}\right) + \phi_0\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \ell(x^2/a) + 2\ell(a/x)1_{a\mathcal{O}}(x) - \ell(1/a) - (\ell(x^2) + 2\ell(1/x)1_{\mathcal{O}}(x)) \\ &= 2\ell(x)(1_{\mathcal{O}}(x) - 1_{a\mathcal{O}}(x)) + 2\ell(a)1_{a\mathcal{O}}(x) \ = \ 2z_\ell(a)(x). \end{aligned}$$

(c) Note that if $\ell = \mathrm{ord}$ then $\phi_0(k) = 0$ for all $k \in K$. Hence for $g = k \cdot \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \in G(F) = KB(F)$ and $h \in K$ we have $(h \cdot \phi_0)(g) = \phi_0(h^{-1}k) + \mathrm{ord}(t_1/t_2) = \mathrm{ord}(t_1/t_2) = \phi_0(g)$, i.e. $(\phi_0, 1)$ is K-invariant. By Frobenius reciprocity we obtain a homomorphism $\Psi : C_c(\mathcal{V}, R) \cong \mathrm{Ind}_K^{G(F)} R \to \mathscr{E}(\ell)$. One can easily verify that Ψ induces an isomorphism $\mathrm{Coker}(\delta \circ \delta_+^*) = \mathrm{Coker}(T - (q+1)\,\mathrm{id}) \cong \mathscr{E}(\ell)$ and that the sequence $C_c^+(\mathcal{E}, R) \xrightarrow{\delta} C_c(\mathcal{V}, R) \xrightarrow{\epsilon\circ\Psi} R \to 0$ is exact, so the assertion follows. $\square$

2.8. **Semi-local theory.** We briefly discuss how to generalize some of the previous constructions to the semi-local case. Let $F_1, \ldots, F_m$ be finite extensions of $\mathbb{Q}_p$ and let q_i be the number of elements of the residue field of F_i for $i = 1, \ldots, m$. We put $F = F_1 \times \ldots \times F_m$ and $F_S = \prod_{i \in S} F_i$ for a subset $S \subseteq \{1, \ldots, m\}$. Let R be a ring and $a_1, \ldots, a_m \in R$ put $\underline{a} = (a_1, \ldots, a_m)$ and define the $R[G(F)]$-module $\mathscr{B}_{\underline{a}}(F, R)$ as the tensor product of $\mathscr{B}_{a_1}(F_1, R), \ldots, \mathscr{B}_{a_m}(F_m, R)$

$$\mathscr{B}_{\underline{a}}(F, R) \;=\; \bigotimes_R \mathscr{B}_{a_i}(F_i, R).$$

To define the semi-local analogues of the maps (26) let $\alpha_1, \ldots, \alpha_m \in R^*$ and assume $a_i = \alpha_i + q_i/\alpha_i$ for $i = 1, \ldots, m$. Let $S_1 = \{i \in \{1, \ldots, m\} \,|\, \alpha_i = 1\}$ and $S_2 = S_1^c := \{1, \ldots, m\} - S_1$. It is easy to see that

$$\begin{aligned}(32) \quad \bigotimes_{i \in S_1} C_c^0(F_i, R) \otimes \bigotimes_{i \in S_2} C_c^0(F_i^*, R) \quad &\longrightarrow \quad C_c^0(F_{S_1} \times F_{S_2}^*, R), \\ \bigotimes_{i \in S_1} f_i \otimes \bigotimes_{i \in S_2} f_i \qquad\qquad &\mapsto \quad ((g_i)_{i=1,\ldots,m} \mapsto \prod_{i=1}^m f_i(g_i))\end{aligned}$$

is an isomorphism. We define the $R[T(F)]$-linear map

$$(33) \qquad \delta_{\underline{\alpha}} : C_c^0(F_{S_1} \times F_{S_2}^*, R) \;\longrightarrow\; \mathscr{B}_{\underline{a}}(F, R)$$

as the composite of the inverse of (32) and $\bigotimes_{i=1,\ldots,m} \delta_{\alpha_i}$.

For a R-module M we define $\mathscr{B}^{\underline{a}}(F, M) = \mathrm{Hom}_R(\mathscr{B}_{\underline{a}}(F, R), M)$ and let

$$(34) \qquad \langle\ ,\ \rangle : \mathscr{B}_{\underline{a}}(F, R) \times \mathscr{B}^{\underline{a}}(F, M) \longrightarrow M$$

be the evaluation pairing. If H is a subgroup of $G(F)$ and M a H-module then we define an H-action on $\mathscr{B}^{\underline{a}}(F, M)$ as before by $\langle \phi, h \cdot c\rangle = h \cdot \langle h^{-1}\phi, c\rangle$ for $h \in H$, $\phi \in \mathscr{B}_{\underline{a}}(F, R)$ and $c \in \mathscr{B}^{\underline{a}}(F, M)$. By passing in (33) to duals we get a $(T(F) \cap H)$-linear map

$$\delta^{\underline{\alpha}} : \mathscr{B}^{\underline{a}}(F, M) \;\longrightarrow\; \mathrm{Dist}(F_{S_1} \times F_{S_2}^*, M).$$

Note that $\mathscr{B}^{\underline{a}}(F, M) \cong \mathscr{B}^{\underline{a}}(F_S, \mathscr{B}^{\underline{a}}(F_{S^c}, M))$ for any subset S of $\{1, \ldots, m\}$. Note also that we have a canonical map $\mathscr{B}^{\underline{a}}(F, R) \otimes_R M \to \mathscr{B}^{\underline{a}}(F, M)$. In particular we get a map $\bigotimes_{i=1}^m \mathscr{B}^{a_i}(F_i, R) \to \mathscr{B}^{a_m}(F_m, \bigotimes_{i=1}^{m-1} \mathscr{B}^{a_i}(F_i, R))$ and by iterating this construction we get a homomorphism

$$(35) \qquad \bigotimes_{i=1}^m \mathscr{B}^{a_i}(F_i, R) \;\longrightarrow\; \mathscr{B}^{\underline{a}}(F, R).$$

Finally, we introduce the semi-local analogues of the maps (20). Let $\chi : F^* \to R^*$ be an unramified quadratic homomorphism, i.e. for each $i = 1, \ldots, m$ the restriction χ_i of χ to the factor F_i is unramified and $\chi^2 = 1$. Put $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_m) = (\chi_1(\varpi_1), \ldots, \chi_m(\varpi_m)) \in \{\pm 1\}^m$ so that $\chi_i = \chi_{\epsilon_i}$ in the notation of section 2.4. Define

$$(36) \qquad \mathfrak{Tw}_\chi = \bigotimes_{i=1,\ldots,m} \mathfrak{Tw}_{\epsilon_i} : \mathscr{B}_{\underline{a}}(F, R) \longrightarrow \mathscr{B}_{\underline{\epsilon a}}(F, R).$$

Again $\chi(\det(g))\,\mathfrak{Tw}_\chi(g\phi) = g\mathfrak{Tw}_\chi(\phi)$ holds for all $g \in G(F)$ and $\phi \in \mathscr{B}_{\underline{a}}(F,R)$. Also for a subgroup H of $G(F)$ and a $R[H]$-module M the isomorphism (36) induces an isomorphism of R-modules

$$\mathfrak{Tw}_\chi : \mathscr{B}^{\underline{\epsilon a}}(F,M) \longrightarrow \mathscr{B}^{\underline{a}}(F,M)$$

which is adjoint to (36) with respect to the pairing (34) and satisfies $h\mathfrak{Tw}_\chi(c) = \chi(\det(h))\mathfrak{Tw}_\chi(hc)$ for all $h \in H$ and $c \in \mathscr{B}^{\underline{a}}(F,M)$.

3. Special zeros of p-adic L-functions

Notation. We introduce the following notation which will be used throughout the rest of this paper. F denotes a totally real number field of degree $d+1$ over $\mathbb{Q}$ with ring of integers $\mathcal{O}_F$. For a non-zero ideal $\mathfrak{a} \subseteq \mathcal{O}_F$ we set $N(\mathfrak{a}) = \sharp(\mathcal{O}_F/\mathfrak{a})$. We denote by $\mathbf{P}_F$ the set of all places of F and by $\mathbf{P}_F^\infty$ (resp. S_∞) the subset of finite (resp. infinite) places. For a prime number ℓ, we shall write S_ℓ for the set of places above ℓ. We denote by $\sigma_0,\ldots,\sigma_d$ the different embeddings of F into $\mathbb{R}$ and let $\infty_0,\ldots,\infty_d$ be the corresponding archimedian places of F. Elements of $\mathbf{P}_F$ will be denoted by v,w or also by $\mathfrak{p},\mathfrak{q}$ if they are finite. If $\mathfrak{p} \in \mathbf{P}_F^\infty$, we denote the corresponding prime ideal of $\mathcal{O}_F$ also by $\mathfrak{p}$. For $v \in \mathbf{P}_F$, we denote by F_v the completion of F at v. If v is finite then $\mathcal{O}_v$ denotes the valuation ring of F_v and ord_v the corresponding normalized (additive) valuation on F_v (so $\mathrm{ord}_v(\varpi) = 1$ if $\varpi \in \mathcal{O}_v$ is a local uniformizer at v). Also for $v \in \mathbf{P}_F$ we let $|\cdot|_v$ be the associated normalized multiplicative valuation on F_v. Thus if $v \in S_\infty$ corresponds to the embedding $\sigma : F \to \mathbb{R}$ then $|x|_v = |\sigma(x)|$ and if $v = \mathfrak{q}$ is finite then $|x|_\mathfrak{q} = N(\mathfrak{q})^{-\mathrm{ord}_\mathfrak{q}(x)}$. For $v \in \mathbf{P}_F$ we put $U_v = \mathbb{R}_+^*$ if v is infinite and $U_v = \mathcal{O}_v^*$ if v is finite. Moreover if $v = \mathfrak{p}$ is finite and $n \geq 0$, then we also put $U_v^{(n)} = \{x \in U_v |\ \mathrm{ord}_v(x-1) \geq n\}$.

Let $\mathbf{A} = \mathbf{A}_F$ be the adele ring of F and $\mathbf{I} = \mathbf{I}_F$ the group of ideles. Let $|\cdot| : \mathbf{I}_F \longrightarrow \mathbb{R}^*$ be the absolute modulus, i.e. $|(x_v)_v| = \prod_v |x_v|_v$ for $(x_v)_v \in \mathbf{I}_F$. For a finite subset $S \subseteq \mathbf{P}_F$ we let $\mathbf{A}^S$ (resp. $\mathbf{I}^S$) denote the S-adeles (resp. S-ideles) and put $F_S = \prod_{v\in S} F_v$. We also define $U^S = \prod_{v\notin S} U_v$ and $U_S = \prod_{v\in S} U_v$. For $T \subseteq \mathbf{P}_\mathbb{Q} = \{2,3,5,\ldots,\infty\}$ and $S = \{v \in \mathbf{P}_F \mid v|_\mathbb{Q} \in T\}$ we often write F_T, $\mathbf{A}^T$, $\mathbf{I}^T$ etc. for F_S, $\mathbf{A}^S$, $\mathbf{I}^S$ etc. We also write U^p, U_p, $U^{p,S}$, $U^{p,\infty}$ etc. for $U^{\{p\}}$, $U_{\{p\}}$, $U^{S_p\cup S}$, $U^{S_p\cup S_\infty}$ etc. and use a similar notation for adeles and ideles. Thus for example for a finite subset S of $\mathbf{P}_F^\infty$, $\mathbf{I}^{S,\infty}$ denotes the set of $S\cup S_\infty$-ideles and for $\ell \in \mathbf{P}_\mathbb{Q}$ we have $F_\ell = F\otimes\mathbb{Q}_\ell = \prod_{v\in S_\ell} F_v$.

We fix an (additive) character $\psi : \mathbf{A} \to \mathbb{C}^*$ which is trivial on F. For $v \in \mathbf{P}_F$ let ψ_v denote the restriction of ψ to $F_v \hookrightarrow \mathbf{A}$. For convenience we choose ψ so that $\mathrm{Ker}(\psi_\mathfrak{p}) = \mathcal{O}_\mathfrak{p}$ for all $\mathfrak{p} \in S_p$. Let dx (resp. dx_v) denote the associated self-dual Haar measure on $\mathbf{A}$ (resp. on F_v). Thus $dx = \prod_v dx_v$. For $v \in \mathbf{P}_F$ we define a normalized Haar measure $dx_v^\times$ on F_v^* by $dx_v^\times = m_v\frac{dx_v}{|x_v|_v}$ where $m_v = (1-\frac{1}{N(v)})^{-1}$ if $v \in \mathbf{P}_F^\infty$ and $m_v = 1$ if $v \in S_\infty$. For a character $\chi : \mathbf{I}/F^* \to \mathbb{C}^*$ and $v \in \mathbf{P}_F$ we denote by χ_v its v-component, i.e. $\chi_v : F_v^* \hookrightarrow \mathbf{I} \xrightarrow{\chi} \mathbb{C}^*$. The *Gauss sum* $\tau(\chi) = \tau(\chi,\psi)$ of χ is then defined as $\tau(\chi) = \prod_{\mathfrak{p}|\mathfrak{f}(\chi)} \tau(\chi_\mathfrak{p})$.

We denote by F_+^* the totally positive elements of F and by $G(F)^+$ (resp. $G(F_\infty)^+$) the subgroup of $G(F)$ (resp. $G(F_\infty)$) of elements with totally positive determinant. The subgroups $B(F)^+ \subseteq B(F)$ and $T(F)^+ \subseteq T(F)$ are defined similarly. Furthermore we define subgroups $K_\infty^+ \subseteq K_\infty \subseteq G(F_\infty)$ as the image of $\mathrm{O}(2)^{\mathrm{Hom}(F,\mathbb{R})} \subseteq \mathrm{GL}_2(F_\infty)$ and $\mathrm{SO}(2)^{\mathrm{Hom}(F,\mathbb{R})}$ under the projection $\mathrm{GL}_2(F_\infty) \to G(F_\infty)$ (thus $K_\infty^+ = K_\infty \cap G(F_\infty)^+$).

There is a canonical $G(F_\infty)^+$-action on $\mathbb{H}^{d+1}$ where $\mathbb{H}\colon = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$; the embeddings $\sigma_0, \dots, \sigma_d$ allow us to identify $G(F_\infty)^+$ with $(G(\mathbb{R})^+)^{d+1}$ and the latter group acts on $\mathbb{H}^{d+1}$ through linear transformations factor-by-factor. For $g = (g_0, \dots, g_d) \in G(F_\infty)^+$ and $\underline{z} = (z_0, \dots, z_d) \in \mathbb{H}^{d+1}$ we define $j(g, \underline{z}) = \prod_{\nu=0}^d j(g_\nu, z_\nu)$ where $j(\gamma, z) = \det(\gamma)^{-1/2}(cz+d)$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{R})^+$, $z \in \mathbb{H}$.

Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}_F$. For $v \in \mathbf{P}_F^\infty$ we put $K_0(\mathfrak{n})_v = \{A \in G(\mathcal{O}_v) |\ A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \mathfrak{n}\mathcal{O}_v\}$ and set $K_0(\mathfrak{n}) = \prod_{v \in \mathbf{P}_F^\infty} K_0(\mathfrak{n})_v$. If $S \subseteq \mathbf{P}_F^\infty$ we also put $K_0(\mathfrak{n})^S = \prod_{v \in \mathbf{P}_F^\infty - S} K_0(\mathfrak{n})_v$.

3.1. **Rings of functions on ideles and adeles.**

The module $C_c^\flat(F_v, K)$. Let v be a finite place of F and let K be a Hausdorff topological field (in the application v will be a place above p and K a p-adic field). We identify $C_c(F_v^*, K)$ with the submodule $\{f \in C_c(F_v, K) \mid f \equiv 0 \text{ near } 0\}$ of $C_c(F_v, K)$ and define

$$C_c^\flat(F_v, K) \quad = \quad C_c^0(F_v, K) + C_c(F_v^*, K).$$

Both $C_c(F_v^*, K)$ and $C_c^\flat(F_v, K)$ are F_v^*-submodules of $C_c(F_v, K)$. For $f \in C_c(F_v^*, K)^{U_v}$ and $x \in F_v^*$ the infinite sum

$$(\sum_{n=0}^\infty \varpi^n f)(x)\colon = \sum_{n=0}^\infty f(\varpi^{-n} x)$$

is finite and one easily checks that $F_v^* \to K$, $x \mapsto (\sum_{n=0}^\infty \varpi^n f)(x)$ extends to a function in $C_c^0(F_v, K)$ which will be denoted by $(1-\varpi)^{-1} f$. For example if $f = 1_{U_v}$ then $(1-\varpi)^{-1} f = 1_{\mathcal{O}_v}$. Thus we obtain a F_v^*-equivariant K-linear monomorphism

$$C_c(F_v^*, K)^{U_v} \longrightarrow C_c^0(F_v, K), \ \ f \mapsto (1-\varpi)^{-1} f. \tag{37}$$

Its image is $C_c^0(F_v, K)^{U_v}$. Hence if we consider the following two-step filtration $\mathcal{F}_v^\bullet$ on $C_c^\flat(F_v, K)$

$$\mathcal{F}_v^0 = C_c^\flat(F_v, K), \quad \mathcal{F}_v^1 = C_c^0(F_v, K)^{U_v}, \quad \mathcal{F}_v^2 = 0 \tag{38}$$

then we have for the associated graded F_v^*-modules $\mathrm{gr}^n_{\mathcal{F}_v} = \mathcal{F}_v^n / \mathcal{F}_v^{n+1}$

$$\mathrm{gr}^n_{\mathcal{F}_v} \ \cong \ \begin{cases} C_c(F_v^*, K)/C_c(F_v^*, K)^{U_v} & \text{if } n = 0, \\ C_c(F_v^*, K)^{U_v} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

Note also that

$$C_c(F_v^*,K)^{U_v} \;\cong\; C_c(F_v^*/U_v,K) \;\cong\; \mathrm{Ind}_{U_v}^{F_v^*} K, \tag{40}$$
$$C_c(F_v^*,K)/C_c(F_v^*,K)^{U_v} \;\cong\; \mathrm{Ind}_{U_v}^{F_v^*}(C_c(U_v^*,K)/K).$$

The module $\mathscr{C}_c^\flat(S_1,S_2,K)$. Consider now two (possibly empty) disjoint subsets S_1, S_2 of S_p and let R be a topological Hausdorff ring. We define

$$\begin{aligned} \mathscr{C}(S_1,S_2,R) &= C(F_{S_1}\times F_{S_2}^*\times \mathbf{I}^{p,\infty}/U^{p,\infty},R),\\ \mathscr{C}_\diamond(S_1,S_2,R) &= C_\diamond(F_{S_1}\times F_{S_2}^*\times \mathbf{I}^{p,\infty}/U^{p,\infty},R),\\ \mathscr{C}_c(S_1,S_2,R) &= C_c(F_{S_1}\times F_{S_2}^*\times \mathbf{I}^{p,\infty}/U^{p,\infty},R),\\ \mathscr{C}^0(S_1,S_2,R) &= C^0(F_{S_1}\times F_{S_2}^*\times \mathbf{I}^{p,\infty}/U^{p,\infty},R). \end{aligned}$$

We have

$$\mathscr{C}_c^0(S_1,S_2,R)\subseteq \mathscr{C}_c(S_1,S_2,R)\subseteq \mathscr{C}_\diamond(S_1,S_2,R)\subseteq \mathscr{C}(S_1,S_2,R).$$

Note that if R carries the discrete topology the first three rings are all equal.

Assume now $S_1\dot\cup S_2=S_p$ and assume that $K=R$ is a field. We define the submodule $\mathscr{C}_c^\flat(S_1,S_2,K)$ of $\mathscr{C}_c(S_1,S_2,K)$ as the image of the embedding

$$\bigotimes_{v\in S_1} C_c^\flat(F_v,K)\otimes C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},K)\longrightarrow \mathscr{C}_c(S_1,S_2,K).$$

We have $\mathscr{C}_c^0(S_1,S_2,K)\subseteq \mathscr{C}_c^\flat(S_1,S_2,K)\subseteq \mathscr{C}_c(S_1,S_2,K)$. The filtrations (38) on $C_c^\flat(F_v,K)$ for all $v\in S_1$ induce a filtration $\mathcal{F}^\bullet$ on $\mathscr{C}_c^\flat(S_1,S_2,K)$. For $\underline{n}=(n_v)_{v\in S_1}\in\mathbb{Z}^{S_1}$ put $|\underline{n}|=\sum_v n_v$. Then $\mathcal{F}^m C_c^\flat(F_v,K)$ is defined as the image of

$$\bigoplus_{\underline{n}\in\mathbb{Z}^{S_1},|\underline{n}|=m}\left(\bigotimes_{v\in S_1}\mathcal{F}_v^{n_v}C_c^\flat(F_v,K)\right)\otimes C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},K)\longrightarrow \mathscr{C}_c(S_1,S_2,K).$$

We get for the associated graded quotients $\mathrm{gr}_{\mathcal{F}}^m=\mathcal{F}^m/\mathcal{F}^{m+1}$

$$\mathrm{gr}_{\mathcal{F}}^m = \bigoplus_{|\underline{n}|=m}\left(\bigotimes_{v\in S_1}\mathrm{gr}_{\mathcal{F}_v}^{n_v}\right)\otimes C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},K). \tag{41}$$

Let E_+ be the group of totally positive units of $\mathcal{O}_F$. We fix a splitting of the exact sequence $1\to E_+\to F_+^*\to\Gamma:=F_+^*/E_+\to 1$, i.e. we fix a subgroup $\mathcal{T}\subseteq F_+^*$ such that $F_+^*=E_+\times\mathcal{T}$.

Proposition 3.1. *$\mathscr{C}_c^\flat(S_1,S_2,K)$ is a free $K[\mathcal{T}]$-module.*

Proof. It is enough to prove that each graded quotient $\mathrm{gr}_{\mathcal{F}}^m$ and therefore each summand in (41) is a free $K[\mathcal{T}]$-module. Since

$$C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},K)\;\cong\;\mathrm{Ind}_{U^{S_1,\infty}}^{\mathbf{I}^{S_1,\infty}} C_c(U^{S_1,\infty}/U^{p,\infty},K)$$

we deduce using (39) and (40) that each summand in (41) is isomorphic to a $K[\mathbf{I}^\infty]$-module of the form $\mathrm{Ind}_{U^\infty}^{\mathbf{I}^\infty}V$ for some $K[U^\infty]$-module V. Hence it is free a $K[\mathcal{T}]$-module. Indeed, since by assumption $\mathcal{T}\cap U^\infty=1$ we have $\mathrm{Ind}_{U^\infty}^{\mathbf{I}^\infty}V=\bigoplus_{i=1}^h\mathrm{Ind}_1^{\mathcal{T}}x_iV$ (as $\mathcal{T}$-modules) where $\{x_1,\dots,x_h\}$ is a set of representatives of $\mathbf{I}^\infty/U^\infty\mathcal{T}=\mathbf{I}^\infty/U^\infty F_+^*$ (cf. [4], Prop. 5.6, p. 69). $\square$

3.2. **Computation of $\partial((\log_p \circ \mathcal{N})^k)$ for $k = 0, \ldots, r$.**

Definition of ∂. Assume again that $S_1 \dot\cup S_2 = S_p$ and that R is a topological Hausdorff ring. Let $\mathcal{G}_p = \mathrm{Gal}(M/F)$ be the Galois group of the maximal abelian extension M/F which is unramified outside p and ∞. We shall now construct a canonical homomorphism

$$\partial : C(\mathcal{G}_p, R) \longrightarrow H_d(F_+^*, \mathscr{C}_c(S_1, S_2, R)). \tag{42}$$

Firstly, there exists an isomorphism

$$C(\mathcal{G}_p, R) \longrightarrow H_0(F_+^*/E_+, H^0(E_+, \mathscr{C}_c(\emptyset, S_p, R))) \tag{43}$$

defined as follows. Let $\overline{E}_+$ be the closure of E_+ in U_p and let $\mathrm{pr} : \mathbf{I}^\infty/U^{p,\infty} \to \mathbf{I}^\infty/(\overline{E}_+ \times U^{p,\infty})$ denote the projection. The map

$$C_c(\mathbf{I}^\infty/(\overline{E}_+ \times U^{p,\infty}), R) \longrightarrow H^0(E_+, C_c(\mathbf{I}^\infty/U^{p,\infty}, R)),\ f \mapsto f \circ \mathrm{pr}$$

is an isomorphism. Hence its inverse induces an isomorphism

$$H_0(F_+^*/E_+, H^0(E_+, C_c(\mathbf{I}^\infty/U^{p,\infty}, R))) \cong H_0(F_+^*/E_+, C_c(\mathbf{I}^\infty/(\overline{E}_+ \times U^{p,\infty}), R)). \tag{44}$$

The reciprocity map of class field theory $\rho : \mathbf{I}/F^* \to \mathcal{G}_p$ induces a surjection $\overline{\rho} : \mathbf{I}^\infty/(\overline{E}_+ \times U^{p,\infty}) \to \mathcal{G}_p$ whose kernel is discrete and $\cong F_+^*/E_+$. It follows that the map

$$\rho^\sharp : H_0(F_+^*/E_+, C_c(\mathbf{I}^\infty/(\overline{E}_+ \times U^{p,\infty}), R)) \longrightarrow C(\mathcal{G}_p, R) \tag{45}$$

defined by $\rho^\sharp([f])(\overline{\rho}(x)) = \sum_{\zeta \in F_+^*/E_+} f(\zeta x)$ is an isomorphism as well. The map (43) is the composite of (44) with (45).

Let A be any F_+^*-module. Next we construct a homomorphism

$$H_0(F_+^*/E_+, H^0(E_+, A)) \longrightarrow H_d(F_+^*, A) \tag{46}$$

Since $E_+ \cong \mathbb{Z}^d$ we have $H_d(E_+, \mathbb{Z}) \cong \mathbb{Z}$. Choose a generator η of $H_d(E_+, \mathbb{Z})$. Since the action of F_+^*/E_+ on $H_d(E_+, \mathbb{Z})$ is trivial, taking the cap product with η yields an F_+^*/E_+-equivariant map $H^0(E_+, A) \to H_d(E_+, A)$ hence

$$H_0(F_+^*/E_+, H^0(E_+, A)) \longrightarrow H_0(F_+^*/E_+, H_d(E_+, A)) \tag{47}$$

We define (46) as the composite of (47) with the edge morphism

$$H_0(F_+^*/E_+, H_d(E_+, A)) \to H_d(F_+^*, A) \tag{48}$$

of the Hochschild-Serre spectral sequence.

There is in fact a canonical choice for η. Consider the action of E_+ on $\mathbb{R}_0^{d+1} = \{(x_0, \ldots, x_d) \in \mathbb{R}^{d+1} | \sum_{i=0}^d x_i = 0\}$ given by $a \cdot (x_0, \ldots, x_d) = (\log(\sigma_0(a)) + x_0, \ldots, \log(\sigma_d(a)) + x_d)$. The d-dimensional manifold $\mathbb{R}_0^{d+1}/E_+$ is oriented and compact. We chose $\eta \in H_d(E_+, \mathbb{Z})$ so that it corresponds to the fundamental class under the canonical isomorphism $H_d(E_+, \mathbb{Z}) \cong H_d(\mathbb{R}_0^{d+1}/E_+, \mathbb{Z})$ (thus η depends on our chosen ordering of the real places of F).

Finally we define (42) is the composite of (43), (46) (for $A = \mathscr{C}_c(\emptyset, S_p, R)$) and the map $H_d(F_+^*, \mathscr{C}_c(\emptyset, S_p, R)) \longrightarrow H_d(F_+^*, \mathscr{C}_c(S_1, S_2, R))$ induced by the inclusion $\mathscr{C}_c(\emptyset, S_p, R) \subseteq \mathscr{C}_c(S_1, S_2, R)$.

Fundamental homology classes. We put $r = \sharp(S_1)$, $m = \sharp(S_p)$ and order the places above p, so that $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ and $S_2 = \{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_m\}$. Beside $\eta \in H_d(E_+, \mathbb{Z})$ we consider two more canonical homology classes ϑ and ϱ. To begin with we introduce the following F_+^*-action on $\mathbb{R}^{d+1}$, $\mathbb{R}^r$ and $\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}$

$$\begin{aligned} a \cdot (x_0, \ldots, x_d) \quad &:= \quad (\log(\sigma_0(a)) + x_0, \ldots, \log(\sigma_d(a)) + x_d), \\ a \cdot (y_1, \ldots, y_r) \quad &:= \quad (\mathrm{ord}_{\mathfrak{p}_1}(a) + y_1, \ldots, \mathrm{ord}_{\mathfrak{p}_r}(a) + y_r), \\ a \cdot (x_v)_{v \notin S_1 \cup S_\infty} \quad &:= \quad (a x_v)_{v \notin S_1 \cup S_\infty}. \end{aligned}$$

Let M be the submanifold of $\mathbb{R}^{d+1} \times \mathbb{R}^r \times \mathbf{I}^{S_1,\infty}/U^{S_1,\infty}$ defined by the equation

$$\sum_{i=0}^{d} x_i - \left(\sum_{j=1}^{r} \log(N(\mathfrak{p}_j)) y_j\right) + \left(\sum_{v \notin S_1 \cup S_\infty} \log(|x_v|_v)\right) = 0$$

and put $M_1 := \mathbb{R}^r \times \mathbf{I}^{S_1,\infty}/U^{S_1,\infty}$. We have

$$H_0(M, \mathbb{Z}) \cong H_0(M_1, \mathbb{Z}) \cong C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}, \mathbb{Z})$$

where the first isomorphism is induced by the projection $M \to M_1$. The group F_+^* (resp. $\Gamma := F_+^*/E_+$) acts properly discontinuously on M (resp. on M_1) and the projection $\pi : M/F_+^* \to M_1/\Gamma$ is a fiber bundle with fiber $\cong \mathbb{R}^{d+1}/E_+$ (in fact it is easy to see that it is trivial i.e. it is homeomorphic to the trivial bundle $M_1/\Gamma \times \mathbb{R}^{d+1}/E_+$ over M_1/Γ). The base M_1/Γ is a compact oriented r-dimensional manifold.

Definition of ϑ. Define ϑ as the image of the fundamental class under the composition

$$\begin{aligned} &H_{d+r}(M/F_+^*, \mathbb{Z}) \cong H_{d+r}(F_+^*, H_0(M, \mathbb{Z})) \cong H_{d+r}(F_+^*, C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}, \mathbb{Z})) \\ (49) \qquad &\longrightarrow H_{d+r}(F_+^*, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty}, \mathbb{Z})) = H_{d+r}(F_+^*, \mathscr{C}_c(\emptyset, S_2, \mathbb{Z})) \end{aligned}$$

where the last map is induced by the projection $\mathbf{I}^{S_1,\infty}/U^{p,\infty} \to \mathbf{I}^{S_1,\infty}/U^{S_1,\infty}$. If R is arbitrary topological Hausdorff ring, then – by abuse of notation – we denote the image of ϑ under the canonical map $H_{d+r}(F_+^*, \mathscr{C}_c(\emptyset, S_2, \mathbb{Z})) \to H_{d+r}(F_+^*, \mathscr{C}_c(\emptyset, S_2, R))$ also by ϑ.

Definition of ϱ. Let $\mathcal{T}$ be any subgroup of F_+^* such that $\mathcal{T} \cap E_+ = \{1\}$ and $\mathcal{T} E_+$ has finite index in F_+^* (we are mainly interested in the case $F_+^* = E_+ \times \mathcal{T}$) so that the group $\mathcal{T}$ acts properly discontinuously on M_1. Let $\varrho_{\mathcal{T}} \in H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty}, \mathbb{Z}))$ be the image of the fundamental class of the oriented r-dimensional compact manifold $M_1/\mathcal{T}$ under the canonical map

$$\begin{aligned} H_r(M_1/\mathcal{T}, \mathbb{Z})) \cong H_r(\mathcal{T}, H_0(M_1, \mathbb{Z})) &\cong H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}, \mathbb{Z}) \\ &\longrightarrow H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty}, \mathbb{Z})) \end{aligned}$$

Remarks 3.2. (a) If $\mathcal{T}$ and $\mathcal{T}'$ are subgroups as above with $\mathcal{T}' \subseteq \mathcal{T}$ then we have $\mathrm{res}(\varrho_{\mathcal{T}}) = \varrho_{\mathcal{T}'}$.

(b) Let $\mathcal{T}_1 = \{x \in \mathcal{T} | \; \mathrm{ord}_{\mathfrak{q}}(x) = 0 \; \forall \, \mathfrak{q} \notin S_1\}$, let $\mathcal{T}_2$ be a subgroup of $\mathcal{T}$ with $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$ and let $\mathcal{F}_2$ be a fundamental domain for the action of $\mathcal{T}_2$ on $\mathbf{I}^{S_1,\infty}/U^{p,\infty}$ such that $U_{S_2}\mathcal{F} = \mathcal{F}$. Then $C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z}) \cong \mathrm{Ind}_{\mathcal{T}_1}^{\mathcal{T}} C(\mathcal{F}_2,\mathbb{Z})$ hence by Shapiro's Lemma

$$H_\bullet(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z})) \;\cong\; H_\bullet(\mathcal{T}_1, C(\mathcal{F}_2,\mathbb{Z})) \tag{50}$$

Let $\varrho_1 \in H_r(\mathcal{T}_1,\mathbb{Z}) \cong H_r(\mathbb{R}^r/\mathcal{T}_1,\mathbb{Z})$ be the fundamental class of $\mathbb{R}^r/\mathcal{T}_1$. Then $\varrho_1 \otimes 1_{\mathcal{F}_2}$ is mapped to $\varrho_{\mathcal{T}}$ under

$$H_r(\mathcal{T}_1,\mathbb{Z}) \otimes C(\mathcal{F}_2,\mathbb{Z})^{\mathcal{T}_1} \xrightarrow{\cap} H_r(\mathcal{T}_1, C(\mathcal{F}_2,\mathbb{Z})) \overset{(50)}{\cong} H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z})).$$

For a subgroup $\mathcal{T} \subseteq F_+^*$ such that $F_+^* = E_+ \times \mathcal{T}$ we shall explain the relation between the homology classes $\varrho_{\mathcal{T}}$, η and ϑ. Consider the Hochschild-Serre spectral sequence

$$E^2_{pq} = H_p(E_+, H_q(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z}))) \Rightarrow H_{p+q}(F_+^*, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z})).$$

Here we have $E^2_{pq} = 0$ if $p > d$ or $q > r$ (the latter follows from (50) above since $\mathcal{T}_1$ is free-abelian of rank r). Thus we get an isomorphism $E^2_{d,r} \cong E_{d+r}$ i.e.

$$H_d(E_+, H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty},\mathbb{Z}))) \cong H_{d+r}(F_+^*, C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty},\mathbb{Z})). \tag{51}$$

The next result follows easily from the definitions of η, $\varrho_{\mathcal{T}}$ and ϑ.

Lemma 3.3. *$\varrho_{\mathcal{T}}$ is mapped to ϑ under the composite*

$$\begin{aligned} H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z}))^{E_+} &\xrightarrow{\cap\eta} H_d(E_+, H_r(\mathcal{T}, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z}))) \\ &\overset{(51)}{\cong} H_{d+r}(F_+^*, C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{Z})). \end{aligned} \tag{52}$$

Recall the definition of cohomology classes defined in 2.11 (b).

Definition 3.4. *Let $\mathfrak{p} \in S_1$, let R be a topological Hausdorff ring and let $\ell : F_\mathfrak{p}^* \to R$ be a continuous homomorphism. We denote by $c_\ell \in H^1(F_\mathfrak{p}^*, C_c(F_\mathfrak{p}, R))$ the cohomology class of the 1-cocycle* (31) *(i.e. of the cocycle $z_\ell(a) := (1-a)(\ell \cdot 1_{\mathcal{O}_\mathfrak{p}})$ for $a \in F_+^*$).*

By abuse of notation we shall write c_ℓ instead of $\mathrm{res}(c_\ell) \in H^1(H, C_c(F_\mathfrak{p}, R))$ for any subgroup H of $F_\mathfrak{p}^*$. We are interested in the case $H = F_+^*$, $R = \mathbb{C}_p$ and either $\ell = \mathrm{ord}_\mathfrak{p}$ or $\ell = \log_p \circ \mathrm{N}_{F_\mathfrak{p}/\mathbb{Q}_p}$ and will derive a formula for $(c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta$ in both cases.

We begin with the first case. Let $H := \{x \in F_+^* \mid \mathrm{ord}_\mathfrak{p}(x) = 0 \;\forall\; \mathfrak{p} \in S_1\}$, $H_1 := \{x \in F_+^* \mid \mathrm{ord}_\mathfrak{p}(x) = 0 \;\forall\; \mathfrak{p} \in S_p\}$ and let $\mathcal{F}_1$ denote a fundamental domain for the action of H_1/E_+ on $\mathbf{I}^{p,\infty}/U^{p,\infty}$. Put $\mathcal{X} := \prod_{\mathfrak{p}\in S_1} \mathcal{O}_\mathfrak{p} \times \prod_{\mathfrak{p}\in S_2} \mathcal{O}_\mathfrak{p}^* \times \mathcal{F}_1 \subseteq F_{S_1} \times F_{S_2}^* \times \mathbf{I}^{p,\infty}/U^{p,\infty}$. The characteristic function of $\mathcal{X}$ clearly lies in $H^0(E_+, \mathscr{C}_c^0(S_1,S_2,\mathbb{Z}))$, hence defines an element $[1_\mathcal{X}] \in H_0(F_+^*/E_+, H^0(E_+, \mathscr{C}_c^0(S_1,S_2,\mathbb{Z})))$.

Proposition 3.5. *For $\mathfrak{p} \in S_p$ put $c_\mathfrak{p} = c_{\mathrm{ord}_\mathfrak{p}} \in H^1(F_+^*, C_c^0(F_\mathfrak{p},\mathbb{Z}))$. We have*

$$\epsilon([1_\mathcal{X}]) \;=\; (-1)^{\binom{r}{2}}\,(c_{\mathfrak{p}_1} \cup \ldots \cup c_{\mathfrak{p}_r}) \cap \vartheta$$

Here ϵ denotes the map (46) *for $A = \mathscr{C}_c^0(S_1,S_2,\mathbb{Z})$.*

Proof. Similar as above we denote by $\varrho_1 \in H_r(F_+^*/H, \mathbb{Z})$ the homology class which corresponds to the fundamental class of $\mathbb{R}^r/F_+^*$ under the natural isomorphism $H_r(F_+^*/H, \mathbb{Z}) \cong H_r(\mathbb{R}^r/F_+^*, \mathbb{Z})$. By taking the cap product with η we can identify $C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}, \mathbb{Z})^{E_+}$ with $H_d(E_+, C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}, \mathbb{Z}))$. Note that $\mathcal{F}\colon = \{1\} \times \mathcal{F}_1 \subseteq F_{S_2}^*/U_{S_2} \times \mathbf{I}^{p,\infty}/U^{p,\infty} = \mathbf{I}^{S_1,\infty}/U^{S_1,\infty}$ is a fundamental domain for the action of $\overline{H} = H/E_+$. Hence if $\mathbb{D}\colon = C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty}, \mathbb{Z})$ then $H_0(\overline{H}, \mathbb{D}) = H_0(\overline{H}, \mathrm{Ind}^{\overline{H}}\ C(\mathcal{F}, \mathbb{Z})) \cong C(\mathcal{F}, \mathbb{Z})$ and $H_q(\overline{H}, \mathbb{D}) = 0$ for $q > 0$. Consider the Hochschild-Serre spectral sequence

$$E_{pq}^2 = H_p(F_+^*/E_+, H_q(E^+, \mathbb{D})) \Rightarrow E_{p+q} = H_{p+q}(F_+^*, \mathbb{D}).$$

We have $E_{pq}^2 = 0$ if $q > d$ and $E_{pd}^2 \cong H_p(F_+^*/E_+, \mathbb{D}) \cong H_p(F_+^*/H, H_0(\overline{H}, \mathbb{D})) = 0$ if $p > r$. It follows $E_{d+r} \cong E_{rd}^2$. Define

$$\begin{aligned} (53) \qquad H_r(F_+^*/E_+, \mathbb{D}) &\xrightarrow{\cap\eta} H_r(F_+^*/E_+, H_d(E^+, \mathbb{D})) \cong H_{d+r}(F_+^*, \mathbb{D}) \\ &\xrightarrow{(49)} H_{d+r}(F_+^*, \mathscr{C}_c(\emptyset, S_2, \mathbb{Z})). \end{aligned}$$

Using Lemma 3.3 it is easy to see that $[1_{\mathcal{F}}] \otimes \varrho_1$ is mapped to ϑ under

$$\begin{aligned} H^0(F_+^*/H, H_0(\overline{H}, \mathbb{D})) \otimes H_r(F_+^*/H, \mathbb{Z}) &\xrightarrow{\cap} H_r(F_+^*/H, H_0(\overline{H}, \mathbb{D})) \\ \cong H_r(F_+^*/E_+, \mathbb{D}) &\xrightarrow{(53)} H_{d+r}(F_+^*, \mathscr{C}_c(\emptyset, S_2, \mathbb{Z})). \end{aligned}$$

Note that we can view $c_{\mathfrak{p}} = c_{\mathrm{ord}_{\mathfrak{p}}}$ as an element of $H^1(F_+^*/H, C_c^0(F_{\mathfrak{p}}, \mathbb{Z})^H)$. The assertion thus follows from

$$(-1)^{\binom{r}{2}}\,(c_{\mathfrak{p}_1} \cup \ldots \cup c_{\mathfrak{p}_r}) \cap \varrho_1 = [1_{\mathcal{O}_{S_1}}] \in H_0(F_+^*/H, C_c^0(F_{S_1}, \mathbb{Z})^H)$$

where $\mathcal{O}_{S_1} = \prod_{\mathfrak{p} \in S_1} \mathcal{O}_{\mathfrak{p}}$. For that put $z_{\mathfrak{p}} = z_{\mathrm{ord}_{\mathfrak{p}}}$ for $\mathfrak{p} \in S_1$ and choose generators $t_1, \ldots, t_r \in F_+^*/H$ such that $\mathrm{ord}_{\mathfrak{p}_i}(t_i) = 1$ and $\mathrm{ord}_{\mathfrak{p}_j}(t_i) = 0$ for all $j \neq i$, $1 \leq j \leq r$. Note that $z_{\mathrm{ord}_{\mathfrak{p}_i}}(t_i) = t_i 1_{\mathcal{O}_{\mathfrak{p}_i}}$ and $z_{\mathrm{ord}_{\mathfrak{p}_i}}(t_j) = 0$ for $j \neq i$. Since the fundamental class of $\mathbb{R}^r/\langle t_1, \ldots, t_r\rangle$ is the cross product of the fundamental classes of $\mathbb{R}/\langle t_1\rangle, \ldots, \mathbb{R}/\langle t_r\rangle$ the r-cycle $\sum_{\sigma \in S_r} \mathrm{sign}(\sigma)\,[t_{\sigma(1)}|\ldots|t_{\sigma(r)}]$ is a representative of ϱ_1 (see [18], Ch. VIII, 8.8). Hence

$$\begin{aligned} &\sum_{\sigma \in S_r} \mathrm{sign}(\sigma)\, z_{\mathfrak{p}_1}(t_{\sigma(1)}) \otimes t_{\sigma(1)} z_{\mathfrak{p}_2}(t_{\sigma(2)}) \otimes \ldots \otimes t_{\sigma(1)} \ldots t_{\sigma(r-1)} z_{\mathfrak{p}_r}(t_{\sigma(r)}) \\ &= z_{\mathfrak{p}_1}(t_1) \otimes \ldots \otimes z_{\mathfrak{p}_r}(t_r) = \prod_{i=1}^r t_i \cdot 1_{\mathcal{O}_{S_1}} \end{aligned}$$

is a representative of $(-1)^{\binom{r}{2}}(c_{\mathfrak{p}_1} \cup \ldots \cup c_{\mathfrak{p}_r}) \cap \varrho_1$. □

We consider (42) for $R = \mathbb{C}_p$, i.e. $\partial : C(\mathcal{G}_p, \mathbb{C}_p) \longrightarrow H_d(F_+^*, \mathscr{C}_c(S_1, S_2, \mathbb{C}_p))$. Let $\mathcal{N} : \mathcal{G}_p \to \mathbb{Z}_p^*$ be defined by $\gamma\zeta = \zeta^{\mathcal{N}(\gamma)}$ for all p-power roots of unity ζ. The following proposition is the key computation of this paper.

Proposition 3.6. *For $\mathfrak{p} \in S_1$ put $\ell_{\mathfrak{p}}\colon = \log_p \circ \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p} : F_{\mathfrak{p}}^* \to \mathbb{C}_p$. We have*

(a) $\partial((\log_p \circ \mathcal{N})^k) = 0$ for all $k = 0, 1, \ldots, r-1$.

(b) $\partial((\log_p \circ \mathcal{N})^r) = (-1)^{\binom{r}{2}}(c_{\ell_1} \cup \ldots \cup c_{\ell_r}) \cap \vartheta$.

Proof. We choose again a subgroup $\mathcal{T} \subseteq F_+^*$ such that $F_+^* = E_+ \times \mathcal{T}$. We denote by $\ell : \mathbf{I} \to \mathbb{Q}_p$ the composite

$$\ell : \mathbf{I} \xrightarrow{\rho} \mathcal{G}_p \xrightarrow{\mathcal{N}} \mathbb{Z}_p^* \xrightarrow{\log_p} \mathbb{Q}_p$$

and for a place v of F let $\ell_v : F_v \hookrightarrow \mathbf{I}_F \xrightarrow{\ell} \mathbb{Q}_p$ be the v-component of ℓ. Note that for $x = (x_v) \in \mathbf{I}_F$ we have $\ell_v(x_v) = 0$ for almost all v and

$$\ell(x) \;= \sum_{\mathfrak{q}} \ell_{\mathfrak{q}}(x_v).$$

Let $\mathcal{F} \subseteq \mathbf{I}^\infty/U^{p,\infty}$ be a compact open fundamental domain for the action of $\mathcal{T}$ such that $U_p\mathcal{F} = \mathcal{F}$. The function $(\log_p \circ \mathcal{N})^k$ is mapped under the inverse of (43) to the class of $\ell^k 1_{\mathcal{F}}$.

Since $z_{\ell_{\mathfrak{p}}}(a) = (1-a)(\ell \cdot 1_{\mathcal{O}_{\mathfrak{p}}})$ is a 1-cocycle with values in $C_c^\flat(F_{\mathfrak{q}}, \mathbb{C}_p)$ we can view $c_{\ell_{\mathfrak{p}}}$ as an element of $H^1(F_+^*, C_c^\flat(F_{\mathfrak{q}}, \mathbb{C}_p))$. Therefore the right hand side of (b) can be viewed as an element of $H_d(F_+^*, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))$. Note also that $\ell^k 1_{\mathcal{F}}$ lies in $H^0(E_+, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))$. Therefore it suffices to show that the class $[\ell^k 1_{\mathcal{F}}] \in H_0(\Gamma, H^0(E_+, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)))$ is mapped to 0 (resp. to $(-1)^{\binom{r}{2}}(c_{\ell_1} \cup \ldots \cup c_{\ell_r})$) under

$$H_0(\Gamma, H^0(E_+, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))) \xrightarrow{(46)} H_d(F_+^*, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))$$

for $k = 0, \ldots, r-1$ (resp. for $k = r$).

After this preliminary remark we prove (a). Consider the commutative diagram

$$\begin{array}{ccc} H_0(\mathcal{T}, H^0(E_+, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))) & \longrightarrow & H^0(E_+, H_0(\mathcal{T}, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))) \\ \big\downarrow{\scriptstyle (46)} & & \big\downarrow{\scriptstyle \cap\eta} \\ H_d(F_+^*, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)) & \xrightarrow{\text{coinf}} & H_d(E_+, H_0(\mathcal{T}, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))) \end{array}$$

where the upper horizontal arrow is the canonical map induced by the inclusion $H^0(E_+, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)) \hookrightarrow \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$. By Prop. 3.1 the coinflation $H_\bullet(F_+^*, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)) \to H_\bullet(E_+, H_0(\mathcal{T}, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)))$ is an isomorphism. Hence it remains to prove that the image of $[\ell^k 1_{\mathcal{F}}]$ under the upper horizontal map vanishes, i.e. we have

$$\ell^k 1_{\mathcal{F}} \in I(\mathcal{T})\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p) \qquad \text{for all } k = 0, 1, \ldots, r-1 \tag{54}$$

where $I(\mathcal{T}) \subseteq \mathbb{C}_p[\mathcal{T}]$ denotes the augmentation ideal.

We may shrink $\mathcal{T}$. In fact if $\mathcal{T}' \subseteq \mathcal{T}$ is a subgroup of finite index then it follows from Prop. 3.1 that $\mathrm{res} : H_0(\mathcal{T}, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)) \to H_0(\mathcal{T}', \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))$ is injective and if $\mathcal{F}' \subseteq \mathbf{I}^\infty/U^{p,\infty}$ is a fundamental domain for the action of $\mathcal{T}'$ then we have $\mathrm{res}([\ell^k 1_{\mathcal{F}}]) = [\ell^k 1_{\mathcal{F}'}]$.

Hence we may assume that

$$\mathcal{T} = \mathcal{T}_p \times \mathcal{T}^p \qquad \text{and} \qquad \mathcal{T}_p = \langle t_1, \ldots, t_m \rangle \tag{55}$$

with $\mathrm{ord}_{\mathfrak{p}_i}(t_i) > 0$ and $\mathrm{ord}_{\mathfrak{q}}(t_i) = 0 = \mathrm{ord}_{\mathfrak{p}_i}(t)$ for all $i \in \{1, \ldots, m\}$, $t \in \mathcal{T}^p$ and all finite places $\mathfrak{q} \neq \mathfrak{p}_i$. Put $F_i := F_{\mathfrak{p}_i}$, $\mathcal{O}_i := \mathcal{O}_{\mathfrak{p}_i}$ and $\mathcal{F}_i := \mathcal{O}_i - t_i\mathcal{O}_i$ for $i = 1, \ldots, m$. Let $\mathcal{F}^p \subseteq \mathbf{I}^{p,\infty}/U^{p,\infty}$ be a fundamental domain for the action of $\mathcal{T}^p$. Then $\mathcal{F}\colon=\prod_{i=1}^m \mathcal{F}_i\times\mathcal{F}^p\subseteq \mathbf{I}^\infty/U^{p,\infty}$ is a fundamental domain for the $\mathcal{T}$-action.

We also denote by ℓ_p resp. ℓ^p the map $\ell_p:\mathbf{I}_F\xrightarrow{\mathrm{pr}}F_p^*\hookrightarrow\mathbf{I}_F\xrightarrow{\ell}\mathbb{Q}_p$ resp. $\ell^p:\mathbf{I}_F\xrightarrow{\mathrm{pr}}\mathbf{I}_F^p\hookrightarrow\mathbf{I}_F\xrightarrow{\ell}\mathbb{Q}_p$ so that $\ell=\ell_p+\ell^p$ and $\ell_p=\sum_{i=1}^m\ell_i$ where $\ell_i=\ell_{\mathfrak{p}_i}$ for $i=1,\ldots,m$. We will show

$$\ell_p^k 1_{\mathcal{F}}\in I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p)\qquad\text{for all } k=0,1,\ldots,r-1 \tag{56}$$

where $I(\mathcal{T}_p)\subseteq\mathbb{C}_p[\mathcal{T}_p]$ denote the augmentation ideal. Since $t\,\ell^p=\ell^p$ for all $t\in\mathcal{T}_p$ this implies

$$\ell_p^k\,(\ell^p)^j\,1_{\mathcal{F}}\in I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p)\subseteq I(\mathcal{T})\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p) \tag{57}$$

for all $k,j\geq 0$ with $k\leq r-1$ and therefore (54).

For $\Xi\subseteq\{1,\ldots,r\}$ we set

$$\mathcal{F}_\Xi\colon=\prod_{i\in\Xi}\mathcal{O}_i\times\prod_{i\in\Xi^c}\mathcal{F}_i\times\prod_{i=r+1}^m\mathcal{F}_i\times\mathcal{F}^p$$

where Ξ^c denotes the complement of Ξ in $\{1,\ldots,r\}$. For $\underline{n}=(n_1,\ldots,n_m)\in\mathbb{N}_0^m$ with $n_i=0$ for all $i\in\Xi$ we let $\lambda(\Xi,\underline{n})\colon=\left(\prod_{i=1}^m\ell_i^{n_i}\right)\cdot 1_{\mathcal{F}_\Xi}\in\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p)$ i.e. $\lambda(\Xi,\underline{n})$ is given by

$$\lambda(\Xi,\underline{n})(x_1,\ldots,x_m,x^{p,\infty})=\begin{cases}\prod_{i=1}^m\ell_i(x_i)^{n_i} & \text{if } (x_1,\ldots,x_m,x^{p,\infty})\in\mathcal{F}_\Xi;\\ 0 & \text{otherwise.}\end{cases}$$

Put $|\underline{n}|:=\sum_{i=1}^m n_i$ and $\underline{n}!:=\prod_{i=1}^m n_i!$. Then,

$$\ell_p^k 1_{\mathcal{F}}=\sum_{|\underline{n}|=k}\frac{k!}{\underline{n}!}\lambda(\emptyset,\underline{n}). \tag{58}$$

Thus (56) follows from

Lemma 3.7. *If $\sharp(\Xi)+|\underline{n}|<r$ then $\lambda(\Xi,\underline{n})\in I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p)$.*

Proof. We first remark that for two functions $f,g:\mathbf{I}^\infty/U^{p,\infty}\to\mathbb{C}_p$ and $t\in F^*$ we have

$$(1-t)(f\cdot g)=((1-t)f)\cdot g+f\cdot((1-t)g)-((1-t)f)\cdot((1-t)g).$$

For $\underline{n},\underline{n}'\in\mathbb{N}_0^m$ write $\underline{n}'<\underline{n}$ if $n_i'\leq n_i$ for all i and $\underline{n}'\neq\underline{n}$. By using $((1-t)\ell_i)(x)=\ell_i(x)-\ell_i(t^{-1}x)=\ell_i(t)$ one can easily show that we have

$$(1-t)\prod_{i=1}^m\ell_i^{n_i}=\sum_{\underline{n}'<\underline{n}}a_{\underline{n}'}\prod_{i=1}^m\ell_i^{n_i'} \tag{59}$$

for some $a_{\underline{n}'}\in\mathbb{C}_p$. In fact if $\underline{n}'\in\mathbb{N}_0^m$ with $\underline{n}'<\underline{n}$ and $|\underline{n}'|=|\underline{n}|-1$ and if $i\in\{1,\ldots,m\}$ with $n_i'=n_i-1$ then

$$a_{\underline{n}'}=n_i\ell_i(t). \tag{60}$$

We prove the assertion by induction on $|\underline{n}|$. Assume first that $\underline{n}=\underline{0}=(0,\ldots,0)$. Let $i\in\Xi^c$ ($\Xi^c\neq\emptyset$ since $\sharp(\Xi)<r$). Then

$$\lambda(\Xi,\underline{0})=(1-t_i)\lambda(\Xi\cup\{i\},\underline{0})\in I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p).$$

Now assume that $|\underline{n}| > 0$. Since $\sharp(\Xi) + \sum_{i\in\Xi^c} n_i \leq \sharp(\Xi) + |\underline{n}| < r$, there exists $j \in \Xi^c$ with $n_j = 0$. Put $\Xi' := \Xi \cup \{j\}$. Modulo $I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$ we obtain

$$\begin{aligned}
(61)\quad \lambda(\Xi, \underline{n}) &= \prod_{i=1}^m \ell_i^{n_i} \cdot 1_{\mathcal{F}_\Xi} = \prod_{i=1}^m \ell_i^{n_i} \cdot (1 - t_j) 1_{\mathcal{F}_{\Xi'}} \\
&= (1 - t_j)\lambda(\Xi', \underline{n}) - ((1 - t_j)\prod_{i=1}^m \ell_i^{n_i}) \cdot 1_{\mathcal{F}_{\Xi'}} + ((1 - t_j)\prod_{i=1}^m \ell_i^{n_i}) \cdot 1_{\mathcal{F}_\Xi} \\
&\equiv -((1 - t_j)\prod_{i=1}^m \ell_i^{n_i}) \cdot 1_{\mathcal{F}_{\Xi'}} + ((1 - t_j)\prod_{i=1}^m \ell_i^{n_i}) \cdot 1_{\mathcal{F}_\Xi}.
\end{aligned}$$

By (59) and the induction hypothesis we have

$$((1 - t_j)\prod_{i=1}^m \ell_i^{n_i}) \cdot 1_{\mathcal{F}_{\Xi'}} \in \sum_{\underline{n}' < \underline{n}} \mathbb{C}_p \lambda(\Xi', \underline{n}') \subseteq I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$$

and

$$((1 - t_j)\prod_{i=1}^m \ell_i^{n_i}) \cdot 1_{\mathcal{F}_\Xi} \in \sum_{\underline{n}' < \underline{n}} \mathbb{C}_p \lambda(\Xi, \underline{n}') \subseteq I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$$

hence $\lambda(\Xi, \underline{n}) \in I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$. □

Next we want to write $(\sum_{i=1}^m \ell_i)^r 1_{\mathcal{F}}$ – modulo $I(\mathcal{T})\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$ – as a linear combination of a particular subset of $\{\lambda(\Xi, \underline{n}) |\ \sharp(\Xi) + |\underline{n}| = r\}$ (this will be used in the proof of Prop. 3.6 (b) below). For that we need to introduce more notation.

For $\Xi \subseteq \{1, \ldots, r\}$ and a map $f : \Xi \to \{1, \ldots, m\}$ we let $\underline{n}(f) := (\sharp(f^{-1}(1)), \ldots, \sharp(f^{-1}(m))) \in \mathbb{N}_0^m$. If in particular $f : \Xi \to \{1, \ldots, m\}$ is the inclusion we write $\underline{n}(\Xi)$ rather than $\underline{n}(f)$. We define

$$\Lambda_\Xi := \lambda(\Xi^c, \underline{n}(\Xi)) = (\prod_{i\in\Xi} \ell_i) \cdot 1_{\mathcal{F}_{\Xi^c}}.$$

Note that $\Lambda_\Xi = \prod_{i\in\Xi} \Lambda_i$ where for $i \in \{1, \ldots, r\}$ we have

$$\Lambda_i := \Lambda_{\{i\}} = \ell_i \cdot 1_{\mathcal{F}_i \times (\prod_{j=1, j\neq i}^r \mathcal{O}_j) \times (\prod_{j=r+1}^m \mathcal{F}_j) \times \mathcal{F}^p}.$$

Lemma 3.8. *Modulo $I(\mathcal{T})\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$ we have*

$$(\sum_{i=1}^m \ell_i)^r 1_{\mathcal{F}} \equiv r! \det \begin{pmatrix} \Lambda_1 + \ell_1(t_1)\Lambda_\emptyset & \ell_1(t_2)\Lambda_\emptyset & \ldots & \ell_1(t_r)\Lambda_\emptyset \\ \ell_2(t_1)\Lambda_\emptyset & \Lambda_2 + \ell_2(t_2)\Lambda_\emptyset & \ldots & \ell_2(t_r)\Lambda_\emptyset \\ \vdots & \vdots & \ddots & \vdots \\ \ell_r(t_1)\Lambda_\emptyset & \ell_r(t_2)\Lambda_\emptyset & \ldots & \Lambda_r + \ell_r(t_r)\Lambda_\emptyset \end{pmatrix}.$$

Proof. For $\Xi \subseteq \{1, \ldots, r\}$ we denote by $M(\Xi) \subseteq \mathrm{Maps}(\Xi, \{1, \ldots, m\})$ the set of maps $f : \Xi \to \{1, \ldots, m\}$ with $f(S) \not\subseteq S$ for all $S \subseteq \Xi$, $S \neq \emptyset$.

Let $\underline{n} \in \mathbb{N}_0^m$ with $\sharp(\Xi)+|\underline{n}| = r$. Firstly, we show that modulo $I(\mathcal{T})\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$ we have

$$\lambda(\Xi, \underline{n}) \quad \equiv \quad \underline{n}! \sum_{(\Upsilon, f)} (-1)^{|\underline{n}(f)|} \left(\prod_{i\in\Upsilon} \ell_{f(i)}(t_i)\right) \Lambda_{\Xi^c-\Upsilon} \tag{62}$$

where the sum runs through all pairs (Υ, f) with $\Upsilon \subseteq \Xi^c$ and $f \in M(\Upsilon)$ such that $\underline{n}(f) + \underline{n}(\Xi^c - \Upsilon) = \underline{n}$.

In the case $n_i > 0$ for all $i \in \Xi^c$ we show that both sides of (62) are $= \Lambda_{\Xi^c}$. Since $|\underline{n}| \geq \sharp(\Xi^c) = r - \sharp(\Xi) = |\underline{n}|$ we have $\underline{n} = \underline{n}(\Xi^c)$ hence $\lambda(\Xi, \underline{n}) = \Lambda_{\Xi^c}$. On the other hand if $\Upsilon \subseteq \Xi^c$ and $f : \Upsilon \to \{1, \ldots, m\}$ is a map with $\underline{n}(f) + \underline{n}(\Xi^c - \Upsilon) = \underline{n}$, then $\underline{n}(f) = \underline{n}(\Xi^c) - \underline{n}(\Xi^c - \Upsilon) = \underline{n}(\Upsilon)$ and therefore $f(\Upsilon) = \Upsilon$, hence $\Upsilon = \emptyset$. Consequently, the left hand side of (62) consists of only one summand Λ_{Ξ^c}.

Suppose $n_j = 0$ for some $j \in \Xi^c$. By Lemma 3.7, (59), (60), (61) we get

$$\begin{aligned}
\lambda(\Xi, \underline{n}) \quad &\equiv \quad -\left((1-t_j)\prod_{i=1}^m \ell_i^{n_i}\right) \cdot 1_{\mathcal{F}_{\Xi\cup\{j\}}} \\
&\equiv \quad - \sum_{\underline{n}'<\underline{n}, |\underline{n}'|=|\underline{n}|-1} a_{\underline{n}'} \ \lambda(\Xi \cup \{j\}, \underline{n}') \\
&= \quad - \sum_{(f, \underline{n}')} n_{f(j)}\, \ell_{f(j)}(t_j) \ \lambda(\Xi \cup \{j\}, \underline{n}')
\end{aligned}$$

where the last sum runs through all pairs $(f, \underline{n}')$ with $f \in M(\{j\})$ and $\underline{n}(f) + \underline{n}' = \underline{n}$. Now (62) can be easily deduced by induction on $\sharp(\Xi^c)$.

By (58) and (62) (for $\Xi = \emptyset$) we have

$$(\sum_{i=1}^m \ell_i)^r 1_{\mathcal{F}} \ \equiv \ r! \sum_{\Xi\subseteq\{1,\ldots,r\}} (-1)^{|\underline{n}(\Xi^c)|} \left(\sum_{f\in M(\Xi^c)} \prod_{i\in\Xi^c} \ell_{f(i)}(t_i)\right) \Lambda_\Xi. \tag{63}$$

On the other hand

$$\det \begin{pmatrix} \Lambda_1 + \ell_1(t_1)\Lambda_\emptyset & \ell_1(t_2)\Lambda_\emptyset & \ldots & \ell_1(t_r)\Lambda_\emptyset \\ \ell_2(t_1)\Lambda_\emptyset & \Lambda_2 + \ell_2(t_2)\Lambda_\emptyset & \ldots & \ell_2(t_r)\Lambda_\emptyset \\ \vdots & \vdots & \ddots & \vdots \\ \ell_r(t_1)\Lambda_\emptyset & \ell_r(t_2)\Lambda_\emptyset & \ldots & \Lambda_r + \ell_r(t_r)\Lambda_\emptyset \end{pmatrix} = \sum_{\Xi\subseteq\{1,\ldots,r\}} a_\Xi \, \Lambda_\Xi$$

where $a_\Xi = \det\left(\ell_i(t_j)\right)_{i,j\in\Xi^c} = \det \begin{pmatrix} \ell_{i_1}(t_{i_1}) & \ldots & \ell_{i_1}(t_{i_k}) \\ \vdots & & \vdots \\ \ell_{i_k}(t_{i_1}) & \ldots & \ell_{i_k}(t_{i_k}) \end{pmatrix}$ if $\Xi^c = \{i_1 < \ldots < i_k\}$.

Note that $\sum_{j=1}^m \ell_j(t_i) = \log_p(\mathrm{N}_{K/\mathbb{Q}}(t_i)) = 0$ since $\mathrm{N}_{K/\mathbb{Q}}(t_i)$ is a power of p (for all $i \in \{1, \ldots, m\}$). The assertion follows from (63) and the following result about determinants. $\square$

Lemma 3.9. *Let $k \leq m$ be positive integers and let $(a_{ij})_{i=1,\ldots,k, j=1,\ldots,m}$ be a $k \times m$–matrix with entries in a commutative ring such that* $\sum_{j=1}^m a_{ij} = 0$

for all $i = 1, \dots, k$. Then,

$$\det (a_{ij})_{i,j=1,\dots,k} = (-1)^k \sum_f \prod_{i=1}^k a_{if(i)}$$

where the sum runs through all maps $f : \{1, \dots, k\} \to \{1, \dots, m\}$ with $f(S) \not\subseteq S$ for all $S \subseteq \{1, \dots, k\}$, $S \neq \emptyset$.

Proof.[2] By replacing a_{ii} with $-\sum_{j\neq i} a_{ij}$ in $\sum_{\sigma\in S_k} \operatorname{sign}(\sigma) \prod_{i=1}^m a_{i\sigma(i)}$ and expanding the sum we get

$$\det (a_{ij})_{i,j=1,\dots,k} \;=\; \sum_{(\Xi,\sigma,g)} (-1)^{k-\sharp(\Xi)} \operatorname{sign}(\sigma) \prod_{i\in\Xi} a_{i\sigma(i)} \prod_{i\in\Xi^c} a_{ig(i)}$$

where the sum ranges over all triples (Ξ, σ, g) with $\Xi \subseteq \{1, \dots, k\}$, σ is a permutation of Ξ without fixed points and g is a map $\Xi^c \colon = \{1, \dots, k\} - \Xi \to \{1, \dots, m\}$ without fixed points.

Let $f : \{1, \dots, k\} \to \{1, \dots, m\}$ be a map without fixed points and let $\Xi \subseteq \{1, \dots, k\}$ be the largest subset with $f(\Xi) = \Xi$. If we decompose the permutation $f|_\Xi$ into disjoint cycles $\sigma_1 \cdots \sigma_t$ of length $l_1, \dots, l_t$, then it is easy to see that the coefficient of $\prod_{i=1}^k a_{if(i)}$ is $(-1)^k \prod_{j=1}^t (1 + \operatorname{sign}(\sigma_j)(-1)^{l_j})$. Thus it is $= 0$ except when $\Xi = \emptyset$. $\square$

Proof Prop. 3.6 (b) We first show

$$[\ell^r 1_{\mathcal{F}}] \;=\; (-1)^{\binom{r}{2}} (c_{\ell_1} \cup \ldots \cup c_{\ell_r}) \cap \varrho_{\mathcal{T}} \tag{64}$$

in $H_0(\mathcal{T}, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))$. As before if $\mathcal{T}'$ is a subgroup of finite index of $\mathcal{T}$ the injectivity of $\operatorname{res} : H_0(\mathcal{T}, \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)) \to H_0(\mathcal{T}', \mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p))$ together with $\operatorname{res}(\varrho_{\mathcal{T}}) = \varrho_{\mathcal{T}'}$ implies that in order to prove (64) we may shrink $\mathcal{T}$ so we can assume that $\mathcal{T}$ is of the form (55).

By (57) we have modulo $I(\mathcal{T}_p)\mathscr{C}_c^\flat(S_1, S_2, \mathbb{C}_p)$

$$\ell^r 1_{\mathcal{F}} \;=\; \sum_{j=0}^r \binom{r}{j} \ell_p^j (\ell^P)^{r-j} 1_{\mathcal{F}} \;\equiv\; \ell_p^r 1_{\mathcal{F}} \;=\; (\sum_{i=1}^m \ell_i)^r 1_{\mathcal{F}}$$

so we may replace $\ell^r 1_{\mathcal{F}}$ by $(\sum_{i=1}^m \ell_i)^r 1_{\mathcal{F}}$ on the left hand side of (64).

We will use the notation of the proof of part (a). Furthermore we put $\mathcal{T}_1 \colon = \langle t_1, \dots, t_r\rangle, \mathcal{T}_2 \colon = \langle t_{r+1}, \dots, t_m\rangle \times \mathcal{T}_0$ and $\mathcal{F}_2 \colon = \prod_{i=r+1}^m \mathcal{F}_i \times \mathcal{F}^p$ so that $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$, $\mathcal{F} = \prod_{i=1}^r \mathcal{F}_i \times \mathcal{F}_2$ and $\mathcal{F}_2$ is a $\mathcal{T}_1$-stable fundamental domain for the action of $\mathcal{T}_2$ on $\mathbf{I}^{S_1,\infty}/U^{p,\infty}$.

Since $\varrho_1 \in H_r(\mathcal{T}_1, \mathbb{Z}) \cong H_r(\mathbb{R}^r/\mathcal{T}_1, \mathbb{Z})$ can be represented by the r-cycle $\sum_{\sigma\in S_r} \operatorname{sign}(\sigma) [t_{\sigma(1)}| \ldots |t_{\sigma(r)}]$ (compare the proof of Prop. 3.5) together with Remark 3.2 (b) we conclude that

$$\sum_{\sigma\in S_r} \operatorname{sign}(\sigma)\, z_{\ell_1}(t_{\sigma(1)}) \otimes t_{\sigma(1)} z_{\ell_2}(t_{\sigma(2)}) \otimes \ldots \otimes t_{\sigma(1)} \ldots t_{\sigma(r-1)} z_{\ell_r}(t_{\sigma(r)}) \otimes 1_{\mathcal{F}_2}$$

[2] Due to V. Paskunas

is a representative of $(-1)^{\binom{r}{2}}(c_{\ell_1}\cup\ldots\cup c_{\ell_r})\cap\varrho\in H_0(\mathcal{T},\mathscr{C}_c^\flat(S_1,S_2,\mathbb{C}_p))$. We have

$$z_{\ell_i}(t_j)\equiv\delta_{ij}\ell_i 1_{\mathcal{F}_i}+\ell_i(t_j)1_{\mathcal{O}_i}\quad\text{and}\quad t_k z_{\ell_i}(t_j)\equiv z_{\ell_i}(t_j)$$

(modulo $K\cdot 1_{\mathcal{F}_i}$) for all $i,j,k\in\{1,\ldots,r\}$ with $j\neq k$. Hence by Lemma 3.7 we obtain

$$\begin{aligned}&z_{\ell_1}(t_{\sigma(1)})\otimes t_{\sigma(1)}z_{\ell_2}(t_{\sigma(2)})\otimes\ldots\otimes t_{\sigma(1)}\ldots t_{\sigma(r-1)}z_{\ell_r}(t_{\sigma(r)})\otimes 1_{\mathcal{F}_2}\\ \equiv\;&(\delta_{1\sigma(1)}\ell_1 1_{\mathcal{F}_1}+\ell_1(t_{\sigma(1)})1_{\mathcal{O}_1})\otimes\ldots\otimes(\delta_{r\sigma(r)}\ell_r 1_{\mathcal{F}_r}+\ell_r(t_{\sigma(r)})1_{\mathcal{O}_r})\otimes 1_{\mathcal{F}_2}\end{aligned}$$

(modulo $I(\mathcal{T})$) for all $\sigma\in S_r$. Now (64) follows from Lemma 3.8.

To finish the proof of Prop. 3.6 (b) consider the commutative diagram

$$\begin{tikzcd} H^r(F_+^*,C_c^\flat(F_{S_1}))\times H_r(\mathcal{T},C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty}))^{E_+} \arrow[r, "\cap\circ(\mathrm{res}\times\mathrm{id})"] \arrow[d, "\cap\circ(\mathrm{id}\times(52))"] & H_0(\mathcal{T},\mathscr{C}_c^\flat(S_1,S_2))^{E_+} \arrow[d, "\cap\eta"] \\ H_d(F_+^*,\mathscr{C}_c^\flat(S_1,S_2)) \arrow[r, "\mathrm{coinf}"] & H_d(E_+,H_0(\mathcal{T},\mathscr{C}_c^\flat(S_1,S_2))) \end{tikzcd}$$

where $C_c^\flat(F_{S_1})=C_c^\flat(F_{S_1},\mathbb{C}_p)$, $C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty}))=C_c(\mathbf{I}^{S_1,\infty}/U^{p,\infty},\mathbb{C}_p)$ etc. and the maps res and coinf denote the restriction and coinflation with respect to $\mathcal{T}\leq F_+^*$ (recall that the latter is an isomorphism by Prop. 3.1). By (64) the image of the pair $(c_{\ell_1}\cup\ldots\cup c_{\ell_r},\varrho_{\mathcal{T}})$ under the composition of the upper horizontal map, the right vertical map and the inverse of coinf is $(-1)^{\binom{r}{2}}\partial((\log_p\circ\mathcal{N})^r)$. On the other hand its image under the first vertical map is $(c_{\ell_1}\cup\ldots\cup c_{\ell_r})\cap\vartheta$. □

Remark 3.10. For $\mu\in\{\pm1\}$ let $e(\mu)\in\mathbb{Z}/2\mathbb{Z}$ be given by $\mu=(-1)^{e(\mu)}$. Let $\Sigma:=\{\pm1\}^{d+1}$. We write elements of Σ in the form $\underline{\mu}=(\mu_0,\ldots,\mu_d)$. Define the pairing $\langle\;,\;\rangle:\Sigma\times\Sigma\longrightarrow\{\pm1\}$ by $\langle\underline{\mu},\underline{\nu}\rangle=(-1)^{\sum_i e(\mu_i)e(\nu_i)}$. Let $\mathbf{C}$ be a field of characteristic zero. For a $\mathbf{C}[\Sigma]$-module V and $\underline{\mu}\in\Sigma$ we put $V_{\underline{\mu}}=\{v\in V\mid \underline{\nu}v=\langle\underline{\nu},\underline{\mu}\rangle\, v\ \ \forall\underline{\nu}\in\Sigma\}$ so that $V=\bigoplus_{\underline{\mu}\in\Sigma}V_{\underline{\mu}}$. For $v\in V$ we denote by $v_{\underline{\mu}}\in V_{\underline{\mu}}$ its projection to $V_{\underline{\mu}}$. If $\underline{\mu}=(+1,\ldots,+1)$ we shall also write v_+ instead of $v_{\underline{\mu}}$.

We identify F^*/F_+^* with Σ via the isomorphism $F^*/F_+^*=F_\infty^*/U_\infty\cong\prod_{i=0}^d\mathbb{R}^*/\mathbb{R}_+^*\cong\Sigma$. Hence for any F^*-module M we obtain an action of Σ on $H_q(F_+^*,M)$. Note that ϑ is Σ-invariant (since Σ acts trivially on $H_{d+r}(F_+^*,C_c(\mathbf{I}^{S_1,\infty}/U^{S_1,\infty},\mathbb{Z}))$) as well as the cohomology classes defined in Def. 3.4. Consequently, the cap-product $(c_{\ell_{\mathfrak{p}_1}}\cup\ldots\cup c_{\ell_{\mathfrak{p}_r}})\cap\vartheta$ for $\ell_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}$ or $\ell_{\mathfrak{p}}=\log_p\circ N_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ is Σ-invariant.

3.3. **p-adic L-functions attached to cohomology classes.** Let S_1,S_2 be arbitrary (possibly empty) disjoint subsets of S_p. For a ring R and an R-module M put

$$\mathscr{D}(S_1,S_2,M)=\mathrm{Dist}(F_{S_1}\times F_{S_2}^*\times\mathbf{I}^{p,\infty}/U^{p,\infty},M)=\mathrm{Hom}_R(\mathscr{C}_c^0(S_1,S_2,R),M).$$

It is easy to see that the functor $M\mapsto\mathscr{D}(S_1,S_2,M)$ is exact. Let

$$\langle\;,\;\rangle:\mathscr{D}(S_1,S_2,M)\times\mathscr{C}_c^0(S_1,S_2,R)\longrightarrow M\tag{65}$$

be the canonical (evaluation) pairing. Also for $\mathfrak{p} \in S_1$ by (7) we obtain a pairing

$$\mathscr{D}(S_1, S_2, M) \times C_c^0(F_\mathfrak{p}, R) \longrightarrow \mathscr{D}(S_1 - \{\mathfrak{p}\}, S_2, M).$$

If H is a subgroup of $\mathbf{I}^\infty$ and M is an $R[H]$-module we define a H-action on $\mathscr{D}(S_1, S_2, M)$ by requiring that $\langle x\lambda, xf\rangle = x\langle \phi, f\rangle$ for all $x \in H$, $f \in \mathscr{C}_c^0(S_1, S_2, R)$ and $\lambda \in \mathscr{D}(S_1, S_2, M)$.

If K is a p-adic field and V a K-Banach space then we denote the subspace of measures of $\mathscr{D}(S_1, S_2, V)$ by

$$\mathscr{D}^b(S_1, S_2, V) \;=\; \mathrm{Dist}^{\mathrm{b}}(F_{S_1} \times F_{S_2}^* \times \mathbf{I}^{p,\infty}/U^{p,\infty}, V).$$

The pairing (65) when restricted to $\mathscr{D}^b(S_1, S_2, V)$ extends canonically to a pairing

$$\langle\ ,\ \rangle : \mathscr{D}^b(S_1, S_2, V) \times \mathscr{C}_\diamond(S_1, S_2, K) \longrightarrow V. \tag{66}$$

Also for $\mathfrak{p} \in S_1$ by (9) we obtain a pairing

$$\mathscr{D}^b(S_1, S_2, V) \times C_\diamond(F_\mathfrak{p}, K) \longrightarrow \mathscr{D}^b(S_1 - \{\mathfrak{p}\}, S_2, V). \tag{67}$$

Assume now that $S_1 \cup S_2 = S_p$, $H = F_+^*$ and that F_+^* acts trivially on M. Again, we order the places above p so that $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ and $S_2 = \{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_m\}$. The pairing (65) yields the bilinear map

$$\cap : H^d(F_+^*, \mathscr{D}(S_1, S_2, M)) \times H_d(F_+^*, \mathscr{C}_c^0(S_1, S_2, R)) \to H_0(F_+^*, M) = M. \tag{68}$$

For $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, M))$ define $\mu_\kappa \in \mathrm{Dist}(\mathcal{G}_p, M)$ by

$$\int_{\mathcal{G}_p} f(\gamma)\, \mu_\kappa(d\gamma) \;=\; \kappa \cap \partial(f) \tag{69}$$

for all $f \in C^0(\mathcal{G}_p, R)$.

Suppose now that $R = K$ is a p-adic field and $M = V$ a finite dimensional K-vector space and let $\kappa \in H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V))$. By abuse of notation we denote its image under $H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V)) \to H^d(F_+^*, \mathscr{D}(S_1, S_2, V))$ also by κ. It is then easy to see that μ_κ is actually a measure. Thus we obtain a map

$$H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V)) \longrightarrow \mathrm{Dist}^{\mathrm{b}}(\mathcal{G}_p, V),\ \kappa \mapsto \mu_\kappa. \tag{70}$$

The integral $C(\mathcal{G}_p, K) \to V, f \mapsto \int_{\mathcal{G}_p} f\mu_\kappa$ admits also a description as a cap-product. More precisely (69) holds more generally for all $f \in C(\mathcal{G}_p, K)$ (where ∂ denotes now the map (42) for $R = K$ equipped with the topology induced by $|\ |_p$ and the cap-product is induced by (66)).

Recall that $\mathcal{N} : \mathcal{G}_p \to \mathbb{Z}_p^*$ is defined by $\gamma\zeta = \zeta^{\mathcal{N}(\gamma)}$ for all p-power roots of unity ζ. For $s \in \mathbb{Z}_p$ and $\gamma \in \mathcal{G}_p$ we put $\langle\gamma\rangle^s := \exp_p(s \log_p(\mathcal{N}(\gamma)))$.

Definition 3.11. *Let K be a p-adic field and V a finite-dimensional K-vector space. We define the p-adic L-function of $\kappa \in H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V))$ by*

$$L_p(s, \kappa) := \int_{\mathcal{G}_p} \langle\gamma\rangle^s \mu_\kappa(d\gamma).$$

The main result of this chapter is the following

Theorem 3.12. *Let $r := \sharp(S_1)$ and let $\kappa \in H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V))$.*

(a) $L_p(s, \kappa)$ is a locally analytic V-valued function on $\mathbb{Z}_p$. We have

$$\operatorname{ord}_{s=0} L_p(s, \kappa) \geq r.$$

(b) For $\mathfrak{p} \in S_1$ put $\ell_{\mathfrak{p}} := \log_p \circ \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p} : F_{\mathfrak{p}}^* \to K$. For the r-th derivative of $L_p(s, \kappa)$ at $s = 0$ we have

$$L_p^{(r)}(0, \kappa) \;=\; (-1)^{\binom{r}{2}}\, r!\, (\kappa \cup c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta.$$

Here the cup-product is induced by (67) *and the cap-product by* (66)*.*

Proof. We have

$$L_p^{(k)}(0, \kappa) = \int_{\mathcal{G}_p} (\log_p \circ \mathcal{N})^k \mu_\kappa(d\gamma) = \kappa \cap \partial((\log_p \circ \mathcal{N})^k)$$

for all $k \in \mathbb{N}$. Hence the assertion follows from Prop. 3.6. □

Remark 3.13. The group $\Sigma \cong F^*/F_+^*$ acts on $H^q(F_+^*, \mathscr{D}(S_1, S_2, M))$ or $H^q(F_+^*, \mathscr{D}^b(S_1, S_2, V))$. Let $\mathcal{G}_p^+$ be the Galois group of the maximal abelian extension of F which is unramified outside p. By class field theory we have an exact sequence $F^*/F_+^* = F_\infty^*/U_\infty \to \mathcal{G}_p \to \mathcal{G}_p^+ \to 1$, which yields an action of Σ on $\mathcal{G}_p$. It is easy to see that (70) is Σ-equivariant. The fact that $\gamma \mapsto \langle\gamma\rangle^s$ factors through $\mathcal{G}_p \to \mathcal{G}_p^+$ implies that $L_p(s, \kappa) = L_p(s, \kappa_+)$ for all $\kappa \in H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V))$. Also by Remark 3.10 we have $\kappa \cap (c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta = \kappa_+ \cap (c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta$.

3.4. **Integral cohomology classes.** For a given cohomology class $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ we will introduce a condition – called *integral* – which guarantees that μ_κ is a p-adic measure (in the sense of section 1.2) and which allows us to apply Theorem 3.12. To begin with we define the module of *periods* of κ.

Definition 3.14. *Let $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ and let R be a subring of $\mathbb{C}$. The image of*

$$H_d(F_+^*, \mathscr{C}_c^0(S_1, S_2, R)) \longrightarrow H_0(F_+^*, \mathbb{C}) = \mathbb{C}, \;\; x \longmapsto \kappa \cap x$$

will be denoted by $L_{\kappa,R}$. If $R \subset \overline{\mathbb{Q}}$ then it is called the R-module of periods of κ.

Lemma 3.15. *Let $R \subseteq \overline{\mathbb{Q}}$ be a Dedekind ring.*

(a) If $R' \supseteq R$ is a subring of $\mathbb{C}$ then $L_{\kappa,R'} = R' L_{\kappa,R}$.

(b) If $\kappa \neq 0$ then $L_{\kappa,R} \neq 0$.

Proof. (a) Since $\mathscr{C}_c^0(S_1, S_2, R') = \mathscr{C}_c^0(S_1, S_2, R) \otimes R'$ and R' is flat R-algebra we have $H_d(F_+^*, \mathscr{C}_c^0(S_1, S_2, R)) \otimes R' = H_d(F_+^*, \mathscr{C}_c^0(S_1, S_2, R'))$.

(b) By (a) it is enough to show $L_{\kappa,\mathbb{C}} \neq 0$. This follows from the fact that the pairing (68) (for $M = R = \mathbb{C}$) is nondegenerate. □

Definition 3.16. *A cohomology class* $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$, $\kappa \neq 0$ *is called integral (or more precisely p-integral) if there exists a Dedekind ring* $R \subset \overline{\mathcal{O}}$ *such that* κ *lies in the image of* $H^d(F_+^*, \mathscr{D}(S_1, S_2, R)) \otimes_R \mathbb{C} \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$. *If in addition there exists a finitely generated* R*-submodule* $M \subseteq H^d(F_+^*, \mathscr{D}(S_1, S_2, R))$ *of rank* ≤ 1 *(i.e.* $\mathrm{rank}_R\, M/M_{\mathrm{tor}} \leq 1$*) such that* κ *lies in the image of* $M \otimes_R \mathbb{C} \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ *then* κ *is called integral of rank* ≤ 1.

Proposition 3.17. *Let* $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$. *The following conditions are equivalent.*

(i) κ *is integral (resp. integral of rank* ≤ 1*).*

(ii) There exists a Dedekind ring $R \subseteq \overline{\mathcal{O}}$ *such that* $L_{\kappa,R}$ *is a finitely generated* R*-module (resp.* $L_{\kappa,R}$ *is either* $= 0$ *or an invertible* R*-module).*

(iii) There exists a Dedekind ring $R \subseteq \overline{\mathcal{O}}$, *a finitely generated* R*-module* M *(resp. an invertible* R*-module* M *of rank 1) and an* R*-linear map* $f : M \to \mathbb{C}$ *such that* κ *lies in the image of the induced map* $f_* : H^d(F_+^*, \mathscr{D}(S_1, S_2, M)) \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$.

Proof. We consider only the case of arbitrary rank and will leave the necessary modifications to the rank ≤ 1 case to the reader.

(i) $\Rightarrow$ (ii) Let R be as in Definition 3.16. If we write κ in the form $\kappa = \sum_{i=1}^n \Omega_i \kappa_i$ with $\kappa_i \in \mathrm{Im}(H^d(F_+^*, \mathscr{D}(S_1, S_2, R)) \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C})))$ and $\Omega_i \in \mathbb{C}$ then $L_{\kappa,R} \subseteq R\Omega_1 + \ldots + R\Omega_n$.

(ii) $\Rightarrow$ (iii) Consider the diagram

$$\begin{tikzcd} H^d(F_+^*, \mathscr{D}(S_1, S_2, L_{\kappa,R})) \arrow[r] \arrow[d] & \mathrm{Hom}_R(H_d(F_+^*, \mathscr{C}_c^0(S_1, S_2, R)), L_{\kappa,R}) \arrow[d] \\ H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C})) \arrow[r] & \mathrm{Hom}_R(H_d(F_+^*, \mathscr{C}_c^0(S_1, S_2, R)), \mathbb{C}) \end{tikzcd}$$

where the horizontal maps are induced by the cap-product and the vertical maps by the inclusion $L_{\kappa,R} \hookrightarrow \mathbb{C}$. By the universal coefficient theorem the lower horizontal map is an isomorphism and the kernel and cokernel of the upper horizontal map are R-torsion. Hence some multiple $a \cdot \kappa$ with $a \in R$, $a \neq 0$ is contained in the image of the left vertical map. Define $f : L_{\kappa,R} \to \mathbb{C}, \Omega \mapsto a^{-1}\Omega$.

(iii) $\Rightarrow$ (i) We may assume that $M = R^n$ (for example replace M by $f(M)$ and f by the inclusion and then enlarge M if necessary). Let $\Omega_1, \ldots, \Omega_n \in \mathbb{C}^n$ be the images of the standard basis under f. It follows

$$\kappa \in \mathrm{Im}(f_*) = \sum_{i=1}^n \Omega_i \cdot \mathrm{Im}(H^d(F_+^*, \mathscr{D}(S_1, S_2, R)) \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))).$$

$\square$

Corollary 3.18. *Assume* $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ *is integral and let* $R \subseteq \overline{\mathcal{O}}$ *be as in Definition 3.16. Then,* *(a)* μ_κ *is a p-adic measure.*

(b) The map $H^d(F_+^*, \mathscr{D}(S_1, S_2, L_{\kappa,R})) \otimes \overline{\mathbb{Q}} \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ *is injective and* κ *lies in its image.*

Proof. (a) The image of $C^0(\mathcal{G}_p, \overline{\mathcal{O}}) \longrightarrow \mathbb{C}, f \mapsto \int f\, d\mu_\kappa$ is contained in $L_{\kappa,\overline{\mathcal{O}}}$. The assertion follows from Proposition 3.17.

(b) follows from the proof of (ii) $\Rightarrow$ (iii) above. $\square$

Let κ, R be as above. By abuse of notation we define the p-adic L-function of κ by $L_p(s, \kappa) := \int_{\mathcal{G}_p} \langle\gamma\rangle^s \mu_\kappa(d\gamma)$. Because of (b) we can view κ as an element of $H^d(F_+^*, \mathscr{D}(S_1, S_2, L_{\kappa,R})) \otimes_R \overline{\mathbb{Q}}$. Put $V_\kappa = L_{\kappa,R} \otimes_R \mathbb{C}_p$ and let $\widetilde{\kappa}$ denote the image of κ under the homomorphism

$$H^d(F_+^*, \mathscr{D}(S_1, S_2, L_{\kappa,R})) \otimes_R \mathbb{C}_p \longrightarrow H^d(F_+^*, \mathscr{D}^b(S_1, S_2, V_\kappa)) \tag{71}$$

induced by the obvious map $\mathscr{D}(S_1, S_2, L_{\kappa,R}) \to \mathscr{D}(S_1, S_2, L_{\kappa,R}) \otimes_R \mathbb{C}_p \to \mathscr{D}^b(S_1, S_2, V_\kappa)$. It is easy to see that the $\widetilde{\kappa}$ does not depend on the choice of R. Since $L_p(s, \kappa) = L_p(s, \widetilde{\kappa})$ we can apply Theorem 3.12.

Corollary 3.19. *Assume* $\kappa \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ *is integral. For* $\mathfrak{p} \in S_1$ *put* $\ell_\mathfrak{p} := \log_p \circ \mathrm{N}_{F_\mathfrak{p}/\mathbb{Q}_p}$. *Then,*

(a) $\mathrm{ord}_{s=0} L_p(s, \kappa) \geq r$.

(b) $L_p^{(r)}(0, \kappa) = (-1)^{\binom{r}{2}}\, r!\, (\widetilde{\kappa}_+ \cup c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta$.

3.5. **Another construction of distributions on $\mathcal{G}_p$.** Let $\mathcal{A}(\mathbb{G}_m)$ be the space of smooth $\mathbb{C}$-valued functions on $\mathbf{I}/F^*$ which are rapidly decreasing as $|x| \to \infty$ or $|x| \to 0$ (i.e. for $f \in \mathcal{A}(\mathbb{G}_m)$ and $N > 0$ there exists a constant $C > 0$ such that $|f(x)| < |x|^{-N}$ for $|x| > C$ and $|f(x)| < |x|^N$ for $|x| < C^{-1}$).

Let S_1, S_2 be disjoint subsets of S_p with $S_1 \cup S_2 = S_p$. We consider maps $\phi : \mathscr{C}_c^0(S_1, S_2, \mathbb{Z}) \times F_\infty^* \to \mathbb{C}$ with the following properties

(i) For $x_\infty \in F_\infty^*$ the map $\phi(\,\cdot\,, x_\infty) : \mathscr{C}_c^0(S_1, S_2, \mathbb{Z}) \to \mathbb{C}, f \mapsto \phi(f, x_\infty)$ lies in $\mathscr{D}(S_1, S_2, \mathbb{C})$.

(ii) For all $f \in \mathscr{C}_c^0(S_1, S_2, \mathbb{Z})$ the function
$$\mathbf{I}^\infty \times F_\infty^* \to \mathbb{C}, \quad x = (x^\infty, x_\infty) \mapsto \phi(x^\infty f, x_\infty)$$
lies in $\mathcal{A}(\mathbb{G}_m)$. In particular we have $\phi(\xi f, \xi x_\infty) = \phi(f, x_\infty)$ for all $\xi \in F^*$.

We denote the space of all ϕ satisfying (i),(ii) by $\mathcal{D}(\mathbb{G}_m, S_1)$. Note that the map

$$C_c^0(F_{S_1} \times F_{S_2}^*, \mathbb{Z}) \times C_c^0(\mathbf{I}^{p,\infty}/U^{p,\infty}, \mathbb{Z}) \longrightarrow \mathscr{C}_c^0(S_1, S_2, \mathbb{Z}), \ (f, g) \mapsto f \otimes g$$

induces an isomorphism $C_c^0(F_{S_1} \times F_{S_2}^*, \mathbb{Z}) \otimes C_c^0(\mathbf{I}^{p,\infty}/U^{p,\infty}, \mathbb{Z}) \cong \mathscr{C}_c^0(S_1, S_2, \mathbb{Z})$ and that any element of $C_c^0(\mathbf{I}^{p,\infty}/U^{p,\infty}, \mathbb{Z})$ can be written as a finite sum of the characteristic functions of elements of $\mathbf{I}^{p,\infty}/U^{p,\infty}$. Hence we can (and will) view an element $\phi \in \mathcal{D}(\mathbb{G}_m, S_1)$ also as a map

$$\phi : C_c^0(F_{S_1} \times F_{S_2}^*, \mathbb{Z}) \times \mathbf{I}^p/U^{p,\infty} \to \mathbb{C}, \ (f, x^p) \mapsto \phi(f, x^p). \tag{72}$$

For $f \in C_c^0(F_{S_1} \times F_{S_2}^*, \mathbb{Z})$ we denote by $\phi_f \in \mathcal{A}(\mathbb{G}_m)$ the map

$$\mathbf{I} = F_p^* \times \mathbf{I}^p \to \mathbb{C}, \quad \phi_f(x_p, x^p) := \phi(x_p f, x^p)$$

In particular for a compact open subset U of $F_{S_1} \times F_{S_2}^*$ we define $\phi_U \in \mathcal{A}(\mathbb{G}_m)$ as $\phi_U = \phi_{1_U}$.

Given $\phi \in \mathcal{D}(\mathbb{G}_m, S_1)$, $f \in C^0(\mathbf{I}/F^*, \mathbb{C})$ and $s \in \mathbb{C}$ we define the integral $\int_{\mathbf{I}/F^*} f(x)|x|^s \phi(dx^\infty, x_\infty) d^\times x_\infty$ as follows. By Lemma 3.20 below, there exists an open subgroup U of U_p such that $f(x_p u, x^p) = f(x_p, x^p)$ for all $(x_p, x^p) \in F_p^* \times \mathbf{I}^p$ and $u \in U$. We set

$$\int_{\mathbf{I}/F^*} f(x)|x|^s \phi(dx^\infty, x_\infty) d^\times x_\infty \; = \; [U_p : U] \int_{\mathbf{I}/F^*} f(x)|x|^s \phi_U(x) d^\times x. \tag{73}$$

It is easy to see that the integral does not depend on the choice of U. Moreover since ϕ_U is rapidly decreasing it is holomorphic in s. Hence there exists a unique distribution $\mu = \mu_\phi$ on $\mathcal{G}_p$ such that

$$\int_{\mathcal{G}_p} f(\gamma) \mu_\phi(d\gamma) = \int_{\mathbf{I}/F^*} f(\rho(x)) \phi(dx_p, x^p) d^\times x^p$$

for all $f \in C^0(\mathcal{G}_p, \mathbb{C})$ (here $\rho : \mathbf{I}/F^* \to \mathcal{G}_p$ denotes the reciprocity map).

Lemma 3.20. *Let X be a set and $f : \mathbf{I}/F^* \to X$ be a locally constant map. Then there exists an open subgroup U of $\mathbf{I}$, such that f factors through $\mathbf{I}/F^* U$.*

Proof. Since $U_\infty = \prod_{v \in S_\infty} \mathbb{R}_+^*$ is connected, f factors as $\mathbf{I}/F^* \to \mathbf{I}/F^* U_\infty \xrightarrow{\bar{f}} X$ and since $\mathbf{I}/F^* U_\infty$ is profinite, $\bar{f}$ factors through a finite quotient of $\mathbf{I}_F/F^* U_\infty$. □

We will construct now a cohomology class $\kappa = \kappa_\phi \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ such that $\mu_\phi = \mu_\kappa$. Put $S^0 = S_\infty^0 = S_\infty - \{\infty_0\} = \{\infty_1, \ldots, \infty_d\}$ and $U_\infty^0 = \prod_{w \in S_\infty^0} \mathbb{R}_+^*$. We write elements of $F_\infty = F_{\infty_0} \times F_{S^0}$ as pairs (x_0, x^0). For $\phi \in \mathcal{D}(\mathbb{G}_m, S_1)$ we denote the function

$$\mathscr{C}_c^0(S_1, S_2, \mathbb{Z}) \times F_{S^0}^* \longrightarrow \mathbb{C}, \; (f, x^0) \mapsto \int_0^\infty \phi(f, x_0, x^0) d^\times x_0$$

by $\int_0^\infty \phi \; d^\times x_0$. It is easy to see that we have $(\int_0^\infty \phi \; d^\times x_0)(\xi f, \xi x^0) = (\int_0^\infty \phi \; d^\times x_0)(f, x^0)$ for all $\xi \in F_+^*$. Therefore we obtain a homomorphism

$$\mathcal{D}(\mathbb{G}_m, S_1) \longrightarrow H^0(F_+^*, \mathscr{D}(S_1, S_2, C^\infty(U_\infty^0))), \; \phi \mapsto \int_0^\infty \phi \; d^\times x_0. \tag{74}$$

Here $C^\infty(U_\infty^0)$ denotes the ring of smooth $\mathbb{C}$-valued functions on U_∞^0 (the homeomorphism $\lambda : U_\infty^0 \to \mathbb{R}^{S_\infty^0} = \mathbb{R}^d, (x_v) \mapsto (\log(x_v))$ provides U_∞^0 with the structure of a real manifold). Note that U_∞^0 carries the canonical d-form $d^\times x_1 \ldots d^\times x_d = \prod_{v \in S_\infty^0} d^\times x_v$ so we obtain a map

$$C^\infty(U_\infty^0) \longrightarrow \Omega^d(U_\infty^0, \mathbb{C}), \; f \mapsto f(x_1, \ldots, x_d) \; d^\times x_1 \ldots d^\times x_d \tag{75}$$

Define

$$\mathcal{D}(\mathbb{G}_m, S_1) \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C})),\ \phi \mapsto \kappa_\phi \tag{76}$$

as follows. Put $C^\bullet :=\ \mathscr{D}(S_1, S_2, \Omega^\bullet(U_\infty^0, \mathbb{C}))$. Since $U_\infty^0 \approx \mathbb{R}^d$, the complex $C^\bullet$ is a resolution of $\mathscr{D}(S_1, S_2, \mathbb{C})$ and we have $C^i = 0$ if $i > d$. The map (76) is the composite of (74) with the map

$$H^0(F_+^*, \mathscr{D}(S_1, S_2, C^\infty(U_\infty^0))) \to H^0(F_+^*, C^d) \to H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C})) \tag{77}$$

where the first arrow is induced by (75) while the second is an edge morphism of the spectral sequence

$$E_1^{pq} = H^q(F_+^*, C^p) \Longrightarrow E^{p+q} = H^{p+q}(F_+^*, C^\bullet) = H^{p+q}(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C})).$$

Proposition 3.21. *For $\phi \in \mathcal{D}(\mathbb{G}_m, S_1)$ and $\kappa = \kappa_\phi \in H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))$ we have $\mu_\phi = \mu_\kappa$.*

Proof. Define a pairing

$$\langle\ ,\ \rangle : \mathcal{D}(\mathbb{G}_m, S_1) \times C^0(\mathcal{G}_p, \mathbb{C}) \longrightarrow \mathbb{C}$$

as the composite of the product of (74) and (43)with the map

$$\begin{aligned} &H^0(F_+^*, \mathscr{D}(S_1, S_2, C^\infty(U_\infty^0))) \times H_0(F_+^*/E_+, H^0(E_+, \mathscr{C})) \\ \xrightarrow{\cap}\ &H_0(F_+^*/E_+, H^0(E_+, C^\infty(U_\infty^0))) \longrightarrow\ H_0(F_+^*/E_+, \mathbb{C}) \cong \mathbb{C} \end{aligned} \tag{78}$$

(where $\mathscr{C} := \mathscr{C}_c^0(S_1, S_2, \mathbb{Z})$). Here the first map is induced by (65) and the second by

$$H^0(E_+, C^\infty(U_\infty^0)) \to \mathbb{C},\ f \mapsto \int_{U_\infty^0/E_+} f(x_1, \ldots, x_d)\, d^\times x_1 \ldots d^\times x_d. \tag{79}$$

A simple computation shows that

$$\langle \phi, f\rangle = \int_{\mathcal{G}_p} f(\gamma)\mu_\phi(d\gamma).$$

for all $f \in C^0(\mathcal{G}_p, \mathbb{C})$. Thus, we need to show $\kappa_\phi \cap \partial(f) = \langle \phi, f\rangle$, i.e. we have to show that the diagram

$$\begin{tikzcd} H^0(F_+^*, \mathscr{D}(S_1, S_2, C^\infty(U_\infty^0))) \times H_0(F_+^*/E_+, H^0(E_+, \mathscr{C})) \arrow[d, "(77)\times\epsilon"] \arrow[dr, "(78)"] & \\ & \mathbb{C} \\ H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C})) \times H_d(F_+^*, \mathscr{C}) \arrow[ur, "\cap"'] & \end{tikzcd} \tag{80}$$

commutes. For that consider the commutative diagram

$$\begin{tikzcd}
H^0(F_+^*, \mathscr{D}(S_1,S_2,C^\infty(U_\infty^0)) \times H_0(F_+^*/E_+, H^0(E_+,\mathscr{C})) \arrow[r] \arrow[d, "\mathrm{id}\times\eta"] & H_0(F_+^*/E_+, H^0(E_+, C^\infty(U_\infty^0))) \arrow[d, "\eta"] \\
H^0(F_+^*, \mathscr{D}(S_1,S_2,C^\infty(U_\infty^0))) \times H_0(F_+^*/E_+, H_d(E_+,\mathscr{C})) \arrow[r] \arrow[d, "3\times\mathrm{id}"] & H_0(F_+^*/E_+, H_d(E_+, C^\infty(U_\infty^0))) \arrow[d, "4"] \\
H^0(F_+^*, \mathscr{D}(S_1,S_2,\Omega^d(U_\infty^0))) \times H_0(F_+^*/E_+, H_d(E_+,\mathscr{C})) \arrow[r] \arrow[d, "\mathrm{id}\times 5"] & H_0(F_+^*/E_+, H_d(E_+, \Omega^d(U_\infty^0))) \arrow[d, "6"] \\
H^0(F_+^*, \mathscr{D}(S_1,S_2,\Omega^d(U_\infty^0))) \times H_d(F_+^*,\mathscr{C})) \arrow[r] \arrow[d, "7\times\mathrm{id}"] & H_d(F_+^*, \Omega^d(U_\infty^0)) \arrow[d, "8"] \\
H^d(F_+^*, \mathscr{D}(S_1,S_2,\mathbb{C})) \times H_d(F_+^*,\mathscr{C}) \arrow[r] & H_0(F_+^*,\mathbb{C}) = \mathbb{C}
\end{tikzcd}$$

Here the horizontal maps are cap-products induced by the pairings (65). The maps 3 and 4 are induced by the map (75), the maps 5 and 6 are edge morphisms in a Hochschild-Serre spectral sequence and 7 and 8 are edge morphisms of a E^1- (resp. E_1-) hyper(co-)homology spectral sequence for the resolution $0 \to \mathbb{C} \to \Omega^0(U_\infty^0) \to \Omega^1(U_\infty^0) \to \ldots$

The commutativity of (80) follows once we have shown that the composition of the right column of vertical maps is induced by the map (79). However this can be easily deduced from the commutativity of the obvious diagram

$$\begin{tikzcd}
H^0(E_+, C^\infty(U_\infty^0)) \arrow[r] \arrow[d, "\cap\eta"] & H^0(E_+, \Omega^d(U_\infty^0)) \arrow[r] \arrow[d, "\cap\eta"] & H^d(E_+,\mathbb{C}) \arrow[d, "\cap\eta"] \\
H_d(E_+, C^\infty(U_\infty^0)) \arrow[r] & H_d(E_+, \Omega^d(U_\infty^0)) \arrow[r] & H_0(E_+,\mathbb{C})
\end{tikzcd}$$

and the fact that the trace map $H^d_{\mathrm{DR}}(M) \to \mathbb{C}, [\omega] \mapsto \int_M \omega$ for a d-dimensional oriented manifold M corresponds under the canonical isomorphism $H^d_{\mathrm{DR}}(M) \cong H^d_{\mathrm{sing}}(M)$ to the map $x \mapsto x \cap \eta_M$ where η_M denotes the fundamental class of M. $\square$

For $\phi \in \mathcal{D}(\mathbb{G}_m, S_1)$ put $\phi_0 = \phi_{U_0}$ where $U_0 = \prod_{\mathfrak{p}\in S_1} \mathcal{O}_\mathfrak{p} \times \prod_{\mathfrak{p}\in S_2} \mathcal{O}_\mathfrak{p}^*$.

Corollary 3.22. *For* $\mathfrak{p} \in S_1$ *let* $c_\mathfrak{p} = c_{\mathrm{ord}_\mathfrak{p}} \in H^1(F_+^*, C_c^0(F_\mathfrak{p},\mathbb{Z}))$ *be the cohomology class of the 1-cocycle* (31) *for* $\ell = \mathrm{ord}_\mathfrak{p}$. *Then we have*

$$\int_{\mathbf{I}/F^*} \phi_0(x)\, d^\times x \;=\; (-1)^{\binom{r}{2}}\, (\kappa_+ \cup c_{\mathfrak{p}_1} \cup \ldots \cup c_{\mathfrak{p}_r}) \cap \vartheta.$$

Here the cup-product is induced by (67) *and the cap-product by* (66).

Proof. We use the notation of Prop. 3.5. Note that $\mathcal{F}_1 \subseteq \mathbf{I}^{p,\infty}/U^{p,\infty}$ is a finite set and $\mathcal{X} = U_0 \times \mathcal{F}_1$ so we have $1_\mathcal{X} = \sum_{x\in\mathcal{F}} x 1_{\mathcal{X}_0}$ (where $\mathcal{F} := \{1\} \times \mathcal{F}_1, \mathcal{X}_0 := U_0 \times \{1\} \subseteq F_p^* \times \mathbf{I}^{p,\infty}/U^{p,\infty} = \mathbf{I}^\infty/U^{p,\infty}$). Because of the commutativity of (80) and Prop. 3.5 it is enough to prove that the pair $(\int_0^\infty \phi\, d^\times x_0, [1_\mathcal{X}])$ is mapped to $\int_{\mathbf{I}/F^*} \phi_0(x) d^\times x$ under the pairing (78). In fact by definition of (78) the pair is mapped to

$$\begin{aligned}
&\int_{U_\infty/E_+} \phi(1_{\mathcal{X}}, x_0, \ldots, x_d)\, d^\times x_0 \ldots d^\times x_d \\
&\sum_{x\in\mathcal{F}} \int_{U_\infty/E_+} \phi(x1_{\mathcal{X}_0}, x_0, \ldots, x_d)\, d^\times x_0 \ldots d^\times x_d \\
&= \int_{\mathbb{R}_+\times\mathcal{E}\times\mathcal{F}} \phi_0(x)\, d^\times x \;=\; \int_{\mathbb{R}_+\times\mathcal{E}\times U_p\times\mathcal{F}} \phi_0(x)\, d^\times x \\
&= \int_{\mathbf{I}/F^*} \phi_0(x)\, d^\times x
\end{aligned}$$

where $\mathcal{E}\subseteq U_\infty^0$ is a fundamental domain for the action E_+. □

4. p-adic L-functions of Hilbert modular forms

4.1. **p-ordinary cuspidal automorphic representations.** Let $\pi = \otimes_v \pi_v$ be a unitary cuspidal automorphic representation of $G(\mathbf{A})$. Thus π is an irreducible direct summand of the right regular representation of $G(\mathbf{A})$ in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbf{A}))$. If the archimedean local representations π_v are discrete series then a p-adic L-function $L_p(s,\pi)$ for π can be defined. The first construction under certain restrictions on π is due to Manin [19]; a construction in most generality (based on earlier work of Panchishkin [23]) is due to Dabrowski [10]; see ([10], sect. 12) for further references.

In this section we shall give another definition of $L_p(s,\pi)$ well-suited for the proof of the weak exceptional zero conjecture. We assume that π has parallel weight $(2,\ldots,2)$ and is p-ordinary. The first condition means that $\pi_v = \mathcal{D}(2)$ is the discrete series representation of $G(\mathbb{R})$ of lowest weight 2 for all $v\in S_\infty$ and the second that $\pi_{\mathfrak{p}}$ is ordinary for all $\mathfrak{p}\in S_p$ in the sense of section 2.2. We shall attach an element $\phi_\pi\in\mathcal{D}(\mathbb{G}_m, S_1)$ to π, show that the corresponding cohomology class $\kappa_\pi = \kappa_{\phi_\pi}$ is integral and define $L_p(s,\pi)$ as the p-adic L-series associated to κ_π (here S_1 denotes the set of $\mathfrak{p}\in S_p$ with $\pi_{\mathfrak{p}} = \mathrm{St}$).

We introduce some notation. Firstly, we denote by $\mathfrak{A}_0(G,\underline{2})$ the set of all unitary cuspidal automorphic representation π of $G(\mathbf{A})$ of parallel weight $(2,\ldots,2)$. For each $\mathfrak{p}\in S_p$ we fix an ordinary parameter $\alpha_{\mathfrak{p}}\in\overline{\mathcal{O}}^*$ and put $a_{\mathfrak{p}} = \alpha_{\mathfrak{p}} + \mathrm{N}(\mathfrak{p})/\alpha_{\mathfrak{p}}$. As before we let $m = \sharp(S_p)$ and r be the number of $\mathfrak{p}\in S_p$ with $\alpha_{\mathfrak{p}} = 1$. We choose an ordering $\mathfrak{p}_1,\ldots,\mathfrak{p}_m$ of the places above p so that $\alpha_{\mathfrak{p}_1} = \ldots = \alpha_{\mathfrak{p}_r} = 1$. We write F_i, α_i and a_i instead of $F_{\mathfrak{p}_i}$, $\alpha_{\mathfrak{p}_i}$ and $a_{\mathfrak{p}_i}$ and put $\underline{\alpha} = (\alpha_1,\ldots,\alpha_m)$ and $\underline{a} = (a_1,\ldots,a_m)$. Moreover we denote by $\mathfrak{A}_0(G,\underline{2},\underline{\alpha})$ the subset of $\pi\in\mathfrak{A}_0(G,\underline{2})$ such that $\pi_{\mathfrak{p}_i} = \pi_{\alpha_i}$ for $i=1,\ldots,m$.

For $\pi\in\mathfrak{A}_0(G,\underline{2},\underline{\alpha})$ and a finite set of places S of F we put $\pi_S = \otimes_{v\in S}\pi_v$ and $\pi^S = \otimes_{v\notin S}\pi_v$. We also write $\pi_\infty, \pi_{p,\infty}, \pi^\infty$ etc. for $\pi_{S_\infty}, \pi_{S_p\cup S_\infty}, \pi^{S_\infty}$ etc. For each finite place $\mathfrak{q}$ of F we denote by $\mathfrak{f}(\pi_{\mathfrak{q}})$ the conductor of $\pi_{\mathfrak{q}}$ and we set $\mathfrak{f}(\pi) := \prod_{\mathfrak{q}}\mathfrak{f}(\pi_{\mathfrak{q}})$. Thus the multiplicity $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}(\pi))$ of $\mathfrak{p}\in S_p$ in $\mathfrak{f}(\pi)$ is $=1$ if $\alpha_{\mathfrak{p}} = \pm 1$ and $=0$ otherwise.

4.2. **Adelic Hilbert modular forms.** In section 4.6 we shall define a certain element $\phi_\pi \in \mathcal{D}(\mathbb{G}_m, S_1)$ for $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{\alpha})$. Firstly, we need to recall the notion of a Hilbert modular cusp form of parallel weight $(2, \ldots, 2)$ (in the adelic setting). It is a function $\Phi: G(\mathbf{A}) \to \mathbb{C}$ with the following properties

(i) For $\gamma \in G(F)$ we have $\Phi(\gamma g) = \Phi(g)$.
(ii) For any $g \in G(\mathbf{A})$, $k_\infty \in K_\infty^+$ we have $\Phi(g k_\infty) = j(k_\infty, \underline{i})^{-2}\Phi(g)$.
(iii) For any $g \in G(\mathbf{A}^\infty)$, $\underline{z} \in \mathbb{H}^{d+1}$ define $\mathbf{f}_\Phi(\underline{z}, g) := j(g_\infty, \underline{i})^2 \Phi(g_\infty, g)$ where $g_\infty \in G(F_\infty)^+$ is such that $g_\infty \underline{i} = \underline{z}$ (by (ii) $\mathbf{f}_\Phi(\underline{z}, g)$ is well defined). Then $\underline{z} \mapsto \mathbf{f}_\Phi(\underline{z}, g)$ is a holomorphic function on $\mathbb{H}^{d+1}$.
(iv) There exists a compact open subgroup K of $G(\mathbf{A}^\infty)$ such that $\Phi(gk) = \Phi(g)$ for all $k \in K$ and $g \in G(\mathbf{A})$.
(v) (Cuspidality) For any $g \in G(\mathbf{A})$ we have
$$\int_{\mathbf{A}/F} \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right) dx = 0.$$

We denote by $\mathcal{A}_0(G, \mathrm{hol}, \underline{2})$ the space of all functions Φ satisfying (i) – (v) above. It is a left $G(\mathbf{A}^\infty)$-module via the right action on $G(\mathbf{A})$. For a compact open subgroup $K \subseteq G(\mathbf{A}^\infty)$ we set $S_2(G, K) = \mathcal{A}_0(G, \mathrm{hol}, \underline{2})^K$. If $K = K_0(\mathfrak{n})$ for an ideal $\mathfrak{n}$ of $\mathcal{O}_F$, we write $S_2(G, \mathfrak{n})$ instead of $S_2(G, K_0(\mathfrak{n}))$.

Let $\Phi \in \mathcal{A}_0(G, \mathrm{hol}, \underline{2})$ and let $\mathbf{f}_\Phi$ be as in (iii) above. We define

$$\int_{\sigma_0(Q)}^{\sigma_0(P)} \mathbf{f}_\Phi(z_0, z_1, \ldots, z_d, g) dz_0 \tag{81}$$

by integrating the function $z_0 \mapsto \mathbf{f}_\Phi(z_0, z_1, \ldots, z_d, g)$ along the geodesic in $\mathbb{H}$ from $\sigma_0(Q)$ to $\sigma_0(P)$. Using the well-known fact that $\mathbf{f}_\Phi(\underline{z}, g)$ for fixed $g \in G(\mathbf{A}^\infty)$ rapidly decreases at the set of cusps $\mathbb{P}^1(F)$ of $\mathbb{H}^{d+1}$ it is easy to see that (81) is well-defined and that

$$\int_{\sigma_0(Q)}^{\sigma_0(P)} \mathbf{f}_\Phi(z_0, \ldots) dz_0 + \int_{\sigma_0(P)}^{\sigma_0(R)} \mathbf{f}_\Phi(z_0, \ldots) dz_0 = \int_{\sigma_0(Q)}^{\sigma_0(R)} \mathbf{f}_\Phi(z_0, \ldots) dz_0$$

for all $P, Q, R \in \mathbb{P}^1(F)$.

Let $\mathrm{Div}(\mathbb{P}^1(F))$ be the free abelian group over $\mathbb{P}^1(F)$ and let $\mathcal{M} = \mathrm{Div}_0(\mathbb{P}^1(F))$ be the subgroup of elements $\sum_{j=1}^r m_i P_i \in \mathrm{Div}(\mathbb{P}^1(F))$ with $\deg(\sum_{j=1}^r m_i P_i) = \sum_{j=1}^r m_i = 0$. The natural $G(F)$-action on $\mathbb{P}^1(F)$ induces a $G(F)$-action on $\mathcal{M}$. For $g \in G(\mathbf{A}^\infty)$ we obtain a homomorphism

$$\mathcal{M} \longrightarrow \mathcal{O}_{\mathrm{hol}}(\mathbb{H}^d), \ \ m \mapsto \int_{\sigma_0(m)} \mathbf{f}_\Phi(z_0, z_1, \ldots, z_d, g) dz_0$$

which coincides with (81) for $m = P - Q$. For $m \in \mathcal{M}$ we define

$$\int_m \omega_\Phi(g) = \left(\int_{\sigma_0(m)} \mathbf{f}_\Phi(z_0, z_1, \ldots, z_d, g) dz_0\right) dz_1 \ldots dz_d \in \Omega^d_{\mathrm{hol}}(\mathbb{H}^d).$$

We let the group $G(F)^+$ act on $\mathbb{H}^d$ via the embedding $G(F)^+ \to (G(\mathbb{R})^+)^d$, $\gamma \mapsto (\sigma_1(\gamma), \ldots, \sigma_d(\gamma))$. A simple computation using (i) shows

$$\gamma^*\left(\int_{\gamma m} \omega_\Phi(\gamma g)\right) = \int_m \omega_\Phi(g) \tag{82}$$

for all $\gamma \in G(F)^+$, $g \in G(\mathbf{A}^\infty)$ and $m \in \mathcal{M}$. The integral $\int_m \omega_\Phi(g)$ will be used in the construction of the *Eichler-Shimura map* (85) in section 4.5.

Definition 4.1. *(a) We denote by $\mathcal{A}_0(G, \mathrm{hol}, \underline{2}, \underline{a})$ the $\mathbb{C}$-vector space of maps $\Phi : G(\mathbf{A}^p) \to \mathscr{B}^{\underline{a}}(F_p, \mathbb{C})$ such that*

(i) *There exists a compact open subgroup K of $G(\mathbf{A}^{p,\infty})$ such that $\Phi(gk) = \Phi(g)$ for all $k \in K$ and $g \in G(\mathbf{A}^p)$.*
(ii) *For $\psi \in \mathscr{B}_{\underline{a}}(F_p, \mathbb{C})$ the map*

$$\langle \Phi, \psi \rangle : G(\mathbf{A}) = G(F_p) \times G(\mathbf{A}^p) \mapsto \mathbb{C}, (g_p, g^p) \mapsto \langle g_p \cdot \psi, \Phi(g^p) \rangle$$

lies in $\mathcal{A}_0(G, \mathrm{hol}, \underline{2})$.

$\mathcal{A}_0(G, \mathrm{hol}, \underline{2}, \underline{a})$ is a left $G(\mathbf{A}^{p,\infty})$-module via the right action on $G(\mathbf{A}^{p,\infty})$.

(b) For a compact open subgroup $K \subseteq G(\mathbf{A}^{p,\infty})$ we set $S_2(G, K, \underline{a}) = \mathcal{A}_0(G, \mathrm{hol}, \underline{2}, \underline{a})^K$. If $K = K_0(\mathfrak{m})^p$ where $\mathfrak{m}$ is an ideal of $\mathcal{O}_F$ which is relatively prime to $p\mathcal{O}_F$, we shall write $S_2(G, \mathfrak{m}, \underline{a})$ instead of $S_2(G, K_0(\mathfrak{m})^p, \underline{a})$.

Remarks 4.2. (a) Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{\alpha})$. It is easy to see that

$$\begin{aligned} \mathcal{A}_0(G, \mathrm{hol}, \underline{2}) &\cong \mathrm{Hom}_{G(F_\infty)}\left(\pi_\infty\, ,\ L^2_0(G(F)\backslash G(\mathbf{A}))\right), \\ \mathcal{A}_0(G, \mathrm{hol}, \underline{2}, \underline{a}) &\cong \mathrm{Hom}_{G(F_p \times F_\infty)}\left(\pi_{p,\infty}\, ,\ L^2_0(G(F)\backslash G(\mathbf{A}))\right) \end{aligned}$$

as representations of $G(\mathbf{A}^\infty)$ and $G(\mathbf{A}^{p,\infty})$ respectively.

(b) Assume that F has narrow class number 1. Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}_F$ and let $\Gamma_0(\mathfrak{n})$ be the subgroup of matrices $A \in \mathrm{SL}_2(\mathcal{O})$ which are upper triangular modulo $\mathfrak{n}$. Then $S_2(G, \mathfrak{n})$ can be identified with the space $S_2(\Gamma_0(\mathfrak{n}))$ of usual Hilbert modular cusp forms of parallel weight $(2, \ldots, 2)$ on $\Gamma_0(\mathfrak{n})$. Moreover if the ideal $\mathfrak{m}$ of $\mathcal{O}_F$ is relatively prime to $p\mathcal{O}_F$ and if $\mathfrak{n}$ is the product of $\mathfrak{m}$ with all $\mathfrak{p} \in S_p$ with $\alpha_\mathfrak{p} = \pm 1$ then $S_2(G, \mathfrak{m}, \underline{a})$ can be identified with the subspace of $f \in S_2(\Gamma_0(\mathfrak{n}))$ which satisfy (i) $T_\mathfrak{p} f = a_\mathfrak{p} f$ for all $\mathfrak{p} \in S_p$ with $\alpha_\mathfrak{p} \neq \pm 1$, (ii) f is $\mathfrak{p}$-new and $U_\mathfrak{p} f = \alpha_\mathfrak{p} f$ for all $\mathfrak{p} \in S_p$ with $\alpha_\mathfrak{p} = \pm 1$. Here, if $\mathfrak{p} \nmid \mathfrak{n}$ (resp. $\mathfrak{p} \mid \mathfrak{n}$) $T_\mathfrak{p}$ (resp. $U_\mathfrak{p}$) denotes the Hecke operator at $\mathfrak{p}$.

4.3. **Hecke Algebra.** We recall here a few facts about the Hecke algebra of $G(\mathbf{A}^S)$ (a reference for what follows is e.g. ([5], 3.4 and 4.2) or ([6], 1.2–4)). Fix a finite set of places S of F containing S_∞. Let dg denote the Haar measure on $G(\mathbf{A}^S)$ normalized such that $\int_K dg = 1$ for $K = \prod_{v \notin S} G(\mathcal{O}_v)$. For a field Ω of characteristic zero we denote by $\mathcal{H}^S_\Omega = \mathcal{H}_{G(\mathbf{A}^S)}$ the Hecke algebra of $G(\mathbf{A}^S)$ with coefficients in Ω, i.e. it is the convolution ring of locally constant compactly supported Ω-valued functions on $G(\mathbf{A}^S)$ (see [5], p. 309). If $K \subseteq G(\mathbf{A}^S)$ is any compact open subgroup then we let $\mathcal{H}^S_{K,\Omega}$ be the subspace of K-biinvariant functions in $\mathcal{H}^S_R$. Let $\mathfrak{q} \in \mathbf{P}_F - S$ and assume $K = K' \times G(\mathcal{O}_\mathfrak{q})$ for some compact open subgroup K' of $G(\mathbf{A}^{S'})$ where $S' = S \cup \{\mathfrak{q}\}$. Then $\mathcal{H}^S_{K,\Omega}$ is isomorphic to the tensor product of $\mathcal{H}^{S'}_{K',\Omega}$ and the Hecke algebra $\mathcal{H}_\Omega(G(F_\mathfrak{q}), G(\mathcal{O}_\mathfrak{q}))$. For the latter we have $\mathcal{H}_\Omega(G(F_\mathfrak{q}), G(\mathcal{O}_\mathfrak{q})) \cong \Omega[T_\mathfrak{p}]$ (see [5], 4.6.5) so in this case $\mathcal{H}^S_{K,\Omega} = \mathcal{H}^{S'}_{K',\Omega}[T_\mathfrak{q}]$.

Recall that the concepts "smooth $\mathcal{H}_\Omega^S$-module" and "smooth Ω-representation of $G(\mathbf{A}^S)$" are interchangeable. In the following we view a smooth $\mathcal{H}_\Omega^S$-module often as a smooth $G(\mathbf{A}^S)$-module and vice versa. A sequence of smooth $\mathcal{H}_\Omega^S$-modules $V_1 \to V_2 \to V_3$ is exact if and only if $V_1^K \to V_2^K \to V_3^K$ is exact for all compact open subgroups K of $G(\mathbf{A}^S)$. We call a smooth representation V of $G(\mathbf{A}^S)$ semisimple if it is isomorphic to a direct sum of smooth irreducible representations of $G(\mathbf{A}^S)$. A smooth representation V of $G(\mathbf{A}^S)$ is irreducible if and only if V^K is either zero or a simple $\mathcal{H}_{K,\Omega}^S$-module for all K. More generally it is easy to see that a smooth representation V of $G(\mathbf{A}^S)$ is semisimple if and only if V^K is a semisimple $\mathcal{H}_{K,\Omega}^S$-module for all K.

Let V and W be smooth $\mathcal{H}_\Omega^S$-modules and assume that V is irreducible and W is semisimple. Let K_0 be a compact open subgroup of $G(\mathbf{A}^S)$ such that $V^{K_0} \neq 0$. Then the canonical map $\mathrm{Hom}_{G(\mathbf{A}^S)}(V, W) \to \mathrm{Hom}_{\mathcal{H}_{K_0}^S}(V^{K_0}, W^{K_0})$ is an isomorphism. For that it is enough to assume that W is irreducible in which case the assertion follows from ([6], Prop. on p. 38).

For $\pi \in \mathfrak{A}_0(G, \underline{2})$ the complex representation π^S of $G(\mathbf{A}^S)$ is an example of a smooth irreducible representation. It is also known that π^S can be defined over a finite extension of $\mathbb{Q}$. More precisely there exists a smallest finite extension $\Omega = \Omega_\pi \subseteq \overline{\mathbb{Q}}$ of $\mathbb{Q}$ (the *field of definition of* π) and a smooth irreducible Ω-representation $G(\mathbf{A}^S) \to \mathrm{GL}(V)$ such that $\pi^S \cong V \otimes_\Omega \mathbb{C}$ as $G(\mathbf{A}^S)$-representations. By abuse of notation we also write π^S (resp. $\pi^{S,K}$) for the $\mathcal{H}_\Omega^S$-module V (resp. for the $\mathcal{H}_K^S$-module V^K). For a field $\mathbf{C}$ containing Ω_π, a compact open subgroup K of $G(\mathbf{A}^S)$ and a smooth semisimple $\mathbf{C}$-representation W of $G(\mathbf{A}^S)$ we write W_π for $\mathrm{Hom}_{G(\mathbf{A}^S)}(\pi^S, W) = \mathrm{Hom}_{G(\mathbf{A}^S)}(V \otimes_\Omega \mathbf{C}, W)$ and W_π^K for $\mathrm{Hom}_{\mathcal{H}_K^S}(V^K, W^K)$. Also if $f : W' \to W$ is a homomorphism of smooth semisimple $G(\mathbf{A}^S)$-representations we denote the induced homomorphism $W'_\pi \to W_\pi$ of $\mathbf{C}$-vector spaces by f_π. We have $W_\pi^K = W_\pi$ if $K \subseteq K_0(\mathfrak{f}(\pi))^S$ and $W_\pi^K = 0$ otherwise. If $K = K_0(\mathfrak{f}(\pi))^S$, then $\pi^{S,K}$ is one-dimensional as a Ω_π-vector space [8]. Thus the action of $\mathcal{H}_K^S$ is given by a homomorphism $\lambda_\pi : \mathcal{H}_K^S \to \Omega_\pi$ (it is known that $\lambda_\pi(T_\mathfrak{q})$ lies in the ring of integers of Ω_π). In this case we have $W_\pi = \{w \in W^K \mid tw = \lambda_\pi(t)w \ \forall t \in \mathcal{H}_K^S\}$. We also remark that if $W' \to W \to W''$ is an exact sequence of smooth semisimple representations of $G(\mathbf{A}^S)$ then $W'_\pi \to W_\pi \to W''_\pi$ is exact as well.

Finally, a representation W of $G(\mathbf{A}^S)$ will be said to be of *automorphic type* if W is smooth and semisimple and the only irreducible subrepresentations of W are either the one dimensional representations or the representations π^S for $\pi \in \mathfrak{A}_0(G, \underline{2})$. By strong multiplicity one, if W is of automorphic type then W_π is independent of the set S in the following sense. Let $S' \supset S$ and $K = \prod_{v \in S' - S} K(\mathfrak{f}(\pi))_v$. Then we have $\mathrm{Hom}_{G(\mathbf{A}^{S'})}(\pi^{S'}, W^K) = \mathrm{Hom}_{G(\mathbf{A}^S)}(\pi^S, W)$ (this fact will be used in the proof of Prop. 4.8 below).

4.4. **Cohomology of** $\mathrm{PGL}_2(F)$**.** In this section we introduce and study the cohomology groups of certain $G(F)^+$-modules $\mathscr{A}(\underline{a}, \mathcal{M}, \mathbb{C})$ on which the group $G(\mathbf{A}^{p,\infty})$ acts smoothly and so that each $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{a})$ occurs with multiplicity 2^{d+1} in $H^d(G(F)^+, \mathscr{A}(\underline{a}, \mathcal{M}, \mathbb{C}))$ (see Prop. 4.8 below).

Let $H \subset G(F)$ be any subgroup and let M be a left $G(F)$-module. Let R is a ring and N an $R[H]$-module. For a finite subset S of $\mathbf{P}_F^\infty$ we denote by $\mathscr{A}(S, M; N)$ the R-module of maps $\Phi : G(\mathbf{A}^{S,\infty}) \times M \to N$ such that $\Phi(g, -) : M \to N$ is a homomorphism and such that there exists a compact open subgroup K of $G(\mathbf{A}^{S,\infty})$ with $\Phi(gk, m) = \Phi(g, m)$ for all $k \in K$, $g \in G(\mathbf{A}^{S,\infty})$ and $m \in M$.

We have commuting $G(\mathbf{A}^{S,\infty})$- and H-actions on $\mathscr{A}(S, M; N)$; the first is induced by right multiplication on $G(\mathbf{A}^{S,\infty})$ and the second is given by $(\gamma \cdot \Phi)(g, m) = \gamma\Phi(\gamma^{-1}g, \gamma^{-1}m)$. For a compact open subgroup $K \subseteq G(\mathbf{A}^{S,\infty})$ we set $\mathscr{A}(K, S, M; N) = \mathscr{A}(S, M; N)^K$. If $K = K_0(\mathfrak{m})^S$ for an ideal $\mathfrak{m}$ of $\mathcal{O}_F$ not divisible by any $\mathfrak{q} \in S$ we write $\mathscr{A}(\mathfrak{m}, S, M; N)$ for $\mathscr{A}(K_0(\mathfrak{m})^S, S, M; N)$. If $S = \emptyset$ we will often drop S from the notation, i.e. we write $\mathscr{A}(M; N)$, $\mathscr{A}(K, M; N)$ etc. for $\mathscr{A}(S, M; N)$, $\mathscr{A}(K, S, M; N)$ etc.

In contrast to our previous notation in this section we denote by S_1, S_2 subsets of S_p with $S_1 \subseteq S_2 \subseteq S_p$. We define the $G(\mathbf{A}^{S_2,\infty})$-$H$-module $\mathscr{A}(\underline{a}_{S_1}, S_2, M; N)$ by

$$\mathscr{A}(\underline{a}_{S_1}, S_2, M; N) = \mathscr{A}(S_2, M; \mathscr{B}^{\underline{a}}(F_{S_1}, N))$$

Also for a compact open subgroup $K \subseteq G(\mathbf{A}^{S_2,\infty})$ we put $\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; N) = \mathscr{A}(\underline{a}_{S_1}, S_2, M; N)^K$ and if $K = K_0(\mathfrak{m})^{S_2}$ for an ideal $\mathfrak{m}$ of $\mathcal{O}_F$ not divisible by any $\mathfrak{p} \in S_2$ we set $\mathscr{A}(\mathfrak{m}, \underline{a}_{S_1}, S_2, M; N)$ for $\mathscr{A}(K_0(\mathfrak{m})^{S_2}, \underline{a}_{S_1}, S_2, M; N)$. If $S = S_1 = S_2$ (resp. $S_1 = S_2 = S_p$) (we deal mostly with the latter case) we shall drop S_2 (resp. S_1 and S_2) again from the notation, i.e. we put

$$\begin{aligned} \mathscr{A}(\underline{a}_S, M; N) &= \mathscr{A}(\underline{a}_S, S, M; N) = \mathscr{A}(S, M; \mathscr{B}^{\underline{a}}(F_S, N)), \\ \mathscr{A}(\underline{a}, M; N) &= \mathscr{A}(\underline{a}_{S_p}, S_p, M; N) = \mathscr{A}(S_p, M; \mathscr{B}^{\underline{a}}(F_p, N)). \end{aligned}$$

So $\mathscr{A}(\underline{a}, M; N)$ can be identified with the R-module of maps $\Phi : G(\mathbf{A}^{p,\infty}) \times M \times \mathscr{B}_{\underline{a}}(F_p, R) \to N$ which are invariant under some compact open subgroup of $G(\mathbf{A}^{p,\infty})$.

The pairing (34) induces a pairing

$$\langle\ ,\ \rangle : \mathscr{A}(\underline{a}_{S_1}, S_2, M; N) \times \mathscr{B}_{\underline{a}}(F_{S_1}, R) \longrightarrow \mathscr{A}(S_2, M; N)$$

hence a homomorphism $\mathscr{A}(\underline{a}_{S_1}, S_2, M; N) \to \mathscr{B}^{\underline{a}}(F_{S_1}, \mathscr{A}(S_2, M; N))$ which becomes an isomorphism when restricting it to K-invariant elements

$$\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; N) \xrightarrow{\cong} \mathscr{B}^{\underline{a}}(F_{S_1}, \mathscr{A}(K, S_2, M; N))$$

(for any compact open subgroup $K \subseteq G(\mathbf{A}^{S_2,\infty})$). Similarly, for any $\mathfrak{p} \in S_1$ and $S_0 := S_1 - \{\mathfrak{p}\}$ we have an isomorphism

$$\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; N) \xrightarrow{\cong} \mathscr{B}^{a_{\mathfrak{p}}}(F_{\mathfrak{p}}, \mathscr{A}(K, \underline{a}_{S_0}, S_2, M; N)) \tag{83}$$

Remark 4.3. Assume that M is free as an abelian group. For a compact open subgroup $K \subseteq G(\mathbf{A}^{S_2,\infty})$ we have

$$\begin{aligned}\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; R) &\cong \operatorname{Coind}_K^{G(\mathbf{A}^{S_2,\infty})} \operatorname{Hom}_{\mathbb{Z}}(M, \mathscr{B}^{\underline{a}}(F_{S_1}, R))\\ &\cong \operatorname{Coind}_K^{G(\mathbf{A}^{S_2,\infty})} \operatorname{Hom}_R(M \otimes_{\mathbb{Z}} \mathscr{B}_{\underline{a}}(F_{S_1}, R), R)\end{aligned}$$

Since $\mathscr{B}_{\underline{a}}(F_{S_1}, R)$ is a free R-module we see that the R-module $\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; R)$ is isomorphic to a product of copies of R.

Assume now that $R = \mathbb{C}_p$, let $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$ so that $L = \mathscr{A}(K, \underline{a}_{S_0}, S_2, M; \mathcal{O})$ is a complete lattice in $V = \mathscr{A}(K, \underline{a}_{S_0}, S_2, M; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p$ (see section 1). Let $\mathfrak{p} \in S_1$ such that $\alpha_{\mathfrak{p}} = 1$ and $S_0 = S_1 - \{\mathfrak{p}\}$. By (83) the vector space $\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p$ can be identified with $\mathrm{Dist}^{\mathrm{b}}(F_{\mathfrak{p}}, V)$ and so the evaluation map (6) extends to a pairing $\mathrm{Dist}^{\mathrm{b}}(F_{\mathfrak{p}}, V) \times C_\diamond(F_{\mathfrak{p}}, \mathbb{C}_p) \longrightarrow V$, i.e. we have a canonical bilinear map

$$\mathscr{A}(K, \underline{a}_{S_1}, S_2, M; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p \times \mathrm{St}(F_{\mathfrak{p}}, \mathbb{C}_p) \to \mathscr{A}(K, \underline{a}_{S_0}, S_2, M; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p.$$

It will be used for the construction of the $\mathcal{L}$-invariants $\mathcal{L}_{\mathfrak{p}}(\pi)$ in the next chapter.

The next result follows immediately from (21) and (22).

Lemma 4.4. *Let $S \subseteq S_p$, let $\mathfrak{p} \in S$ and let $S_0 := S - \{\mathfrak{p}\}$. Let $K \subseteq G(\mathbf{A}^{S,\infty})$ be a compact open subgroup and put $K_0 = K \times G(\mathcal{O}_{\mathfrak{p}})$, $K_1 = K \times K_0(\mathfrak{p})_{\mathfrak{p}}$.*

(a) If $\alpha_{\mathfrak{p}} \neq \pm 1$ then the following sequence is exact

$$0 \to \mathscr{A}(K, \underline{a}_S, M; N) \to \mathscr{A}(K_0, \underline{a}_{S_0}, M; N) \xrightarrow{(T_{\mathfrak{p}} - a_{\mathfrak{p}})} \mathscr{A}(K_0, \underline{a}_{S_0}, M; N) \to 0$$

(b) If $\alpha_{\mathfrak{p}} = \pm 1$ then there exists a short exact sequence

$$0 \to \mathscr{A}(K, \underline{a}_S, M; N) \to \mathscr{A}(K_1, \underline{a}_{S_0}, M; N)^{W = \mp 1} \to \mathscr{A}(K_0, \underline{a}_{S_0}, M; N) \to 0$$

where $W = W_{\mathfrak{p}}$ is a certain involution acting on $\mathscr{A}(K_1, \underline{a}_{S_0}, M; N)$.

Remark 4.5. The involution in part (b) above induces an involution – also denoted by $W_{\mathfrak{p}}$ – on the cohomology groups $H^\bullet(G(F)^+, \mathscr{A}(K_1, \underline{a}_{S_0}, M; N))$. In particular if $\mathfrak{n}$ is an ideal of $\mathcal{O}_F$ such that $\mathfrak{p}$ devides $\mathfrak{n}$ exactly once we have an involution $W_{\mathfrak{p}}$ acting on $H^{d+1}(G(F)^+, \mathscr{A}(\mathfrak{n}, R))$. This is the analogue of the Atkin-Lehner involution. As in the classical case we have $W_{\mathfrak{p}} = -U_{\mathfrak{p}}$ on $H^{d+1}(G(F)^+, \mathscr{A}(\mathfrak{n}, R))$.

Proposition 4.6. *Let $S_1 \subseteq S_2 \subseteq S_p$ and let K be a compact open subgroup of $G(\mathbf{A}^{S_2,\infty})$.*

(a) Let N be a flat R-module (equipped with the trivial $G(F)$-action). Then the canonical map

$$H^q(G(F)^+, \mathscr{A}(K, \underline{a}_{S_1}, S_2, \mathcal{M}; R)) \otimes_R N \to H^q(G(F)^+, \mathscr{A}(K, \underline{a}_{S_1}, S_2, \mathcal{M}; N))$$

is an isomorphism for all $q \geq 0$.

(b) If R is noetherian then $H^q(G(F)^+, \mathscr{A}(K, \underline{a}_{S_1}, S_2, \mathcal{M}; R))$ is a finitely generated R-module.

Proof. (a) The sequence $0 \to \mathcal{M} \to \mathrm{Div}(\mathbb{P}^1(F)) \cong \mathrm{Ind}_{B(F)}^{G(F)} \mathbb{Z} \to \mathbb{Z} \to 0$ yields a short exact sequence

$$0 \longrightarrow \mathscr{A}(K, \underline{a}_{S_1}, S_2; N) \longrightarrow \mathrm{Coind}_{B(F)^+}^{G(F)^+} \mathscr{A}(K, \underline{a}_{S_1}, S_2; N) \longrightarrow \mathscr{A}(K, \underline{a}_{S_1}, S_2, \mathcal{M}; N) \longrightarrow 0 \tag{84}$$

(where $\mathscr{A}(K, \underline{a}_{S_1}, S_2; N) := \mathscr{A}(K, \underline{a}_{S_1}, S_2, \mathbb{Z}; N)$). By considering the associated long exact cohomology sequences it is enough to prove the assertion when we replace the coefficients $\mathscr{A}(K, \underline{a}_{S_1}, S_2, \mathcal{M}; \cdot)$ with $\mathscr{A}(K, \underline{a}_{S_1}, S_2; \cdot)$ or $\mathrm{Coind}_{B(F)^+}^{G(F)^+} \mathscr{A}(K, \underline{a}_{S_1}, S_2; \cdot)$ (here we use the flatness assumption). Furthermore using Lemma 4.4 it is enough to consider the case $S_1 = \emptyset$, $S = S_2$. Since $\mathscr{A}(K, S, \mathbb{Z}; N) \cong \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N$ it suffices to show that

$$H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} R) \otimes_R N \longrightarrow H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N)$$
$$H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} R) \otimes_R N \longrightarrow H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N)$$

are isomorphism for all $q \geq 0$ and all R-modules N. For that it is enough to prove that the functors $N \mapsto H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N)$ and $N \mapsto H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N)$ commute with direct limits (since any module is the direct limit of free modules of finite rank). For $g \in G(\mathbf{A}^{S,\infty})$ put $\Gamma_g = G(F)^+ \cap gKg^{-1}$. By the strong approximation theorem there are only finitely many double cosets $G(F)^+ gK$ in $G(\mathbf{A}^{S,\infty})$. If $g_1, \ldots, g_n \in G(\mathbf{A}^{S,\infty})$ is a system of representatives then

$$H^q(G(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N) \;=\; \bigoplus_{i=1}^{n} H^q(\Gamma_{g_i}, N).$$

Since the group Γ_g is S-arithmetic, hence of type (VFL), the functor $N \mapsto H^q(\Gamma_g, N)$ commutes with direct limits (see [24], p. 101). The same proof works for $H^q(B(F)^+, \mathrm{Coind}_K^{G(\mathbf{A}^{S,\infty})} N)$ as well. Indeed using the Iwasawa decomposition $G(\mathbf{A}^{S,\infty}) = B(\mathbf{A}^{S,\infty}) \prod_{v \nmid \infty} G(\mathcal{O}_v)$ one can easily see that $B(F)^+ \backslash G(\mathbf{A}^{S,\infty})/K$ is finite.

(b) can be deduced using similar arguments and the fact that the groups Γ_g are S-arithmetic and ([24], remarque on p. 101). $\square$

Let $S_1 \subset S_2 \subseteq S_p$ be as before, let R be a ring and let M be a left $G(F)$-module. We define

$$H^q_*(G(F)^+, \mathscr{A}(\underline{a}_{S_1}, S_2, M; R)) = \varinjlim H^q(G(F)^+, \mathscr{A}(K, \underline{a}_{S_1}, S_2, M; R))$$

where K runs through all compact open subgroups of $G(\mathbf{A}^{S_2,\infty})$. By 4.6 we have

Corollary 4.7. *Let $S \subseteq S_p$ and let $R \to R'$ be a flat ring homomorphism. Then the canonical map*

$$H^q_*(G(F)^+, \mathscr{A}(\underline{a}_S, \mathcal{M}; R)) \otimes_R R' \longrightarrow H^q_*(G(F)^+, \mathscr{A}(\underline{a}_S, \mathcal{M}; R'))$$

is an isomorphism for all $q \geq 0$.

If $R = \mathbf{C}$ is a field of characteristic zero then $H^q_*(G(F)^+, \mathscr{A}(\underline{a}_{S_1}, S_2, M; \mathbf{C}))$ is a smooth $G(\mathbf{A}^{S_2,\infty})$-module and it is easy to see that we have

$$H^q_*(G(F)^+, \mathscr{A}(\underline{a}_{S_1}, S_2, M; \mathbf{C}))^K = H^q(G(F)^+, \mathscr{A}(K, \underline{a}_{S_1}, S_2, M; \mathbf{C})).$$

We identify $G(F)/G(F)^+$ with the group $\Sigma = \{\pm 1\}^{d+1}$ via the isomophism $G(F)/G(F)^+ \xrightarrow{\det} F^*/F^*_+ \cong \Sigma$ (compare Remark 3.13). Hence Σ acts on $H^q_*(G(F)^+, \mathscr{A}(\underline{a}_{S_1}, S_2, M; \mathbf{C}))$ and $H^q(G(F)^+, \mathscr{A}(K, \underline{a}_{S_1}, S_2, M; \mathbf{C}))$ by conjugation.

Proposition 4.8. *Let $S \subseteq S_p$ and let $\mathbf{C}$ be a field containing the field of definition of $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{\alpha})$.*

(a) The $G(\mathbf{A}^{S,\infty})$-representation $H^q_*(G(F)^+, \mathscr{A}(\underline{a}_S, \mathcal{M}; \mathbf{C}))$ is of automorphic type for all $q \in \mathbb{Z}$.

(b) Let $\underline{\mu} \in \Sigma$ and $q \in \{0, 1, \ldots, d\}$. Then

$$H^q_*(G(F)^+, \mathscr{A}(\underline{a}_S, \mathcal{M}; \mathbf{C}))_{\pi,\underline{\mu}} \;=\; \begin{cases} \mathbf{C} & \text{if } q = d; \\ 0 & \text{if } q \leq d-1. \end{cases}$$

Proof. Firstly, we assume $S = \emptyset$. Consider the long exact sequence

$$\begin{aligned} \ldots \longrightarrow H^q_*(G(F)^+, \mathscr{A}(\mathbf{C})) \longrightarrow H^q_*(B(F)^+, \mathscr{A}(\mathbf{C})) \longrightarrow \\ \longrightarrow H^q_*(G(F)^+, \mathscr{A}(\mathcal{M}; \mathbf{C})) \longrightarrow H^{q+1}_*(G(F)^+, \mathscr{A}(\mathbf{C})) \ldots \end{aligned}$$

associated to (84) (the second group is defined similarly as the direct limit of the groups $H^q(B(F)^+, \mathscr{A}(K, \mathbf{C}))$). The action of $G(\mathbf{A}^\infty)$ on $H^q_*(G(F)^+, \mathscr{A}(\mathbf{C}))$ and $H^q_*(B(F)^+, \mathscr{A}(\mathbf{C}))$ has been determined in [16]. As a $G(\mathbf{A}^\infty)$-module $H^q_*(G(F)^+, \mathscr{A}(\mathbf{C}))$ is a direct sum of one dimensional representations except for $q = d+1$ in which case there exists a $G(\mathbf{A}^\infty)$-stable decomposition

$$H^{d+1}_*(G(F)^+, \mathscr{A}(\mathbf{C})) \;=\; H^{d+1}_{\text{cusp}} \oplus H^{d+1}_{\text{res}} \oplus H^{d+1}_{\text{Eis}}.$$

Again, the action of $G(\mathbf{A}^\infty)$ on the second and third summand is direct sum of one dimensional representations. On the first factor it is of automorphic type and we have ([16], 3.6.2.2)

$$H^{d+1}_{\text{cusp}}(G(F)^+, \mathscr{A}(\mathbf{C}))_{\pi,\underline{\mu}} \;=\; \mathbf{C}.$$

Using 4.4, (a) can now be easily deduced from the case $S = \emptyset$. For (b) we may pass to the $K_0(\mathfrak{m})^S$-invariant part $H^q(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}_S, \mathcal{M}; \mathbf{C}))_{\underline{\mu}}$ where $\mathfrak{m}$ denotes the maximal prime-to-S divisor of $\mathfrak{f}(\pi)$. By keeping in mind that the Hecke operator $T_\mathfrak{p}$ (resp. $U_\mathfrak{p} = -W_\mathfrak{p}$) acts by multiplication with $a_\mathfrak{p}$ (resp. ± 1) on $H^d_*(G(F)^+, \mathscr{A}(\underline{a}_{S'}, \mathcal{M}; \mathbf{C}))_\pi$ for $\mathfrak{p} \in S$ with $\alpha_\mathfrak{p} \neq \pm 1$ (resp. $\alpha_\mathfrak{p} = \pm 1$) and $S' \subseteq S - \{\mathfrak{p}\}$, (b) can also be deduced from the case $S = \emptyset$ using Lemma 4.4. $\square$

4.5. **Eichler-Shimura map.** For the rest of this chapter we change the notation slightly again and denote by $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ the subset of $\mathfrak{p} \in S_p$ such that $\alpha_\mathfrak{p} = 1$ (i.e. $\pi_\mathfrak{p} = \mathrm{St}$) and put $S_2 = S_p - S_1$ (thus with our previous notation we have $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ and $S_2 = \{\mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_m\}$).

Let $K \subseteq G(\mathbf{A}^{p,\infty})$ be a compact open subgroup. Our aim is to define a $\mathcal{H}_K^{p,\infty}$-equivariant homomorphism (Eichler-Shimura map)

$$S_2(G, K, \underline{a}) \to H^d(G(F)^+, \mathscr{A}(K, \underline{a}, \mathcal{M}; \mathbb{C})), \;\; \Phi \mapsto \kappa_\Phi. \tag{85}$$

Its definition is similar to (76) (the role of the manifold U_∞ with its natural F_+^*-action is replaced by $\mathbb{H}^{d+1}$ with $G(F)^+$ acting on it). Firstly, define

$$I_0 : S_2(G, K, \underline{a}) \longrightarrow H^0(G(F)^+, \mathscr{A}(K, \underline{a}, \mathcal{M}; \Omega^d_{\mathrm{hol}}(\mathbb{H}^d))) \tag{86}$$

by

$$\langle I_0(\Phi), \psi\rangle(g, m) = \int_m \omega_{\langle \Phi, \psi\rangle}(1, g)$$

for $\psi \in C_{\underline{a}}(F_p, \mathbb{C})$, $g \in G(\mathbf{A}^{p,\infty})$ and $m \in \mathcal{M}$ (here $(1, g)$ denotes the element $(1_{G(F_p)}, g) \in G(F_p) \times G(\mathbf{A}^{p,\infty}) = G(\mathbf{A}^\infty)$). That the image of I_0 is $G(F)^+$-invariant follows from (82) by a tedious but straightforward computation. Note that the complex $C^\bullet := \mathscr{A}(K, \underline{a}, \mathcal{M}; \Omega^\bullet_{\mathrm{hol}}(\mathbb{H}^d))$ is a resolution of $\mathscr{A}(K, \underline{a}, \mathcal{M}; \mathbb{C})$ and we have $C^q = 0$ for $q > d$. We define (85) to be the composite of (86) with the edge morphism

$$H^0(G(F)^+, C^d) \longrightarrow H^d(G(F)^+, C^\bullet) \cong H^d(G(F)^+, \mathscr{A}(K, \underline{a}, \mathcal{M}; \mathbb{C}))$$

of the spectral sequence $E_1^{pq} = H^q(G(F)^+, C^p) \Rightarrow E^{p+q} = H^{p+q}(G(F)^+, C^\bullet)$.

Next we define two maps

$$\Delta^{\underline{\alpha}} : S_2(G, \mathfrak{m}, \underline{a}) \longrightarrow \mathcal{D}(\mathbb{G}_m, S_1) \tag{87}$$
$$\Delta^{\underline{\alpha}} : \mathscr{A}(\mathfrak{m}, \underline{a}_{S_0}, S_p, \mathcal{M}; N) \longrightarrow \mathscr{D}(S_0 \cap S_1, S_0 \cap S_2, N) \tag{88}$$

(for S_0 any subset of S_p) which are global analogues of the map $\delta^{\underline{\alpha}}$ defined in section 2.5. The first is given by

$$\Delta^{\underline{\alpha}}(\Phi)(f, x^p) \;=\; \delta^{\underline{\alpha}}\left(\Phi\begin{pmatrix} x^p & 0 \\ 0 & 1\end{pmatrix}\right)(f) \;=\; \langle\left(\Phi\begin{pmatrix} x^p & 0 \\ 0 & 1\end{pmatrix}\right), \delta_{\underline{\alpha}}(f)\rangle$$

for $f \in C_c^0(F_{S_1} \times F_{S_2}^*, \mathbb{Z})$ and $x^p \in \mathbf{I}^p$ and the second by

$$\begin{aligned}\Delta^{\underline{\alpha}}(\Phi)(f, x^{p,\infty}) &= \delta^{\underline{\alpha}}\left(\Phi\left(\begin{pmatrix} x^{p,\infty} & 0 \\ 0 & 1\end{pmatrix}, \infty - 0\right)\right)(f) \\ &= \langle\left(\Phi\left(\begin{pmatrix} x^{p,\infty} & 0 \\ 0 & 1\end{pmatrix}, \infty - 0\right)\right), \delta_{\underline{\alpha}}(f)\rangle\end{aligned}$$

for $f \in C_c^0(F_{S_0\cap S_1} \times F^*_{S_0 \cap S_2}, \mathbb{Z})$ and $x^{p,\infty} \in \mathbf{I}^{p,\infty}$ (as usual $\infty, 0$ denote the points $[1:0]$ and $[0:1]$ of $\mathbb{P}^1(F)$ so $\infty - 0 \in \mathcal{M}$).

One checks easily that (88) is $T(F)$-equivariant (we let $T(F)$ act on $\mathscr{D}(S_0 \cap S_1, S_0 \cap S_2, N)$ via the identification $T = \mathbb{G}_m$). Hence the maps $F_+^* = T(F)^+ \hookrightarrow G(F)^+$, $\Delta^{\underline{\alpha}} : \mathscr{A}(\underline{a}_{S_0}, \mathcal{M}; N) \to \mathscr{D}(S_0 \cap S_1, S_0 \cap S_2, N)$ induce a Σ-equivariant homomorphism

$$H^q(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}_{S_1}, S_p, \mathcal{M}; N)) \longrightarrow H^q(F_+^*, \mathscr{D}(S_0 \cap S_1, S_0 \cap S_2, N)). \tag{89}$$

The proof of the following lemma is straightforward and will be left to the reader.

Lemma 4.9. *The following diagram commutes*

$$\begin{array}{ccc} S_2(G,\mathfrak{m},\underline{a}) & \xrightarrow{(85)} & H^d(G(F)^+,\mathscr{A}(\mathfrak{m},\underline{a},\mathcal{M};\mathbb{C})) \\ \Big\downarrow \Delta^{\underline{\alpha}} & & \Big\downarrow (89) \\ \mathcal{D}(\mathbb{G}_m,S_1) & \xrightarrow{(76)} & H^d(F_+^*,\mathscr{D}(S_1,S_2,\mathbb{C})) \end{array}$$

4.6. **p-adic measures attached to Hilbert modular forms.** Let $\pi \in \mathfrak{A}_0(G,\underline{2},\underline{\alpha})$. As in section 2.6 let μ_{α_i} denote the distribution $\chi_{\alpha_i}(x)\psi_{\mathfrak{p}_i}(x)dx$ on F_i (resp. F_i^*) if $i \le r$ (resp. $i > r$) and let $\mu_{\pi_p} = \mu_{\alpha_1} \times \ldots \times \mu_{\alpha_m}$ be the product distribution on $F_{S_1} \times F_{S_2}^*$.

For $v \in \mathbf{P}_F$ let $\mathcal{W}_v$ denote the Whittaker model of π_v and let $\mathcal{W} = \mathcal{W}(\pi)$ be the global Whittaker model. We can choose $W_v \in \mathcal{W}_v$ such that the local zeta function

$$\zeta(s,W_v,\chi_v) = \int_{F_v^*} W_v\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\chi_v(x)\,|x|^{s-\frac{1}{2}}d^\times x$$

is equal to the local L-factor $L(s,\pi_v\otimes\chi_v)$ for any unramified quasicharacter $\chi_v : F_v^* \to \mathbb{C}^*$ and $\mathrm{Re}(s) \gg 0$. In fact if $v \in S_\infty$ we can choose W_v such that $W_v(gk) = j(k,i)^{-2}W_v(g)$ for all $g \in \mathrm{GL}_2(\mathbb{R})$ and $k \in \mathrm{SO}(2)$. If v is finite we can (and will) take W_v to be $K_0(\mathfrak{f}(\pi_v))_v$-invariant. It is then uniquely determined ([8], Theorem 1). If π_v is spherical (i.e. $\mathrm{ord}_v(\mathfrak{f}(\pi)) = 0$), then $W_v = W_{v,0}$ is the unique $G(\mathcal{O}_v)$-invariant element of $\mathcal{W}_v$ with $W_{v,0}(g) = 1$ for all $g \in G(\mathcal{O}_v)$. Put $W^p(g) = \prod_{v\nmid p} W_v(g_v)$ for $g = (g_v) \in G(\mathbf{A}^p)$.

We define $\phi = \phi_\pi : C_c^0(F_{S_1}\times F_{S_2}^*,\mathbb{Z}) \times \mathbf{I}^p/U^{p,\infty} \to \mathbb{C}$ in $\mathcal{D}(\mathbb{G}_m,S_1)$ by

$$\phi(f,x^p) := \sum_{\zeta\in F^*} \mu_{\pi_p}(\zeta f)\; W^p\begin{pmatrix}\zeta x^p & 0\\ 0 & 1\end{pmatrix}.$$

That ϕ_π is well-defined follows from 2.10 (b). In fact for $f \in C_c^0(F_{S_1}\times F_{S_2}^*,\mathbb{Z})$ there exists an element W_f of the Whittaker model $\mathcal{W}_p$ of $\pi_p = \otimes_{v\in S_p}\pi_v$ such that

$$\phi_f(x) := \phi(x_p f, x^p) = \sum_{\zeta\in F^*} W\begin{pmatrix}\zeta x & 0\\ 0 & 1\end{pmatrix}$$

where $W(g) := W_f(g_p)W^p(g^p)$ for $g = (g_p,g^p) \in G(\mathbf{A})$. It follows $\phi_\pi \in \mathcal{D}(\mathbb{G}_m,S_1)$. Let $\mu_\pi := \mu_{\phi_\pi}$ be the corresponding distribution on $\mathcal{G}_p$ and $\kappa_\pi := \kappa_{\phi_\pi} \in H^d(F_+^*,\mathscr{D}(S_1,S_2,\mathbb{C}))$. We have

Proposition 4.10. *(a) (Interpolation property) Let $\chi : \mathcal{G}_p \to \mathbb{C}^*$ be a character of finite order with conductor $\mathfrak{f}(\chi)$. Then*

$$\int_{\mathcal{G}_p}\chi(\gamma)\mu_\pi(d\gamma) \;=\; \tau(\chi)\prod_{\mathfrak{p}\in S_p} e(\alpha_\mathfrak{p},\chi_\mathfrak{p})\,\cdot\, L(\tfrac{1}{2},\pi\otimes\chi)$$

where

$$e(\alpha_\mathfrak{p},\chi_\mathfrak{p}) = \begin{cases} (1-\alpha_\mathfrak{p}\chi(\varpi)^{-1}) & \textit{if } \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0, \alpha_\mathfrak{p} = \pm 1;\\ (1-\frac{\chi(\varpi)}{\alpha_\mathfrak{p}})(1-\frac{1}{\alpha_\mathfrak{p}\chi(\varpi)}) & \textit{if } \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) = 0, \alpha_\mathfrak{p} \ne \pm 1;\\ \alpha_\mathfrak{p}^{-\mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi))} & \textit{if } \mathrm{ord}_\mathfrak{p}(\mathfrak{f}(\chi)) > 0. \end{cases}$$

(b) Let $U_0 = \prod_{\mathfrak{p}\in S_1} \mathcal{O}_{\mathfrak{p}} \times \prod_{\mathfrak{p}\in S_2} \mathcal{O}_{\mathfrak{p}}^*$ and put $\phi_0 := (\phi_\pi)_{U_0}$. Then,

$$\int_{\mathbf{I}/F^*} \phi_0(x)\, d^\times x \;=\; \prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}}, 1)\,\cdot\, L(\tfrac{1}{2}, \pi)$$

(c) κ_π is integral (compare Def. 3.16). For $\underline{\mu} \in \Sigma$ let $\kappa_{\pi,\underline{\mu}}$ denote the projection of κ_π onto $H^d(F_+^*, \mathscr{D}(S_1, S_2, \mathbb{C}))_{\underline{\mu}}$. Then $\kappa_{\pi,\underline{\mu}}$ is integral of rank ≤ 1.

Proof. (a) We view χ as a character of $\mathbf{I}/F^*$ and choose an open subgroup U of U_p which lies in the kernel of $\chi_p = \chi|_{F_p^*}$. Let $W_U := W_{1_U} \in \mathcal{W}_p$ be as above and $W(g) := W_U(g_p)W^p(g^p) \in \mathcal{W}$. It suffices to show

$$\begin{aligned}[U_p : U] \int_{\mathbf{I}/F^*} &\chi(x)|x|^s \phi_U(x)\, d^\times x \\ &= \mathrm{N}(\mathfrak{f}(\chi))^s \tau(\chi) \prod_{\mathfrak{p}\in S_p} e(\alpha_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)\,\cdot\, L(s+\tfrac{1}{2}, \pi\otimes\chi)\end{aligned}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > -1$. Since both sides are holomorphic functions it is enough to prove this for $\mathrm{Re}(s) \gg 0$. Using 2.10 (c) and 2.9 we obtain

$$\begin{aligned}&[U_p : U] \textstyle\int_{\mathbf{I}/F^*} \chi(x)|x|^s \phi_U(x)\, d^\times x = [U_p : U] \int_{\mathbf{I}} \chi(x)|x|^s W\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} d^\times x \\ &= [U_p : U] \textstyle\int_{F_p^*} \chi_p(x)|x|^s W_U\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} d^\times x \int_{\mathbf{I}^p} \chi^p(y)|y|^s W^p\begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} d^\times y \\ &= \textstyle\prod_{\mathfrak{p}\in S_p} \int_{F_{\mathfrak{p}}^*} \chi_{\mathfrak{p}}(x)|x|_{\mathfrak{p}}^s\, \mu_{\alpha_{\mathfrak{p}}}(dx)\;\cdot L_{S_p}(s+\tfrac{1}{2}, \pi\otimes\chi) \\ &= \mathrm{N}(\mathfrak{f}(\chi))^s\tau(\chi) \textstyle\prod_{\mathfrak{p}\in S_p} e(\alpha_{\mathfrak{p}}, \chi_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)\,\cdot\, L(s+\tfrac{1}{2}, \pi\otimes\chi).\end{aligned}$$

(b) Again it suffices to show that

$$\int_{\mathbf{I}/F^*} |x|^s\phi_0(x) d^\times x \;=\; \prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}}, |\cdot|_{\mathfrak{p}}^s)\,\cdot\, L(s+\tfrac{1}{2}, \pi)$$

for $\mathrm{Re}(s) \gg 0$. A similar computation as above yields

$$\begin{aligned}&\textstyle\int_{\mathbf{I}/F^*} |x|^s\phi_0(x) d^\times x \;= \\ &\textstyle\prod_{\mathfrak{p}\in S_1}\int_{F_{\mathfrak{p}}^*} |x|_{\mathfrak{p}}^s \mu_1(x\mathcal{O}_{\mathfrak{p}}) d^\times x\,\cdot\,\prod_{\mathfrak{p}\in S_2}\int_{F_{\mathfrak{p}}^*} |x|_{\mathfrak{p}}^s \mu_{\alpha_{\mathfrak{p}}}(x\mathcal{O}_{\mathfrak{p}}^*) d^\times x\,\cdot\, L_{S_p}(s+\tfrac{1}{2}, \pi).\end{aligned}$$

For $\mathfrak{p} \in S_1$ we have $\mu_1(x\mathcal{O}_{\mathfrak{p}}) = \int_{F_{\mathfrak{p}}} 1_{x\mathcal{O}_{\mathfrak{p}}}(y)\psi_{\mathfrak{p}}(y)dy = |x|_{\mathfrak{p}} 1_{\mathcal{O}_{\mathfrak{p}}}(x)$ hence

$$\begin{aligned}\int_{F_{\mathfrak{p}}^*} |x|_{\mathfrak{p}}^s \mu_1(x\mathcal{O}_{\mathfrak{p}}) d^\times x \;&=\; \int_{F_{\mathfrak{p}}^*} |x|_{\mathfrak{p}}^{s+1} 1_{\mathcal{O}_{\mathfrak{p}}}(x) d^\times x \\ &=\; (1-\mathrm{N}(\mathfrak{p})^{-(s+1)})^{-1} \;=\; L(s+\tfrac{1}{2}, \pi_{\mathfrak{p}}).\end{aligned}$$

On the other hand by Prop. 2.9 we get for $\mathfrak{p} \in S_2$

$$\int_{F_{\mathfrak{p}}^*} |x|_{\mathfrak{p}}^s\; \mu_{\alpha_{\mathfrak{p}}}(x\mathcal{O}_{\mathfrak{p}}^*) d^\times x = \int_{F_{\mathfrak{p}}^*} |x|_{\mathfrak{p}}^s\; \mu_{\alpha_{\mathfrak{p}}}(dx) \;=\; e(\alpha_{\mathfrak{p}}, |\cdot|_{\mathfrak{p}}^s)\,\cdot\, L(s+\tfrac{1}{2}, \pi_\alpha).$$

The assertion follows.

(c) Let $\lambda_{\underline{a}} \in \mathscr{B}^{\underline{a}}(F_p, \overline{\mathbb{Q}})$ be the image of $\otimes_{i=1}^m \lambda_{a_i}$ under (35) and define $\Phi_\pi \in \mathcal{A}_0(G, \mathrm{hol}, \underline{2}, \underline{a})$ by

$$\langle \psi, \Phi_\pi(g^p)\rangle \;=\; \sum_{\zeta\in F^*} \langle \begin{pmatrix} \zeta & 0\\ 0 & 1\end{pmatrix} \psi, \lambda_{\underline{a}}\rangle\, W^p\left(\begin{pmatrix} \zeta & 0\\ 0 & 1\end{pmatrix} g^p\right)$$

for $g^p \in G(\mathbf{A}^p)$ and $\psi \in \mathscr{B}_{\underline{a}}(F_p, \mathbb{C})$. To see that Φ_π satisfies property (ii) of Def. 4.1, let $\psi \in \mathscr{B}_{\underline{a}}(F_p, \mathbb{C})$ and define $W_\psi \in \mathcal{W}(\pi)$ by $W_\psi(g_p, g^p) \colon = \langle g_p\psi, \lambda_{\underline{a}}\rangle\, W^p(g_p)$. Then

$$\langle \psi, \Phi_\pi\rangle(g) \;=\; \sum_{\zeta\in F^*} W_\psi\left(\begin{pmatrix} \zeta & 0\\ 0 & 1\end{pmatrix} g\right) \in \mathcal{A}_0(G, \mathrm{hol}, \underline{2}).$$

Let $\mathfrak{m}$ be the maximal prime-to-p divisor of $\mathfrak{f}(\pi)$. Because W^p is $K_0(\mathfrak{m})$-invariant we have $\Phi_\pi \in S_2(G, \mathfrak{m}, \underline{a})$. Since $\Delta^{\underline{\alpha}}(\Phi_\pi) = \phi_\pi$ by 2.10 (a), we can apply Lemma 4.9 to conclude that κ_π lies in the image of (89). That κ_π is integral now follows from the fact that $R \mapsto H^d(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; R))$ commutes with flat base change by Prop. 4.6. The second assertion is a consequence of Prop. 4.6 and 4.8. $\square$

Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{\alpha})$. By 3.18 and 3.21 the distribution μ_π is a p-adic measure. We define the *p-adic L-function of* π by

$$L_p(s, \pi) \colon = L_p(s, \kappa_\pi) = L_p(s, \kappa_{\pi,+}) = \int_{\mathcal{G}_p} \langle\gamma\rangle^s \mu_\pi(d\gamma) \quad \text{ for } s \in \mathbb{Z}_p.$$

It is a locally analytic function with values in the vector space $L_{\kappa_{\pi,+}} \otimes_{\overline{\mathcal{O}}} \mathbb{C}_p$ of dimension ≤ 1 (compare Remark 3.13).

5. Exceptional zero conjecture

5.1. **Automorphic $\mathcal{L}$-invariants.** We keep the notation and assumptions from the end of last section. In this section we define for each $\mathfrak{p} \in S_1$ a certain number $\mathcal{L}_\mathfrak{p}(\pi) \in \mathbb{C}_p$, the *$\mathcal{L}$-invariant of* π *at* $\mathfrak{p}$. It has the property that it does not change under suitable quadratic twists (see Lemma 5.5 below).

Let $\mathcal{O}$ denote the valuation ring of $\mathbb{C}_p$. Fix $\mathfrak{p} \in S_1$ and put $S_0 = S_p - \{\mathfrak{p}\}$. Recall that by Remark 4.3) there exists a canonical pairing

$$\begin{aligned}(90)\qquad & \mathscr{A}(K, \underline{a}, \mathcal{M}; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p \times \mathrm{St}(F_\mathfrak{p}, \mathbb{C}_p)\\ &\longrightarrow\; \mathscr{A}(K, \underline{a}_{S_0}, S_p, \mathcal{M}; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p \subseteq \mathscr{A}(K, \underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p).\end{aligned}$$

It induces a cup-product pairing on $G(F)^+$-cohomology. Together with 4.6 this yields a pairing

$$\begin{aligned}(91)\qquad & \cup : H^p(G(F)^+, \mathscr{A}(K, \underline{a}, \mathcal{M}; \mathbb{C}_p)) \times H^q(G(F)^+, \mathrm{St}(F_\mathfrak{p}, \mathbb{C}_p))\\ &\qquad\qquad \longrightarrow H^{p+q}(G(F)^+, \mathscr{A}(K, \underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p))\end{aligned}$$

Hence by passing to the direct limit over all K we obtain a homomorphism of smooth $G(\mathbf{A}^{p,\infty})$-representations

$$\cdot \cup b : H^p_*(G(F)^+, \mathscr{A}(\underline{a}, \mathcal{M}; \mathbb{C}_p)) \to H^{p+q}_*(G(F)^+, \mathscr{A}(\underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p))$$

for all $b \in H^q(G(F)^+, \mathrm{St}(F_{\mathfrak{p}}, \mathbb{C}_p))$.

Remarks 5.1. (a) Note that in (90) we cannot replace $\mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p$ by $\mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathbb{C}_p)$. Therefore the compatibility of $R \mapsto H^{q_1}(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; R))$ with flat base change is crucial in the definition of (91).

(b) Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_F$ which is relatively prime to $p\mathcal{O}_F$. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p \times \mathrm{St}(F_{\mathfrak{p}}, \mathbb{C}_p) & \longrightarrow & \mathscr{A}(\mathfrak{m}, \underline{a}_{S_0}, S_p, \mathcal{M}; \mathcal{O}) \otimes_{\mathcal{O}} \mathbb{C}_p \\ \Big\downarrow {\scriptstyle \Delta^{\underline{\alpha}} \times \delta_1^{-1}} & & \Big\downarrow {\scriptstyle \Delta^{\underline{\alpha}}} \\ \mathscr{D}^b(S_1, S_2, \mathbb{C}_p) \times C_{\diamond}(F_{\mathfrak{p}}, \mathbb{C}_p) & \longrightarrow & \mathscr{D}^b(S_0 \cap S_1, S_0 \cap S_2, \mathbb{C}_p) \end{array}$$

(the top horizontal arrow is the map (90)). Hence for $b \in H^q(G(F)^+, \mathrm{St}(F_{\mathfrak{p}}, \mathbb{C}_p))$ the diagram

$$\begin{array}{ccc} H^p(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathbb{C}_p)) & \xrightarrow{\cdot \cup b} & H^{p+q}(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p)) \\ \Big\downarrow {\scriptstyle (89)} & & \Big\downarrow {\scriptstyle (89)} \\ H^p(F_+^*, \mathscr{D}^b(S_1, S_2, \mathbb{C}_p)) & \xrightarrow{\cdot \cup \delta^*(b)} & H^{p+q}(F_+^*, \mathscr{D}^b(S_0 \cap S_1, S_0 \cap S_2, \mathbb{C}_p)) \end{array}$$

commutes as well. Here $\delta^* : H^q(G(F), \mathrm{St}(F_{\mathfrak{p}}, \mathbb{C}_p)) \to H^q(F^*, C_{\diamond}(F_{\mathfrak{p}}, \mathbb{C}_p))$ is the canonical map induced by δ_1^{-1} (compare 2.11 (b)) and the first vertical map is given by

$$\begin{aligned} H^p(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathbb{C}_p)) &\cong H^p(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathcal{O})) \otimes_{\mathcal{O}} \mathbb{C}_p \\ &\xrightarrow{(89)} H^p(F_+^*, \mathscr{D}(S_1, S_2, \mathcal{O})) \otimes_{\mathcal{O}} \mathbb{C}_p \longrightarrow H^p(F_+^*, \mathscr{D}^b(S_1, S_2, \mathbb{C}_p)) \end{aligned}$$

The definition of the second vertical map is analogous.

The extension classes (30) associated to the homomorphisms $\mathrm{ord}_{\mathfrak{p}} : F_{\mathfrak{p}}^* \to \mathbb{C}_p$ and $\ell_{\mathfrak{p}} = \log_p \circ \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p} : F^* \to \mathbb{C}_p$ define two cohomology classes $b_{\mathrm{ord},\mathfrak{p}}$, $b_{\log,\mathfrak{p}} \in H^1(G(F)^+, \mathrm{St}(F_{\mathfrak{p}}, \mathbb{C}_p))$.

Lemma 5.2. *(a) The $G(\mathbf{A}^{p,\infty})$-representation $H_*^q(G(F)^+, \mathscr{A}(\underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p))$ is of automorphic type for all $q \in \mathbb{Z}$.*

(b) For $\underline{\mu} \in \Sigma$ the map $\cdot \cup b_{\mathrm{ord},\mathfrak{p}}$ induces an isomorphism

(92) $$H_*^d(G(F)^+, \mathscr{A}(\underline{a}, \mathcal{M}; \mathbb{C}_p))_{\pi,\underline{\mu}} \longrightarrow H_*^{d+1}(G(F)^+, \mathscr{A}(\underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p))_{\pi,\underline{\mu}}$$

of $\mathbb{C}_p$-vector spaces.

Proof. Let K be a compact open subgroup of $G(\mathbf{A}^{p,\infty})$ and put $K_0 = K \times G(\mathcal{O}_{\mathfrak{p}}) \subseteq G(\mathbf{A}^{S_0,\infty})$. We define the $\mathcal{H}_K^{S_p \cup S_\infty}$-module $\mathscr{Q}_K$ by

$$0 \longrightarrow \mathscr{Q}_K \longrightarrow \mathscr{A}(K_0, \underline{a}_{S_0}, \mathcal{M}; \mathbb{C}_p) \xrightarrow{T_{\mathfrak{p}} - (q+1)} \mathscr{A}(K_0, \underline{a}_{S_0}, \mathcal{M}; \mathbb{C}_p) \longrightarrow 0.$$

By considering the corresponding long exact cohomology sequence we obtain that $H^\bullet(G(F)^+, \mathscr{Q}_K)$ is of automorphic type and $H^\bullet(G(F)^+, \mathscr{Q}_K)_\pi = 0$. For the latter note that $H^\bullet(G(F)^+, \mathscr{A}(K_0, \underline{a}_{S_0}, \mathcal{M}; \mathbb{C}_p))_\pi = 0$ since $K_0 \not\subseteq K_0(\mathfrak{f}(\pi))^{S_0 \cup S_\infty}$. By (17) there exists an exact sequence

$$0 \longrightarrow \mathscr{A}(K, \underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p) \longrightarrow \mathscr{Q}_K \longrightarrow \mathscr{A}(K, \underline{a}, \mathcal{M}; \mathbb{C}_p) \longrightarrow 0.$$

It follows from 2.11 (c) that (92) is equal to the connecting homomorphism in the corresponding long exact sequence in degree d (up to sign). Hence the assertions follow from Prop. 4.8. $\square$

Definition 5.3. *For $\underline{\mu} \in \Sigma$ there exists a unique $\mathcal{L}_\mathfrak{p}(\pi, \underline{\mu}) \in \mathbb{C}_p$ – called the $\mathcal{L}$-invariant of π at $\mathfrak{p}$ – such that*

$$(\,\cdot \cup b_{\log,\mathfrak{p}})_{\pi,\underline{\mu}} \;=\; \mathcal{L}_\mathfrak{p}(\pi, \underline{\mu})\; (\,\cdot \cup b_{\mathrm{ord},\mathfrak{p}})_{\pi,\underline{\mu}}.$$

If $\underline{\mu} = (1, \ldots, 1)$ then we write $\mathcal{L}_\mathfrak{p}(\pi) = \mathcal{L}_\mathfrak{p}(\pi, \underline{\mu})$.

Conjecture 5.4. *$\mathcal{L}_\mathfrak{p}(\pi, \underline{\mu})$ is independent of the choice of $\underline{\mu} \in \Sigma$, i.e. we have $\mathcal{L}_\mathfrak{p}(\pi, \underline{\mu}) = \mathcal{L}_\mathfrak{p}(\pi)$ for all $\underline{\mu} \in \Sigma$.*

Lemma 5.5. *Let $\chi : \mathbf{I}/F^* \to \{\pm 1\}$ be a quadratic character whose conductor is prime to p and such that $\chi_\mathfrak{p} = 1$. Then*

$$\mathcal{L}_\mathfrak{p}(\pi, \underline{\mu}) \;=\; \mathcal{L}_\mathfrak{p}(\pi \otimes \chi, \mathrm{sign}(\chi)\underline{\mu})$$

where $\mathrm{sign}(\chi) = \chi_\infty(-1, \ldots, -1) \in \Sigma$.

Proof. For a smooth semisimple representation V of $G(\mathbf{A}^{p,\infty})$ we denote by V_χ the representation $V \otimes \det \circ \chi^{p,\infty}$. Note that $(V_\chi)_\pi = V_{\pi\otimes\chi}$. Put $\underline{\epsilon} := \underline{\epsilon}(\chi_p) = (\chi_{\mathfrak{p}_1}(\varpi_1), \ldots, \chi_{\mathfrak{p}_m}(\varpi_m))$ (compare section 2.8). We define a *twisting operator*

$$\mathfrak{Tw}_\chi : \mathscr{A}(\underline{a}, S_p, \mathcal{M}; \mathbb{C}_p) \longrightarrow \mathscr{A}(\underline{\epsilon a}, S_p, \mathcal{M}; \mathbb{C}_p)$$

by $\mathfrak{Tw}_\chi(\Phi)(g, m) = \chi^{p,\infty}(\det(g))\mathfrak{Tw}_{\chi_p}(\Phi(g, m))$ for all $g \in G(\mathbf{A}^{p,\infty})$, $m \in \mathcal{M}$ and define $\mathfrak{Tw}_\chi : \mathscr{A}(\underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p) \to \mathscr{A}((\underline{\epsilon a})_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p)$ analogously. Note that $\mathfrak{Tw}_\chi$ is $G(F)^+$-linear and maps $\mathscr{A}(K, \underline{a}, S_p, \mathcal{M}; \mathbb{C}_p)$ onto $\mathscr{A}(K, \underline{\epsilon a}, S_p, \mathcal{M}; \mathbb{C}_p)$ as long as $\det(K)$ is contained in the kernel of $\chi^{p,\infty}$. Note also that $\mathfrak{Tw}_\chi \circ \mathfrak{Tw}_\chi = \mathrm{id}$. For $b \in H^q(G(F)^+, \mathrm{St}(F_\mathfrak{p}, \mathbb{C}_p))$ we obtain a diagram

$$\begin{tikzcd}
H^d_*(G(F)^+, \mathscr{A}(\underline{a}, \mathcal{M}; \mathbb{C}_p))_\pi \arrow[r, "(\,\cdot\, \cup b)_\pi"] \arrow[d, "\cong"] & H^{d+1}_*(G(F)^+, \mathscr{A}(\underline{a}_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p))_\pi \arrow[d, "\cong"] \\
H^d_*(G(F)^+, \mathscr{A}(\underline{\epsilon a}, \mathcal{M}; \mathbb{C}_p))_{\pi\otimes\chi} \arrow[r, "(\,\cdot\, \cup b)_{\pi\otimes\chi}"] & H^{d+1}_*(G(F)^+, \mathscr{A}((\underline{\epsilon a})_{S_0}, S_p, \mathcal{M}; \mathbb{C}_p))_{\pi\otimes\chi}
\end{tikzcd}$$

where the vertical maps are induced by $\mathfrak{Tw}_\chi$. The commutativity of the diagram for $b = b_{\log,\mathfrak{p}}$ and $b = b_{\mathrm{ord},\mathfrak{p}}$ implies the assertion. $\square$

Remarks 5.6. (a) Conjecture 5.4 is known in the case $F = \mathbb{Q}$ ([2], Theorem 6.8).

(b) Let D be a quaternion algebra over F and let $G' = D^*$ (viewed as an algebraic group). Let π be an automorphic representation of $G'(\mathbf{A})$ whose components π_v are discrete series for all $v \in S_\infty$. By a similar construction as above one should be able to define an $\mathcal{L}$-invariant $\mathcal{L}_\mathfrak{p}(\pi)$ whenever $\mathfrak{p}$ does not divide the discriminant of D and we have $\pi_\mathfrak{p} \cong \mathrm{St}$. These $\mathcal{L}$-invariants have been defined in the case $F = \mathbb{Q}$ and D definite by Teitelbaum [28], for $F = \mathbb{Q}$, $D = M_2(\mathbb{Q})$ by Darmon [9] (for weight 2) and Orton [22] (higher even weight) and if F has narrow class number 1 by Greenberg [13] (in the case of parallel weight $(2,\ldots,2)$) (it not difficult to see that for $D = M_2(F)$ the $\mathcal{L}$-invariant defined in [13] match with the one defined above). An interesting and difficult problem is to show that $\mathcal{L}_{\mathfrak{p}}(\pi)$ is invariant under the Jacquet-Langlands correspondence (this has been proved for $F = \mathbb{Q}$ in certain cases in [2], Cor. 6.9) or under base change. The author hopes to return to the study of these $\mathcal{L}$-invariants in the future.

5.2. **Main results.** Our first main result is

Theorem 5.7. *Let $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{\alpha})$. The vanishing order of $L_p(s,\pi)$ at $s = 0$ is at least equal to r (i.e. to the number of places $\mathfrak{p}$ of F above p with $\pi_{\mathfrak{p}} \cong \mathrm{St}$). Moreover we have*

$$L_p^{(r)}(0,\pi) \;=\; r! \prod_{\mathfrak{p}\in S_1} \mathcal{L}_{\mathfrak{p}}(\pi) \cdot \prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}},1) \cdot L(\tfrac{1}{2},\pi) \tag{93}$$

where $e(\alpha_{\mathfrak{p}},1) = \begin{cases} 2 & \text{if } \alpha_{\mathfrak{p}} = -1; \\ (1-\frac{1}{\alpha_{\mathfrak{p}}})^2 & \text{if } \alpha_{\mathfrak{p}} \neq \pm 1. \end{cases}$

Proof. The first statement follows from 3.19 (a). By 3.22, 4.10 (b) and 3.19 (b) we have

$$\begin{aligned} \textstyle\prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}},1) \cdot L(\tfrac{1}{2},\pi) &= (-1)^{\binom{r}{2}} \, (\kappa_{\pi,+} \cup c_{\mathfrak{p}_1} \cup \ldots \cup c_{\mathfrak{p}_r}) \cap \vartheta \\ L_p^{(r)}(0,\pi) &= (-1)^{\binom{r}{2}} \, r! \, (\widetilde{\kappa}_{\pi,+} \cup c_{\ell_{\mathfrak{p}_1}} \cup \ldots \cup c_{\ell_{\mathfrak{p}_r}}) \cap \vartheta \end{aligned}$$

Thus it suffices to show $\widetilde{\kappa}_{\pi,+} \cup c_{\ell_{\mathfrak{p}}} = \mathcal{L}_{\mathfrak{p}}(\pi)\, \widetilde{\kappa}_{\pi,+} \cup c_{\mathfrak{p}}$ for $\mathfrak{p} \in S_1$. Let $\mathfrak{m}$ be the maximal prime-to-p divisor of $\mathfrak{f}(\pi)$. The proof of 4.10 (c) and 4.8 shows that there exists an element $\beta \in H^d(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \overline{\mathbb{Q}}))_{\pi}$ which is mapped under (89) to some non-zero multiple of κ_{π}. Also by Lemma 2.11 (b) we have $2c_{\mathfrak{p}} = \delta^*(b_{\mathrm{ord},\mathfrak{p}})$ and $2c_{\ell_{\mathfrak{p}}} = \delta^*(b_{\log,\mathfrak{p}})$. Hence the assertion follows from $\beta_+ \cup b_{\mathrm{ord},\mathfrak{p}} = \mathcal{L}_{\mathfrak{p}}(\pi)\, \beta_+ \cup b_{\log,\mathfrak{p}}$ (here we view β as an element of $H^d(G(F)^+, \mathscr{A}(\mathfrak{m}, \underline{a}, \mathcal{M}; \mathbb{C}_p))$). □

Now assume that E/F is an elliptic curve which is *p-ordinary*, i.e. it has either good ordinary or multiplicative reduction at all places above p. We also assume that E is *modular* by which we mean that for some prime number ℓ, the ℓ-adic Tate module of E is isomorphic (as a Galois representation) to the ℓ-adic representation associated to some $\pi = \pi_E \in \mathfrak{A}_0(G, \underline{2})$ (this holds then for any ℓ; compare [30]). Then it is known ([7], [27]) that the local L-factors of E and π all match up. In particular we have $\Lambda(E,\chi,s) = L(s-\frac{1}{2}, \pi\otimes\chi)$ for any character $\chi: \mathbf{I}/F^* \to \mathbb{C}^*$ (where $\Lambda(E,\chi,s)$ denotes the completed Hasse-Weil L-function) and the conductor of E is $= \mathfrak{f}(\pi)$. Moreover π is p-ordinary and E has split multiplicative reduction at $\mathfrak{p}$ if and only if $\pi_{\mathfrak{p}} = \mathrm{St}$. Thus $\pi \in \mathfrak{A}_0(G, \underline{2}, \underline{\alpha})$ for some ordinary parameters $\underline{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and if $S_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ denotes the set of $\mathfrak{p} \in S_p$ where E has split multiplicative reduction then $\alpha_1 = \ldots = \alpha_r = 1$. For $\mathfrak{p} \in S_1$ recall that $\mathcal{L}_{\mathfrak{p}}(E) = \frac{\log_p(\mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(q_{E/F_{\mathfrak{p}}}))}{\mathrm{ord}_{\mathfrak{p}}(q_{E/F_{\mathfrak{p}}})}$ where $q_{E/F_{\mathfrak{p}}}$ denotes the Tate period associated to $E/F_{\mathfrak{p}}$. The p-adic L-function of E is defined as $L_p(E,s) := L_p(s,\pi)$. Recall Hida's conjecture [17]

Conjecture 5.8. *(a)* $\mathrm{ord}_{s=0} L_p(E,s) \geq r$.

(b) $L_p^{(r)}(E,0) \;=\; r! \,\prod_{\mathfrak{p}\in S_1} \mathcal{L}_{\mathfrak{p}}(E) \,\cdot\, \prod_{\mathfrak{p}\in S_2} e(\alpha_{\mathfrak{p}},1) \,\cdot\, \Lambda(E,1)$.

In the case $r=1$ this has been proved by Mok [21] (under the additional assumption that p is unramified in F and ≥ 5). We prove the first assertion unconditionally and deduce the second from Mok's result under some further (mild) restrictions using Theorem 5.7 and a non-vanishing result for twisted L-values [29], [12]. Let $w(\pi)$ denote the root number of π. We first show

Proposition 5.9. *Assume that $p \geq 5$ is unramified in F. If (i) E has multiplicative reduction at some place $\mathfrak{q} \nmid p$ or (ii) $r + w(\pi) \equiv 1 \mod 2$ then $\mathcal{L}_{\mathfrak{p}}(E) = \mathcal{L}_{\mathfrak{p}}(\pi)$ for all $\mathfrak{p} \in S_1$.*

Proof. Suppose that there exists a quadratic character $\chi : \mathbf{I}/F^* \to \{\pm 1\}$ whose conductor is relatively prime to $\mathfrak{f}(\pi)$ with the following properties

- $\chi_v = 1$ for all $v \in \{\mathfrak{p}\} \cup S_2 \cup S_\infty$ (where $S_2 = S_p - S_1$);
- $\chi_{\mathfrak{q}} \neq 1$ for all $\mathfrak{q} \in S_1 - \{\mathfrak{p}\}$;
- $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$.

Let E_χ denote the twist of E by χ. The first property and 5.5 imply $\mathcal{L}_{\mathfrak{p}}(E) = \mathcal{L}_{\mathfrak{p}}(E_\chi)$ and $\mathcal{L}_{\mathfrak{p}}(\pi) = \mathcal{L}_{\mathfrak{p}}(\pi\otimes\chi)$. It also follows from the first two properties that E_χ is p-ordinary and $\mathfrak{p}$ is the only place above p where E_χ has good ordinary reduction. Thus by ([21], Theorem 1.1) Conj. 5.8 (b) holds for E_χ. Since the formula coincides with (93) with the possible exception of the $\mathcal{L}$-invariants and because of the non-vanishing of $L(\frac{1}{2}, \pi\otimes\chi)$ we deduce $\mathcal{L}_{\mathfrak{p}}(E) = \mathcal{L}_{\mathfrak{p}}(E_\chi) = \mathcal{L}_{\mathfrak{p}}(\pi\otimes\chi) = \mathcal{L}_{\mathfrak{p}}(\pi)$.

Therefore it remains to show that under the assumptions (i) or (ii) there exists a quadratic character with the above properties. If χ is any quadratic character with $\mathfrak{f}(\chi)$ relatively prime to $\mathfrak{f}(\pi)$ then it is well known that $w(\pi \otimes \chi) = \prod_{v\in S_\infty} \chi_v(-1)\, \chi(\mathfrak{f}(\pi))\, w(\pi)$. Hence under the assumptions (i) or (ii) it is clear that there exists χ so that at least the first two properties are satisfied and such that $w(\pi\otimes\chi) = 1$. Then a theorem of Waldspurger ([29]; see also [12], Thm. B) implies that we can choose χ so that $L(\frac{1}{2}, \pi\otimes\chi) \neq 0$ holds as well. □

Theorem 5.10. *(a)* $\mathrm{ord}_{s=0} L_p(E,s) \geq r$.

(b) Assume $p \geq 5$ is unramified in F. If (i) E has multiplicative reduction at some place $\mathfrak{q} \nmid p$ or (ii) r is odd or (iii) $w(\pi) = -1$ then Conj. 5.8 (b) holds.

Proof. (a) is part of Theorem 5.7. For (b) assume first $w(\pi) = -1$. Then both sides of the equation vanish (the left hand side by (93)). If $w(\pi) = 1$ and (i) or (ii) hold then (b) follows from Thm. 5.7 and Prop. 5.9. □

Remarks 5.11. (a) In the case $F = \mathbb{Q}$, Thm. 5.7 is due to Darmon [9].

(b) If Conjecture 5.4 holds then it is not necessary to assume (i) or (ii) in Prop. 5.9 above. In fact, obviously, there exists a quadratic character χ with $\chi_v = 1 \neq \chi_{\mathfrak{q}}$ for all $v \in \{\mathfrak{p}\} \cup S_2$ and $\mathfrak{q} \in S_1 - \{\mathfrak{p}\}$ and so that $w(\pi \otimes \chi) = 1$. Hence by Waldspurger's theorem we can choose χ which satisfies also $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$. In the forthcoming work [14] we shall give a different and unconditional proof of the equality $\mathcal{L}_{\mathfrak{p}}(E) = \mathcal{L}_{\mathfrak{p}}(\pi)$ (hence of Hida's conjecture) without using Mok's result.

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