SOLUTION OF A UNIQUENESS PROBLEM IN THE DISCRETE TOMOGRAPHY OF ALGEBRAIC DELONE SETS

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Abstract. We consider algebraic Delone sets Λ in the Euclidean plane and address the problem of distinguishing convex subsets of Λ by X-rays in prescribed Λ-directions, i.e. directions parallel to lines through two different points of Λ. Here, an X-ray in direction u of a finite set gives the number of points in the set on each line parallel to u. It is shown that for any algebraic Delone set Λ there are four prescribed Λ-directions such that any two convex subsets of Λ can be distinguished by the corresponding X-rays. We further prove the existence of a natural number c_Λ such that any two convex subsets of Λ can be distinguished by their X-rays in any set of c_Λ prescribed Λ-directions. In particular, this extends a well-known result of Gardner and Gritzmann on the corresponding problem for planar lattices to nonperiodic cases that are relevant in quasicrystallography.

1. Introduction

Discrete tomography is concerned with the inverse problem of retrieving information about some *finite* set in Euclidean space from (generally noisy) information about its slices. One important problem is the *unique reconstruction* of a finite point set in Euclidean 3-space from its *(discrete parallel) X-rays* in a small number of directions, where the *X-ray* of the finite set in a certain direction is the *line sum function* giving the number of points in the set on each line parallel to this direction.

The interest in the discrete tomography of planar Delone sets Λ with long-range order is motivated by the requirement in materials science for the unique reconstruction of solid state materials like *quasicrystals* slice by slice from their images under quantitative *high resolution transmission electron microscopy* (HRTEM). In fact, in [29], [36] a technique is described, which can, for certain crystals, effectively measure the number of atoms lying on densely occupied columns. It is reasonable to expect that future developments in technology will extend this situation to other solid state materials. The aforementioned density condition forces us to consider only Λ-directions, i.e. directions parallel to lines through two different points of Λ. Further, since typical objects may be damaged or even destroyed by the radiation energy after about 3 to 5 images taken by HRTEM, applicable results may only use a small number of X-rays. It is actually this restriction to few high-density directions that makes the problems of discrete tomography mathematically challenging, even if one assumes the absence of noise.

In the traditional setting, motivated by *crystals*, the positions to be determined form a finite subset of a three-dimensional lattice, the latter allowing a slicing into equally spaced congruent copies of a planar lattice. In the crystallographic setting, by the affine nature of the problem, it therefore suffices to study the discrete tomography of the square lattice; cf. [15], [16], [17], [19], [21], [22], [23] for an overview.

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For the *quasicrystallographic* setting, the positions to be determined form a finite subset of a *nonperiodic Delone set* with *long-range order* (more precisely, a *mathematical quasicrystal* or *model set* [8], [30]) which on the other hand is contained in a free additive subgroup of $\mathbb{R}^3$ of finite rank $r > 3$. These model sets possess, as is the case for lattices, a dimensional hierarchy, i.e. they allow a slicing into planar model sets. However, the slices are in general no longer pairwise congruent or equally spaced in 3-space; cf. [32]. Still, most of the model sets that describe real quasicrystallographic structures allow a slicing such that each slice is, when seen from a common perpendicular viewpoint, a (planar) *n-cyclotomic model set*, where $n = 5$, $n = 8$ and $n = 12$, respectively (Example 3.11); cf. [24, Section 1.2], [26], [28, Section 4.5] and [37] for details. These cyclotomic model sets thus take over the role played by the planar lattices in the crystallographic case. In the present text, we shall focus on the larger class of *algebraic Delone sets* (Definition 3.1).

Since different finite subsets of a Delone set $\varLambda$ may have the same X-rays in several $\varLambda$-directions (in other words, the above problem of uniquely reconstructing a finite point set from its X-rays is an *ill-posed* problem in general), one is naturally interested in conditions to be imposed on the set of $\varLambda$-directions together with restrictions on the possible finite subsets of $\varLambda$ such that the latter phenomenon cannot occur. Here, we consider the *convex subsets* of $\varLambda$ (i.e. bounded subsets of $\varLambda$ with the property that their convex hull contains no new points of $\varLambda$) and show that for any algebraic Delone set $\varLambda$ there are four prescribed $\varLambda$-directions such that any two convex subsets of $\varLambda$ can be distinguished by the corresponding X-rays, whereas less than four $\varLambda$-directions never suffice for this purpose (Theorem 5.10(a)). We further prove the existence of a finite number $c_\varLambda$ such that any two convex subsets of $\varLambda$ can be distinguished by their X-rays in *any* set of $c_\varLambda$ prescribed $\varLambda$-directions (Theorem 5.10(b)). Moreover, we demonstrate that the least possible numbers $c_\varLambda$ in the case of the practically most relevant examples of n-cyclotomic model sets $\varLambda$ with $n = 5$, $n = 8$ and $n = 12$ are (in that very order) 11, 9 and 13 (Theorem 5.11(b) and Remark 5.12). This extends a well-known result of Gardner and Gritzmann (cf. [15, Theorem. 5.7]) on the corresponding problem for planar lattices $\varLambda$ ($c_\varLambda = 7$) to cases that are relevant in quasicrystallography and in particular solves Problem 4.34 of [28]. The above results and their continuous analogue (Theorem 6.2) follow from deep insights into the existence of certain *U-polygons* in the plane (cf. Section 2). We believe that our main result on these polygons (Theorem 5.6) is of independent interest from a purely geometrical point of view. For the algorithmic reconstruction problem in the quasicrystallographic setting, we refer the reader to [3], [26].

2. Preliminaries and notation

Natural numbers are always assumed to be positive. For a natural number n and a prime number p, we denote the exponent of the highest power of p dividing n by $\mathrm{ord}_p(n)$. We denote the norm in Euclidean d-space by $\|\cdot\|$. The Euclidean plane will occasionally be identified with the complex numbers. For $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate of z and $|z| = \sqrt{z\bar{z}}$ its modulus. The unit circle in $\mathbb{C}$ is denoted by $\mathbb{S}^1$ and its elements are also called *directions*. For a nonzero complex number z, we denote by $\mathrm{sl}(z)$ the slope of z, i.e. $\mathrm{sl}(z) = -i(z-\bar{z})/(z+\bar{z}) \in \mathbb{R}\cup\{\infty\}$. For $r > 0$ and $z \in \mathbb{C}$, $B_r(z)$ is the open ball of radius r about z. Recall that an ($\mathbb{R}$-)*linear endomorphism* (resp., *affine endomorphism*) Ψ of $\mathbb{C}$ is given by $z \mapsto az + b\bar{z}$ (resp., $z \mapsto az + b\bar{z} + t$), where $a, b, t \in \mathbb{C}$. In both cases, it is an automorphism if and only if $\det \Psi = a\bar{a} - b\bar{b} \neq 0$. A *homothety* $h : \mathbb{C} \to \mathbb{C}$ is given by $z \mapsto \lambda z + t$, where $\lambda \in \mathbb{R}$ is positive and $t \in \mathbb{C}$. In the following, let $\varLambda$ be a subset of $\mathbb{C}$. A direction $u \in \mathbb{S}^1$ is called a *$\varLambda$-direction* if it is parallel to a nonzero element of the difference set $\varLambda - \varLambda = \{v - w \,|\, v, w \in \varLambda\}$ of $\varLambda$. A *convex polygon* is the convex hull of a finite set of points in $\mathbb{C}$. A *polygon in* $\varLambda$ is a convex polygon with all vertices in $\varLambda$. Further, a bounded subset C of $\varLambda$ is called a *convex subset of* $\varLambda$ if $C = (\operatorname{conv} C) \cap \varLambda$, where $\operatorname{conv} C$ denotes the convex hull of C. Let $U \subset \mathbb{S}^1$ be a finite set of directions. A nondegenerate convex polygon P is called a *U-polygon* if it has the property that whenever v is a vertex of P and $u \in U$, the line in the complex plane in direction u which passes through v also meets another vertex v' of P. By a *regular polygon* we shall always mean a nondegenerate convex regular polygon. An *affinely regular polygon* is the image of a regular polygon under an affine automorphism of the complex plane. $\varLambda$ is called *uniformly discrete* if there is a radius $r > 0$ such that every ball $B_r(z)$ with $z \in \mathbb{C}$ contains at most one point of $\varLambda$. Note that the bounded subsets of a uniformly discrete set $\varLambda$ are precisely the finite subsets of $\varLambda$. $\varLambda$ is called *relatively dense* if there is a radius $R > 0$ such that every ball $B_R(z)$ with $z \in \mathbb{C}$ contains at least one point of $\varLambda$. $\varLambda$ is called a *Delone set* if it is both uniformly discrete and relatively dense. $\varLambda$ is said to be of *finite local complexity* if $\varLambda - \varLambda$ is discrete and closed. Note that $\varLambda$ is of finite local complexity if and only if for every $r > 0$ there are, up to translation, only finitely many *patches of radius* r, i.e. sets of the form $\varLambda \cap B_r(z)$, where $z \in \mathbb{C}$; cf. [30]. A Delone set $\varLambda$ is a *Meyer set* if $\varLambda - \varLambda$ is uniformly discrete. Trivially, any Meyer set is of finite local complexity. $\varLambda$ is called *periodic* if it has nonzero translation symmetries. Finally, we denote by $K_\varLambda$ the intermediate field of $\mathbb{C}/\mathbb{Q}$ that is given by

$$K_\varLambda \;=\; \mathbb{Q}\left((\varLambda - \varLambda) \cup \left(\overline{\varLambda - \varLambda}\right)\right) .$$

2.1. **Recollections from the theory of cyclotomic fields.** Let $K \subset \mathbb{C}$ be a field and let μ be the group of roots of unity in $\mathbb{C}$. We denote the maximal real subfield $K \cap \mathbb{R}$ of K by K^+ and set $\mu(K) = \mu \cap K$. As usual, let $K^* = K \setminus \{0\}$. For $n \in \mathbb{N}$, we always let $\zeta_n = e^{2\pi i/n}$, a primitive nth root of unity in $\mathbb{C}$. Then, $\mathbb{Q}(\zeta_n)$ is the nth cyclotomic field. Further, ϕ will always denote Euler's totient function, i.e.

$$\phi(n) = \operatorname{card}\left(\left\{k \in \mathbb{N} \,\middle|\, 1 \le k \le n \text{ and } \gcd(k,n) = 1\right\}\right) .$$

Recall that ϕ is multiplicative with $\phi(p^r) = p^{r-1}(p-1)$ for p prime and $r \in \mathbb{N}$.

Fact 2.1 (Gauss)**.** [38, Theorem 2.5] $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \phi(n)$ *and the field extension* $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ *is a Galois extension with Abelian Galois group* $G(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^\times$, *with* $a (\operatorname{mod} n)$ *corresponding to the automorphism given by* $\zeta_n \mapsto \zeta_n^a$. □

Note that the composition $\mathbb{Q}(\zeta_n)\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_n, \zeta_m)$ of cyclotomic fields is equal to the cyclotomic field $\mathbb{Q}(\zeta_{\operatorname{lcm}(n,m)})$. Further, the intersection $\mathbb{Q}(\zeta_n) \cap \mathbb{Q}(\zeta_m)$ of cyclotomic fields is equal to the cyclotomic field $\mathbb{Q}(\zeta_{\gcd(n,m)})$. Note that $\mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_n + \bar{\zeta}_n) = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. Clearly, if n divides m then $\mathbb{Q}(\zeta_n)$ is a subfield of $\mathbb{Q}(\zeta_m)$. Since $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{2n})$ for odd n by Fact 2.1, we may sometimes restrict ourselves to $n \in \mathbb{N}$ with $n \not\equiv 2 \pmod 4$.

2.2. **Cross ratios.** Let (t_1, t_2, t_3, t_4) be an ordered tuple of four distinct elements of $\mathbb{R} \cup \{\infty\}$. Then, its *cross ratio* $\langle t_1, t_2, t_3, t_4 \rangle$ is the nonzero real number defined by

$$\langle t_1, t_2, t_3, t_4 \rangle = \frac{(t_3 - t_1)(t_4 - t_2)}{(t_3 - t_2)(t_4 - t_1)} ,$$

with the usual conventions if one of the t_i equals ∞. We need the following invariance property of cross ratios of slopes which is usually stated in the framework of projective geometry; cf. [11, Corollary 96.11]. For the reader's convenience, we give a reformulation and also include a proof.

Lemma 2.2. *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}^*$ be pairwise nonparallel and let Ψ be a linear automorphism of the complex plane. Then, one has*

$$\big\langle\, \mathrm{sl}(z_1), \mathrm{sl}(z_2), \mathrm{sl}(z_3), \mathrm{sl}(z_4)\big\rangle = \big\langle\, \mathrm{sl}(\Psi(z_1)), \mathrm{sl}(\Psi(z_2)), \mathrm{sl}(\Psi(z_3)), \mathrm{sl}(\Psi(z_4))\big\rangle\,.$$

Proof. The assertion follows from the identities

$$\big\langle\, \mathrm{sl}(z_1), \mathrm{sl}(z_2), \mathrm{sl}(z_3), \mathrm{sl}(z_4)\big\rangle = \frac{(z_3\bar{z}_1 - \bar{z}_3 z_1)(z_4\bar{z}_2 - \bar{z}_4 z_2)}{(z_3\bar{z}_2 - \bar{z}_3 z_2)(z_4\bar{z}_1 - \bar{z}_4 z_1)} \tag{2.1}$$

and $\Psi(v)\overline{\Psi(w)} - \overline{\Psi(v)}\Psi(w) = (\det\Psi)(v\bar{w} - \bar{v}w)$, where $v, w \in \mathbb{C}$. □

Lemma 2.3. *Let $\Lambda \subset \mathbb{C}$. Then the cross ratio of slopes of four pairwise nonparallel Λ-directions is an element of K_Λ^+.*

Proof. This follows immediately from (2.1). □

3. Algebraic Delone sets

The following notions will be useful; see also [24], [25], [27], [28] for generalisations and for results related to those presented below.

Definition 3.1. A Delone set $\Lambda \subset \mathbb{C}$ is called an *algebraic Delone set* if it satisfies the following properties:

(Alg) $[K_\Lambda : \mathbb{Q}] < \infty$.

(Hom) For any finite subset F of K_Λ, there is a homothety h of the complex plane such that $h(F) \subset \Lambda$.

Moreover, Λ is called an *n-cyclotomic Delone set* if it satisfies the property

$$(n\text{-Cyc}) \qquad K_\Lambda \subset \mathbb{Q}(\zeta_n)$$

for some $n \geq 3$ and has property (Hom). Further, Λ is called a *cyclotomic Delone set* if it is an n-cyclotomic Delone set for a suitable $n \geq 3$.

Remark 3.2. Algebraic Delone sets were already introduced in [28, Definition 4.1]. Clearly, for every algebraic Delone set Λ, the field extension $K_\Lambda/\mathbb{Q}$ is an imaginary extension (due to Λ being relatively dense) with $\overline{K_\Lambda} = K_\Lambda$. By the Kronecker-Weber theorem (cf. [38, Theorem 14.1]) and Fact 2.1, the cyclotomic Delone sets are precisely the algebraic Delone sets Λ with the additional property that $K_\Lambda/\mathbb{Q}$ is an Abelian extension.

Following Moody [30], modified along the lines of the algebraic setting of Pleasants [31], we make the following definition.

Definition 3.3. Let $K \subset \mathbb{C}$ be an imaginary quadratic extension of a real algebraic number field (necessarily, this real algebraic number field is K^+) of degree $[K : \mathbb{Q}] = d$ over $\mathbb{Q}$ (in particular, d is even). Let $\mathcal{O}_K$ be the ring of integers in K and let $.^\star : \mathcal{O}_K \to \mathbb{C}^{s-1} \times \mathbb{R}^t$ be any map of the form $z \mapsto (\sigma_2(z), \dots, \sigma_s(z), \sigma_{s+1}(z), \dots, \sigma_{s+t}(z))$, where $\sigma_{s+1}, \dots, \sigma_{s+t}$ are the real embeddings of $K/\mathbb{Q}$ into $\mathbb{C}/\mathbb{Q}$ and $\sigma_2, \dots, \sigma_s$ arise from the complex embeddings of $K/\mathbb{Q}$ into $\mathbb{C}/\mathbb{Q}$ except the identity and the complex conjugation by choosing exactly one embedding from each pair of complex conjugate ones (in particular, $d = 2s + t$ and $s \geq 1$). Then, for any such choice, each translate Λ of

$$\Lambda(W) = \{z \in \mathcal{O}_K \mid z^\star \in W\}\,,$$

where $W \subset \mathbb{C}^{s-1} \times \mathbb{R}^t \simeq \mathbb{R}^{d-2}$ is a relatively compact set with nonempty interior, is called a *K-algebraic model set.* Moreover, $.^\star$ and W are called the *star map* and the *window* of Λ, respectively.

Remark 3.4. Algebraic number fields K as above may be obtained by starting with a real algebraic number field L and adjoining the square root of a negative number from L. Note that, in the situation of Definition 3.3, the quadratic extension K/K^+ is a Galois extension with $G(K/K^+)$ containing the identity and the complex conjugation (in particular, one has $\overline{K} = K$). We use the convention that for $d = 2$ (meaning that $s = 1$ and $t = 0$), $\mathbb{C}^{s-1} \times \mathbb{R}^t$ is the trivial group $\{0\}$ and the star map is the zero map. Due to the Minkowski representation $\{(z, z^\star) \,|\, z \in \mathcal{O}_K\}$ of the maximal order $\mathcal{O}_K$ of K being a (full) lattice in $\mathbb{C} \times \mathbb{C}^{s-1} \times \mathbb{R}^t \simeq \mathbb{R}^d$ (cf. [10, Chapter 2, Section 3]) that is in one-to-one correspondence with $\mathcal{O}_K$ via the canonical projection on the first factor and due to $\mathcal{O}_K^\star$ being a dense subset of $\mathbb{C}^{s-1} \times \mathbb{R}^t$ (see Lemma 3.7 below), K-algebraic model sets are indeed *model sets* and thus are Meyer sets; cf. [8], [9], [30], [34], [35] for the general setting and further properties of model sets. Since the star map is a monomorphism of Abelian groups for $d > 2$ and since the window is a bounded set, a K-algebraic model set $\varLambda$ is periodic if and only if $d = 2$, in which case $\varLambda$ is a translate of the planar lattice $\mathcal{O}_K$.

A real algebraic integer λ is called a *Pisot-Vijayaraghavan number* (*PV-number*) if $\lambda > 1$ while all other conjugates of λ have moduli strictly less than 1.

Fact 3.5. [33, Chapter 1, Theorem 2] *Every real algebraic number field contains a primitive element that is a PV-number.* □

Before we can show that K-algebraic model sets are algebraic Delone sets, we need the following lemmas.

Lemma 3.6. *Let $\varLambda$ be a nonperiodic K-algebraic model set with star map $.^\star$. Then, there is an algebraic integer $\lambda \in K^+$ such that a suitable power of the $\mathbb{Z}$-module endomorphism $m_\lambda^\star$ of $\mathcal{O}_K^\star$, defined by $m_\lambda^\star(z^\star) = (\lambda z)^\star$, is contractive, i.e. there is an $l \in \mathbb{N}$ and a real number $c \in (0,1)$ such that $\|(m_\lambda^\star)^l(z^\star)\| \le c\,\|z^\star\|$ holds for all $z \in \mathcal{O}_K$.*

Proof. By Fact 3.5, we may choose a PV-number λ of degree $d/2 = [K^+ : \mathbb{Q}]$ in K^+, where $d = [K : \mathbb{Q}] \ge 4$ due to the nonperiodicity; see Remark 3.4. Since all norms on $\mathbb{C}^{s-1} \times \mathbb{R}^t \simeq \mathbb{R}^{d-2}$ are equivalent, it suffices to prove the assertion in case of the maximum norm on $\mathbb{C}^{s-1} \times \mathbb{R}^t$ with respect to the absolute value on $\mathbb{C}$ and $\mathbb{R}$, respectively, rather than considering the Euclidean norm itself. But in that case, the assertion follows immediately with $l = 1$ and

$$c = \max\left\{|\sigma_j(\lambda)| \,\middle|\, j \in \{2, \dots, s+t\}\right\},$$

since the set $\{\sigma_2(\lambda), \dots, \sigma_{s+t}(\lambda)\}$ of conjugates of λ does not contain λ itself. To see this, note that $\sigma_j(\lambda) = \lambda$, where $j \in \{2, \dots, s+t\}$, implies that σ_j fixes K^+ whence σ_j is the identity or the complex conjugation, a contradiction; see Definition 3.3 and Remark 3.4. □

Lemma 3.7. *Let $\varLambda$ be a K-algebraic model set with star map $.^\star$ and let $d = [K : \mathbb{Q}]$. Then $\mathcal{O}_K^\star$ is dense in $\mathbb{C}^{s-1} \times \mathbb{R}^t \simeq \mathbb{R}^{d-2}$.*

Proof. If $d = 2$, one even has $\mathcal{O}_K^\star = \mathbb{C}^{s-1} \times \mathbb{R}^t = \{0\}$. Otherwise, choose a PV-number λ of degree $d/2$ in K^+; cf. Fact 3.5. Since $\mathcal{O}_K$ is a full $\mathbb{Z}$-module in K, the set $\{\lambda^k z \,|\, z \in \mathcal{O}_K\}$ is a full $\mathbb{Z}$-module in K for any $k \in \mathbb{N}$. Thus the set

$$\{(\lambda^k z, (m_\lambda^\star)^k(z^\star)) \,|\, z \in \mathcal{O}_K\},$$

is a (full) lattice in $\mathbb{C}^s \times \mathbb{R}^t \simeq \mathbb{R}^d$ for any $k \in \mathbb{N}$, where $m_\lambda^\star$ is the $\mathbb{Z}$-module endomorphism of $\mathcal{O}_K^\star$ from Lemma 3.6; cf. [10, Chapter 2, Section 3]. In conjunction with Lemma 3.6, this implies that, for any $\varepsilon > 0$, the $\mathbb{Z}$-module $\mathcal{O}_K^\star$ contains an $\mathbb{R}$-basis of $\mathbb{C}^{s-1} \times \mathbb{R}^t$ whose elements have norms $\le \varepsilon$. The assertion follows. □

Lemma 3.8. *Let $\varLambda$ be a K-algebraic model set. Then, for any finite set $F \subset K$, there is a homothety h of the complex plane such that $h(F) \subset \varLambda$. Moreover, h can be chosen such that $h(z) = \kappa z + v$, where $\kappa \in K^+$ is an algebraic integer with $\kappa \geq 1$ and $v \in \varLambda$.*

Proof. Without loss of generality, we may assume that $\varLambda$ is of the form $\varLambda(W)$ (see Definition 3.3) and that $F \neq \varnothing$. Note that there is an $l \in \mathbb{N}$ such that $\{lz \,|\, z \in F\} \subset \mathcal{O}_K$. Let $d = [K : \mathbb{Q}]$ and let $.^\star$ be the star map of $\varLambda$. If $d = 2$, we are done by setting $h(z) = lz$. Otherwise, since W has nonempty interior, Lemma 3.7 shows the existence of a suitable $z_0 \in \mathcal{O}_K$ with $z_0^\star \in \operatorname{int} W$. Consider the open neighbourhood $V = (\operatorname{int} W) - z_0^\star$ of 0 in $\mathbb{C}^{s-1} \times \mathbb{R}^t$ and choose a PV-number λ of degree $d/2$ in K^+; cf. Fact 3.5. By virtue of Lemma 3.6, there is a $k \in \mathbb{N}$ such that

$$(m_\lambda^\star)^k\big((lF)^\star\big) \subset V\,.$$

It follows that $\{(\lambda^k z + z_0)^\star \,|\, z \in lF\} \subset \operatorname{int} W$ and, further, that $h(F) \subset \varLambda$, where h is the homothety given by $z \mapsto (l\lambda^k)z + z_0$. The additional statement follows immediately from the observation that $z_0 \in \varLambda$. □

Proposition 3.9. *K-algebraic model sets are algebraic Delone sets. Moreover, any K-algebraic model set $\varLambda$ satisfies $K_\varLambda = K$.*

Proof. Since $K_\varLambda = K_{t+\varLambda}$ for any $t \in \mathbb{C}$, we may assume that $\varLambda$ is of the form $\varLambda(W)$ (see Definition 3.3). Any K-algebraic model set $\varLambda$ is a Delone set by Remark 3.4. Property (Alg) follows from the observation that $K_\varLambda \subset K$ (recall that $\varLambda - \varLambda \subset \mathcal{O}_K$ and that $\overline{K} = K$). Further, property (Hom) is an immediate consequence of Lemma 3.8. Let $\{\alpha_1, \dots, \alpha_d\}$ be a $\mathbb{Q}$-basis of $K/\mathbb{Q}$. By the additional statement of Lemma 3.8 there is a nonzero element $\kappa \in K^+$ and a point $v \in \varLambda$ such that the $\mathbb{Q}$-linear independent set $\{\kappa\alpha_1, \dots, \kappa\alpha_d\}$ is contained in $\varLambda - \{v\} \subset K_\varLambda$. Since $K_\varLambda \subset K$, this shows that $K_\varLambda = K$. □

Remark 3.10. As another immediate consequence of Lemma 3.8, one verifies that, for any K-algebraic model set $\varLambda$, the set of $\varLambda$-directions is precisely the set of $\mathcal{O}_K$-directions.

Example 3.11. Standard examples of n-cyclotomic Delone sets are the $\mathbb{Q}(\zeta_n)$-algebraic model sets, where $n \geq 3$, which from now on are called *n-cyclotomic model sets*; cf. Fact 2.1 and Proposition 3.9 (note also that $\mathbb{Q}(\zeta_n)$ is obtained from $\mathbb{Q}(\zeta_n)^+$ by adjoining the square root of the negative number $\zeta_n^2 + \zeta_n^{-2} - 2 \in \mathbb{Q}(\zeta_n)^+$, the latter being the discriminant of $X^2 - (\zeta_n + \zeta_n^{-1})X + 1$). These sets were also called *cyclotomic model sets with underlying $\mathbb{Z}$-module* $\mathbb{Z}[\zeta_n]$ in [28, Section 4.5], since $\mathbb{Z}[\zeta_n]$ is the ring of integers in the nth cyclotomic field; cf. [38, Theorem 2.6]. The latter range from periodic examples like the fourfold square lattice ($n = 4$) or the sixfold triangular lattice ($n = 3$) to nonperiodic examples like the vertex set of the tenfold Tübingen triangle tiling [6], [7] ($n = 5$), the eightfold Ammann-Beenker tiling of the plane [1], [5], [20] ($n = 8$) or the twelvefold shield tiling [20] ($n = 12$); see [27, Figure 1], [28, Figure 2] and Figure 1 below for illustrations. In general, for any divisor m of $\operatorname{lcm}(n, 2)$, one can choose the window such that the corresponding n-cyclotomic model sets have m-fold cyclic symmetry in the sense of symmetries of LI-classes, meaning that a discrete structure has a certain symmetry if the original and the transformed structure are locally indistinguishable; cf. [2] for details. Note that the vertex sets of the famous Penrose tilings of the plane fail to be 5-cyclotomic model sets but can still be seen to be 5-cyclotomic Delone sets; see [4] and references therein.

4. A cyclotomic theorem

Definition 4.1. Let $m \geq 4$ be a natural number. Set

$$D_m = \big\{(k_1,k_2,k_3,k_4) \in \mathbb{N}^4 \,\big|\, k_3 < k_1 \leq k_2 < k_4 \leq m-1 \text{ and } k_1+k_2 = k_3+k_4\big\}$$

and define the function $f_m : D_m \to \mathbb{Q}(\zeta_m)^*$ by

$$f_m(k_1,k_2,k_3,k_4) = \frac{(1-\zeta_m^{k_1})(1-\zeta_m^{k_2})}{(1-\zeta_m^{k_3})(1-\zeta_m^{k_4})}. \tag{4.1}$$

We further set $\mathcal{C}_m = f_m(D_m)$ (note that $\mathcal{C}_m \subset \mathcal{C}_{m'}$ for any multiple m' of m) and $\mathcal{C} = \bigcup_{m\geq 4} \mathcal{C}_m$. Moreover, for a subset K of $\mathbb{C}$, we set $\mathcal{C}(K) = \mathcal{C} \cap K$ and $\mathcal{C}_m(K) = \mathcal{C}_m \cap K$.

Fact 4.2. [15, Lemma 3.1] *Let $m \geq 4$. The function f_m is real-valued. Moreover, one has $f_m(d) > 1$ for all $d \in D_m$.* □

For our application to discrete tomography, we shall show below the *finiteness* of the set $\mathcal{C}(L)$ for all real algebraic number fields L and provide explicit results in the three cases $\mathbb{Q}(\zeta_5)^+ = \mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\zeta_8)^+ = \mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\zeta_{12})^+ = \mathbb{Q}(\sqrt{3})$. Gardner and Gritzmann showed the following result for the field $\mathbb{Q} = \mathbb{Q}(\zeta_3)^+ = \mathbb{Q}(\zeta_4)^+$.

Theorem 4.3. [15, Lemma 3.8, Lemma 3.9 and Theorem 3.10]

$$\mathcal{C}(\mathbb{Q}) = \mathcal{C}_{12}(\mathbb{Q}) = \big\{\tfrac{4}{3}, \tfrac{3}{2}, 2, 3, 4\big\}.$$

Moreover, all solutions of $f_m(d) = q \in \mathbb{Q}$, where $m \geq 4$ and $d \in D_m$, are either given, up to multiplication of m and d by the same factor, by $m = 12$ and one of the following

(i)	$d = (6,6,4,8), q = \frac{4}{3}$;	(ii)	$d = (6,6,2,10), q = 4$;
(iii)	$d = (4,8,3,9), q = \frac{3}{2}$;	(iv)	$d = (4,8,2,10), q = 3$;
(v)	$d = (4,4,2,6), q = \frac{3}{2}$;	(vi)	$d = (8,8,6,10), q = \frac{3}{2}$;
(vii)	$d = (4,4,1,7), q = 3$;	(viii)	$d = (8,8,5,11), q = 3$;
(ix)	$d = (3,9,2,10), q = 2$;	(x)	$d = (3,3,1,5), q = 2$;
(xi)	$d = (9,9,7,11), q = 2$;		

or by one of the following

(xii) $d = (2k, s, k, k+s), q = 2,$ *where* $s \geq 2, m = 2s$ *and* $1 \leq k \leq \frac{s}{2}$;

(xiii) $d = (s, 2k, k, k+s), q = 2,$ *where* $s \geq 2, m = 2s$ *and* $\frac{s}{2} \leq k < s$. □

The next three lemmas are the key tools for our approach.

Lemma 4.4. *Let $a \in \mathbb{R}^*$. If $a = \frac{1+x}{1+y}$ for $x, y \in \mu \cup \{0\}$ with $y \neq -1$ then $a \in \{\frac{1}{2}, 1, 2\}$.*

Proof. It suffices to consider the cases $a = 1 + \omega$ and $a = \frac{1+\omega_1}{1+\omega_2}$ with $\omega, \omega_1, \omega_2 \in \mu$ and $\omega_2 \neq -1$. In the first case, one has $\omega = a - 1 \in \mu(\mathbb{R}) = \{\pm 1\}$ whence $\omega = 1$ (due to $a \neq 0$) and $a = 2$. In the second case, one has

$$a = \bar{a} = \frac{1+\bar{\omega}_1}{1+\bar{\omega}_2} = \omega_2\omega_1^{-1}\frac{1+\omega_1}{1+\omega_2} = \omega_2\omega_1^{-1}a\,,$$

thus $\omega_1 = \omega_2$ and $a = 1$. □

Lemma 4.5 (Comparison of coefficients)**.** *Let $K \subset \mathbb{C}$ be a field, let $m \in \mathbb{N}$, and let $\zeta \in \mu$ with $\zeta^m \in K$. Let $a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1} \in K$ with*

$$\sum_{i=0}^{m-1} a_i\zeta^i = \sum_{i=0}^{m-1} b_i\zeta^i\,.$$

Then one has $a_i = b_i$ for all $i = 0, \dots, m-1$ if one of the following conditions holds.

(a) $[K(\zeta):K]=m$.
(b) $[K(\zeta):K]=m-1$ *and at most* $m-1$ *of* $a_0,\dots,a_{m-1},b_0,\dots,b_{m-1}$ *are nonzero.*

Moreover, if $[K(\zeta):K]=m-1$ *and* $a_k-b_k\neq 0$ *for some* k *then* $|a_i-b_i|=|a_j-b_j|\neq 0$ *for all* i,j.

Proof. In case (a), the assertion follows immediately from the linear independence of $1,\zeta,\dots,\zeta^{m-1}$ over K. If $[K(\zeta):K]=m-1$, set $\omega=\zeta^m\in K$. The minimum polynomial $f\in K[X]$ of ζ over K has degree $m-1$ and one has $X^m-\omega=(X-\epsilon)f$ with $\epsilon\in K$, hence $\omega=\epsilon^m$ (in particular, $\epsilon\in\mu(K)$) and

$$f=\frac{X^m-\epsilon^m}{X-\epsilon}=\sum_{i=0}^{m-1}\epsilon^{m-1-i}X^i\,.$$

If $\sum_{i=0}^{m-1}(a_i-b_i)\zeta^i=0$ then there is an element $c\in K$ with $a_i=b_i+c\epsilon^{m-1-i}$ for all $i=0,\dots,m-1$. By assumption (b) one has $a_i=0=b_i$ for some i. This implies $c=0$ and therefore the assertion. For the additional statement, first observe that due to $a_k\neq b_k$ for some k one has $c\neq 0$. Thus $|a_i-b_i|=|c\epsilon^{m-1-i}|=|c|=|c\epsilon^{m-1-j}|=|a_j-b_j|\neq 0$ for all i,j. □

Lemma 4.6. *Let* $K\subset\mathbb{C}$ *be a field, let* $m\in\mathbb{N}$, *and let* $\zeta\in\mu$ *with* $\zeta^m\in K$. *Further, let* $\omega_1,\omega_2,\omega_3,\omega_4\in\mu(K)$ *and* $k_1,k_2,k_3,k_4\in\{0,\dots,m-1\}$ *satisfy the following conditions.*

(i) $\gcd(k_i,m)=1$ *for some* $i\in\{1,2,3,4\}$.
(ii) $k_1+k_2\equiv k_3+k_4 \pmod m$
(iii) $\omega_3\zeta^{k_3},\omega_4\zeta^{k_4}\neq 1$ *and* $a=\frac{(1-\omega_1\zeta^{k_1})(1-\omega_2\zeta^{k_2})}{(1-\omega_3\zeta^{k_3})(1-\omega_4\zeta^{k_4})}\in K\cap(\mathbb{R}^*\setminus\{\pm1\})$.

Then $a\in\{\frac12,2\}$ *if one of the following conditions holds.*

(a) $[K(\zeta):K]=m$ *and* $m\geq 3$.
(b) $[K(\zeta):K]=m-1$ *and* $m\geq 5$.

Proof. Without restriction, we may assume that $\gcd(k_1,m)=1$. Then, for $i=2,3,4$, there are $a_i,b_i\in\mathbb{Z}$ such that $k_i=a_ik_1+b_im$ and, with $\zeta'=\zeta^{k_1}$, $\zeta^{k_i}=(\zeta')^{a_i}(\zeta^m)^{b_i}$. Since one has $(\zeta')^m\in K$, $K(\zeta')=K(\zeta)$ and

$$\frac{(1-\omega_1\zeta^{k_1})(1-\omega_2\zeta^{k_2})}{(1-\omega_3\zeta^{k_3})(1-\omega_4\zeta^{k_4})}=\frac{(1-\omega_1'\zeta')(1-\omega_2'\zeta'^{k_2'})}{(1-\omega_3'\zeta'^{k_3'})(1-\omega_4'\zeta'^{k_4'})}$$

for suitable $\omega_1',\omega_2',\omega_3',\omega_4'\in\mu(K)$ and $k_2',k_3',k_4'\in\{0,\dots,m-1\}$ with $1+k_2'\equiv k_3'+k_4' \pmod m$, we may further assume that $k_1=1$. We thus obtain

$$1-\omega_1\zeta-\omega_2\zeta^{k_2}+\omega_1\omega_2\zeta^{k_2+1}=a-a\omega_3\zeta^{k_3}-a\omega_4\zeta^{k_4}+a\omega_3\omega_4\zeta^{k_3+k_4}\,,$$

where, without restriction, $k_3\leq k_4$. From now on, let $[k]\in\{0,\dots,m-1\}$ denote the canonical representative of the equivalence class of $k\in\mathbb{Z}$ modulo m. We may finally write

$$1-\omega_1\zeta-\omega_2\zeta^{k_2}+\omega_1\omega_2\omega\zeta^{[k_2+1]}=a-a\omega_3\zeta^{k_3}-a\omega_4\zeta^{[1+k_2-k_3]}+a\omega_3\omega_4\omega'\zeta^{[k_2+1]}$$

with $k_3\leq[1+k_2-k_3]$ and suitable $\omega,\omega'\in\mu(K)$.

Case 1. $k_2=0$. Then

$$1-\omega_1\zeta-\omega_2+\omega_1\omega_2\omega\zeta=a-a\omega_3\zeta^{k_3}-a\omega_4\zeta^{[1-k_3]}+a\omega_3\omega_4\omega'\zeta\,.$$

If $k_3=0$ then $a=\frac{1-\omega_2}{1-\omega_3}$ by Lemma 4.5 and the assertion follows from Lemma 4.4. The case $k_3=1$ cannot occur (due to $k_3\leq[1-k_3]$), whereas $k_3\geq 2$ implies $a=1-\omega_2$ by Lemma 4.5. The assertion follows from Lemma 4.4.

Case 2. $k_2=1$. Then

$$1-(\omega_1+\omega_2)\zeta+\omega_1\omega_2\omega\zeta^2=a-a\omega_3\zeta^{k_3}-a\omega_4\zeta^{[2-k_3]}+a\omega_3\omega_4\omega'\zeta^2\,.$$

If $k_3 = 0$ then $a = \frac{1}{1-\omega_3}$ by Lemma 4.5 and the assertion follows from Lemma 4.4. If $k_3 = 1$ then Lemma 4.5 implies $a = 1$, which is excluded by assumption. The case $k_3 = 2$ is impossible (due to $k_3 \leq [2-k_3]$). Let $k_3 \geq 3$ (hence $m \geq 4$). Under condition (a), this implies $a = 1$ by Lemma 4.5, which is excluded by assumption. Under condition (b), $k_3 = 3$ implies

$$1 - (\omega_1 + \omega_2)\zeta + \omega_1\omega_2\omega\zeta^2 = a - a\omega_3\zeta^3 - a\omega_4\zeta^{m-1} + a\omega_3\omega_4\omega'\zeta^2$$

with $m - 1 \geq 4$ (due to $m \geq 5$). The additional statement of Lemma 4.5 implies $m = 5$ and $|1-a| = |a\omega_4| = |a|$, thus $a = 1/2$. If $k_3 \geq 4$ then $m \geq 6$ (due to $k_3 \leq [2-k_3]$) and Lemma 4.5 implies $a = 1$, which is excluded by assumption.

Case 3. $k_2 \in \{2, \dots, m-2\}$ (hence $m \geq 4$ and $2 \leq k_2 < k_2 + 1 \leq m-1$). Then

$$(1-a) - \omega_1\zeta - \omega_2\zeta^{k_2} + (\omega_1\omega_2\omega - a\omega_3\omega_4\omega')\zeta^{k_2+1} = -a\omega_3\zeta^{k_3} - a\omega_4\zeta^{[1+k_2-k_3]}\,.$$

Under condition (a), Lemma 4.5 shows that this is impossible, since there are at least three nontrivial coefficients on the left-hand side and at most two nontrivial coefficients on the right-hand side of this equation. Under condition (b) (hence $m \geq 5$), $k_3 = 0$ implies $a - 1 = a\omega_3$ by Lemma 4.5, thus $a = \frac{1}{1-\omega_3}$ and the assertion follows from Lemma 4.4. If $k_3 = 1$ then $a = 1$ by Lemma 4.5, which is excluded by assumption. If $k_3 \geq 2$ and $m \geq 7$ then $a = 1$ by Lemma 4.5, which is excluded by assumption. Employing the additional statement of Lemma 4.5, we shall now see that the missing cases ($k_3 \geq 2$ and $m \in \{5,6\}$) are either impossible or yield $|1-a| = 1$ and thus $a = 2$ (due to $a \neq 0$). In fact, $m = 5$ and $k_3 = 2$ imply $k_2 = 3$ (due to $k_3 \leq [1+k_2-k_3]$) and, further, $|1-a| = |\omega_1| = 1$. The case $m = 5$ and $k_3 = 3$ cannot occur (due to $k_3 \leq [1+k_2-k_3]$). If $m = 5$ and $k_3 = 4$ then $k_2 = 2$ (due to $k_3 \leq [1+k_2-k_3]$) and, further, $|1-a| = |\omega_1| = 1$. If $m = 6$ and $k_3 = 2$ then $k_2 \in \{3,4\}$ (due to $k_3 \leq [1+k_2-k_3]$). The case $k_2 = 3$ is impossible, whereas the case $k_2 = 4$ yields $|1-a| = |\omega_1| = 1$. The case $m = 6$ and $k_3 = 3$ is impossible (due to $k_3 \leq [1+k_2-k_3]$). The case $m = 6$ and $k_3 = 4$ implies $k_2 = 2$ (due to $k_3 \leq [1+k_2-k_3]$) and, further, $|1-a| = |\omega_1| = 1$. Finally, the case $m = 6$ and $k_3 = 5$ implies $k_2 = 3$ (due to $k_3 \leq [1+k_2-k_3]$) and, once again, $|1-a| = |\omega_1| = 1$.

Case 4. $k_2 = m-1$. Then

$$(1 + \omega_1\omega_2\omega) - \omega_1\zeta - \omega_2\zeta^{m-1} = a(1 + \omega_3\omega_4\omega') - a\omega_3\zeta^{k_3} - a\omega_4\zeta^{[m-k_3]}\,.$$

Under condition (a), Lemma 4.5 implies $\{k_3, [m-k_3]\} = \{1, m-1\}$, thus $k_3 = 1$ and $[m-k_3] = m-1$ (due to $k_3 \leq [m-k_3]$). Further, Lemma 4.5 yields $a = \omega_2/\omega_4$, a contradiction (due to $|a| \neq 1$). By the additional statement of Lemma 4.5, condition (b) (hence $m \geq 5$) implies $m = 5$, $k_3 = 2$ and, further, $|a\omega_3| = |\omega_2| = 1$, a contradiction (due to $|a| \neq 1$). □

We are now in a position to prove the following extension of Theorem 4.3.

Theorem 4.7. *For $n \in \mathbb{N}$, one has*

$$\mathcal{C}(\mathbb{Q}(\zeta_n)^+) = \mathcal{C}_{\mathrm{lcm}(2n,12)}(\mathbb{Q}(\zeta_n)^+)\,.$$

In particular, the last set is finite. Moreover, all solutions of $f_m(d) \in \mathbb{Q}(\zeta_n)^+$, where $m \geq 4$ and $d \in D_m$, are either of the form (xii) or (xiii) of Theorem 4.3 or are given, up to multiplication of m and d by the same factor, by $m = \mathrm{lcm}(2n, 12)$ and d from a finite list.

Proof. Since $\mathbb{Q}(\zeta_n)^+ = \mathbb{Q}(\zeta_{2n})^+$ for odd n it suffices to consider the case where n is even (hence $\mathrm{lcm}(2n,12) = \mathrm{lcm}(2n,3)$). Let $m \geq 4$ and $d = (k_1,k_2,k_3,k_4) \in D_m$ such that

$$a = f_m(d) = \frac{(1-\zeta_m^{k_1})(1-\zeta_m^{k_2})}{(1-\zeta_m^{k_3})(1-\zeta_m^{k_4})} \in \mathbb{Q}(\zeta_n)^+\,.$$

Recall that $a > 1$ by Fact 4.2. We may assume that $\gcd(m, k_1, k_2, k_3, k_4) = 1$. By virtue of Theorem 4.3, we may also assume that $a \notin \mathbb{Q}$. Observe that

$$a \in \mathbb{Q}(\zeta_n)^+ \cap \mathbb{Q}(\zeta_m)^+ = \mathbb{Q}(\zeta_{\gcd(m,n)})^+ .$$

Claims 1 and 2 below show that $\mathrm{lcm}(2n, 3)$ is a multiple of m, hence the assertion.

Claim 1. Let p be an odd prime number and assume that $\mathrm{ord}_p(n) < \mathrm{ord}_p(m)$. Then one has $p = 3$, $\mathrm{ord}_p(m) = 1$ and $\mathrm{ord}_p(n) = 0$.

To see this, set $r = \mathrm{ord}_p(m) \geq 1$, $K = \mathbb{Q}(\zeta_{m/p})$ and note that $\mathbb{Q}(\zeta_{\gcd(m,n)}) \subset K$. Let $m = p^r m'$, where $\gcd(p, m') = 1$. Then, for $i = 1, 2, 3, 4$, there are $a_i, b_i \in \mathbb{Z}$ such that $k_i = a_i p + b_i m'$ and, further, $\zeta_m^{k_i} = \zeta_{m/p}^{a_i} \zeta_{p^r}^{b_i}$. Since $\zeta_{p^r}^p = \zeta_{p^{r-1}} \in K$ one has

$$a = \frac{(1 - \omega_1 \zeta_{p^r}^{l_1})(1 - \omega_2 \zeta_{p^r}^{l_2})}{(1 - \omega_3 \zeta_{p^r}^{l_3})(1 - \omega_4 \zeta_{p^r}^{l_4})}$$

for suitable $\omega_1, \omega_2, \omega_3, \omega_4 \in \mu(K)$ and $l_1, l_2, l_3, l_4 \in \{0, \dots, p-1\}$ with $\gcd(l_i, p) = 1$ for some $i \in \{1, 2, 3, 4\}$ and $l_1 + l_2 \equiv l_3 + l_4 \pmod{p}$. Further, by Fact 2.1, one has

$$[K(\zeta_{p^r}) : K] = [\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_{m/p})] = \frac{\phi(p^r)}{\phi(p^{r-1})} = \begin{cases} p - 1 & \text{if } r = 1; \\ p & \text{if } r \geq 2. \end{cases}$$

Lemma 4.6 implies both for $p \geq 5$ and $r \geq 2$ that $a = 2$, a contradiction. Therefore $p = 3$, $r = \mathrm{ord}_p(m) = 1$ and consequently $\mathrm{ord}_p(n) = 0$.

Claim 2. $\mathrm{ord}_2(m) \leq \mathrm{ord}_2(n) + 1$.

Assume that $r = \mathrm{ord}_2(m) \geq \mathrm{ord}_2(n) + 2 \geq 3$. Set $K = \mathbb{Q}(\zeta_{m/4})$ and note that $\mathbb{Q}(\zeta_{\gcd(m,n)}) \subset K$. As above, since $\zeta_{2^r}^4 = \zeta_{2^{r-2}} \in K$, one has

$$a = \frac{(1 - \omega_1 \zeta_{2^r}^{l_1})(1 - \omega_2 \zeta_{2^r}^{l_2})}{(1 - \omega_3 \zeta_{2^r}^{l_3})(1 - \omega_4 \zeta_{2^r}^{l_4})}$$

for suitable $\omega_1, \omega_2, \omega_3, \omega_4 \in \mu(K)$ and $l_1, l_2, l_3, l_4 \in \{0, 1, 2, 3\}$ with $\gcd(l_i, 4) = 1$ for some $i \in \{1, 2, 3, 4\}$ and $l_1 + l_2 \equiv l_3 + l_4 \pmod{4}$. Further, by Fact 2.1, one has

$$[K(\zeta_{2^s}) : K] = [\mathbb{Q}(\zeta_m) : \mathbb{Q}(\zeta_{m/4})] = \frac{\phi(2^r)}{\phi(2^{r-2})} = 4 .$$

Lemma 4.6 now implies $a = 2$, a contradiction. This proves the claim. □

Remark 4.8. Similar to the proof of Theorem 4.7, one can also use Lemma 4.6 to give another proof of the fact shown in [15] that all solutions of $f_m(d) \in \mathbb{Q} \setminus \{2\}$, where $m \geq 4$ and $d \in D_m$, are given, up to multiplication of m and d by the same factor, by $m = 12$. Thus the number 2 plays a special role in this context. Indeed this number leads to infinite families of solutions (see Theorem 4.3(xii)-(xiii) above) that can be found by using the 2-adic valuation; cf. [15] for details.

One even has the following result, which improves [28, Theorem 4.19].

Theorem 4.9. *For any real algebraic number field L, the set $\mathcal{C}(L)$ is finite. Moreover, there is a number $m_L \in \mathbb{N}$ such that all solutions of $f_m(d) \in L$, where $m \geq 4$ and $d \in D_m$, are either of the form (xii) or (xiii) of Theorem 4.3 or are given, up to multiplication of m and d by the same factor, by $m = m_L$ and d from a finite list.*

Proof. The finiteness of $L/\mathbb{Q}$ together with the identity

$$\mathbb{Q}(\mu)^+ = \bigcup_{n \in \mathbb{N}} \mathbb{Q}(\zeta_n)^+$$

implies that $L \cap \mathbb{Q}(\mu)^+ = L \cap \mathbb{Q}(\zeta_n)^+$ for some $n \in \mathbb{N}$. Since $\mathcal{C} \subset \mathbb{Q}(\mu)^+$ by Fact 4.2 it follows that

$$\mathcal{C}(L) = L \cap \mathcal{C} = L \cap \mathcal{C} \cap \mathbb{Q}(\mu)^+ = L \cap \mathcal{C} \cap \mathbb{Q}(\zeta_n)^+ \subset \mathcal{C}(\mathbb{Q}(\zeta_n)^+) .$$

By virtue of Theorem 4.7, the assertion follows with $m_L = \mathrm{lcm}(2n, 12)$. $\square$

Corollary 4.10. (a)

$$\begin{aligned}\mathcal{C}(\mathbb{Q}(\sqrt{5})) &= \mathcal{C}_{60}(\mathbb{Q}(\sqrt{5}))\\ &= \big\{\tfrac{10-2\sqrt{5}}{5}, \tfrac{\sqrt{5}}{2}, \tfrac{9-3\sqrt{5}}{2}, \tfrac{5+3\sqrt{5}}{10}, \tfrac{5+\sqrt{5}}{6}, -1+\sqrt{5}, \tfrac{3+\sqrt{5}}{4}, \tfrac{4}{3}, \tfrac{5-\sqrt{5}}{2}, \tfrac{5+\sqrt{5}}{5},\\ &\quad \tfrac{3}{2}, 6-2\sqrt{5}, \tfrac{1+\sqrt{5}}{2}, \tfrac{5+\sqrt{5}}{4}, \tfrac{-3+3\sqrt{5}}{2}, \tfrac{5+2\sqrt{5}}{5}, 2, \tfrac{2+\sqrt{5}}{2}, \tfrac{15+3\sqrt{5}}{10}, \sqrt{5},\\ &\quad \tfrac{3+\sqrt{5}}{2}, \tfrac{10+2\sqrt{5}}{5}, 3, 1+\sqrt{5}, \tfrac{5+\sqrt{5}}{2}, 4, 2+\sqrt{5}, 3+\sqrt{5}, \tfrac{5+3\sqrt{5}}{2}, \tfrac{7+3\sqrt{5}}{2},\\ &\quad \tfrac{9+3\sqrt{5}}{2}, 5+2\sqrt{5}, 6+2\sqrt{5}\big\}.\end{aligned}$$

Moreover, all solutions of $f_m(d) \in \mathbb{Q}(\sqrt{5})$, where $m \geq 4$ and $d \in D_m$, are either of the form (xii) or (xiii) of Theorem 4.3 or are given, up to multiplication of m and d by the same factor, by $m = 60$ and d from the *following list.*

1	$(12,36,6,42)$	2	$(24,24,9,39)$	3	$(24,48,18,54)$	4	$(36,36,21,51)$
5	$(24,24,18,30)$	6	$(36,36,30,42)$	7	$(4,8,2,10)$	8	$(5,25,3,27)$
9	$(6,42,4,44)$	10	$(8,14,4,18)$	11	$(8,32,5,35)$	12	$(8,50,6,52)$
13	$(9,21,5,25)$	14	$(9,39,6,42)$	15	$(10,10,4,16)$	16	$(10,28,6,32)$
17	$(10,52,8,54)$	18	$(12,18,6,24)$	19	$(14,26,8,32)$	20	$(14,34,9,39)$
21	$(14,42,10,46)$	22	$(16,32,10,38)$	23	$(18,18,8,28)$	24	$(18,26,10,34)$
25	$(18,36,12,42)$	26	$(18,46,14,50)$	27	$(18,54,16,56)$	28	$(21,51,18,54)$
29	$(24,24,12,36)$	30	$(24,42,18,48)$	31	$(26,32,16,42)$	32	$(26,46,21,51)$
33	$(28,34,18,44)$	34	$(28,44,22,50)$	35	$(28,52,25,55)$	36	$(32,50,28,54)$
37	$(34,42,26,50)$	38	$(34,46,28,52)$	39	$(35,55,33,57)$	40	$(36,36,24,48)$
41	$(39,51,35,55)$	42	$(42,42,32,52)$	43	$(42,48,36,54)$	44	$(46,52,42,56)$
45	$(50,50,44,56)$	46	$(52,56,50,58)$	47	$(18,18,6,30)$	48	$(42,42,30,54)$
49	$(4,52,2,54)$	50	$(5,35,2,38)$	51	$(6,18,2,22)$	52	$(8,10,2,16)$
53	$(8,28,3,33)$	54	$(8,46,4,50)$	55	$(8,56,6,58)$	56	$(9,21,3,27)$
57	$(9,39,4,44)$	58	$(10,32,4,38)$	59	$(10,50,6,54)$	60	$(12,42,6,48)$
61	$(14,18,4,28)$	62	$(14,26,5,35)$	63	$(14,34,6,42)$	64	$(14,52,10,56)$
65	$(16,28,6,38)$	66	$(18,24,6,36)$	67	$(18,34,8,44)$	68	$(18,42,10,50)$
69	$(18,48,12,54)$	70	$(21,51,16,56)$	71	$(24,36,12,48)$	72	$(25,55,22,58)$
73	$(26,28,10,44)$	74	$(26,42,16,52)$	75	$(26,46,18,54)$	76	$(28,50,22,56)$
77	$(32,34,16,50)$	78	$(32,44,22,54)$	79	$(32,52,27,57)$	80	$(34,46,25,55)$
81	$(36,42,24,54)$	82	$(39,51,33,57)$	83	$(42,46,32,56)$	84	$(42,54,38,58)$
85	$(50,52,44,58)$	86	$(12,12,2,22)$	87	$(12,24,3,33)$	88	$(12,36,4,44)$
89	$(12,48,6,54)$	90	$(24,24,6,42)$	91	$(24,36,10,50)$	92	$(24,48,16,56)$
93	$(36,36,18,54)$	94	$(36,48,27,57)$	95	$(48,48,38,58)$	96	$(8,28,6,30)$
97	$(14,26,10,30)$	98	$(18,24,12,30)$	99	$(18,42,15,45)$	100	$(32,52,30,54)$
101	$(34,46,30,50)$	102	$(36,42,30,48)$	103	$(12,24,6,30)$	104	$(24,36,15,45)$
105	$(36,48,30,54)$	106	$(15,45,12,48)$	107	$(18,30,12,36)$	108	$(30,42,24,48)$
109	$(24,36,20,40)$	110	$(10,30,8,32)$	111	$(15,15,9,21)$	112	$(18,30,14,34)$
113	$(24,30,18,36)$	114	$(30,36,24,42)$	115	$(30,42,26,46)$	116	$(30,50,28,52)$
117	$(45,45,39,51)$	118	$(15,15,3,27)$	119	$(18,30,6,42)$	120	$(30,42,18,54)$
121	$(45,45,33,57)$	122	$(8,32,2,38)$	123	$(14,34,4,44)$	124	$(18,18,3,33)$
125	$(18,36,6,48)$	126	$(24,42,12,54)$	127	$(26,46,16,56)$	128	$(28,52,22,58)$
129	$(42,42,27,57)$	130	$(10,30,2,38)$	131	$(15,45,6,54)$	132	$(18,30,4,44)$
133	$(24,30,6,48)$	134	$(30,36,12,54)$	135	$(30,42,16,56)$	136	$(30,50,22,58)$
137	$(24,36,6,54)$	138	$(30,30,6,54)$	139	$(30,30,18,42)$	140	$(12,12,6,18)$
141	$(12,24,8,28)$	142	$(12,36,9,39)$	143	$(12,48,10,50)$	144	$(24,24,14,34)$
145	$(24,36,18,42)$	146	$(24,48,21,51)$	147	$(36,36,26,46)$	148	$(36,48,32,52)$
149	$(48,48,42,54)$	150	$(14,26,2,38)$	151	$(18,42,6,54)$	152	$(34,46,22,58)$
153	$(14,34,12,36)$	154	$(18,18,12,24)$	155	$(26,46,24,48)$	156	$(42,42,36,48)$
157	$(18,42,12,48)$	158	$(20,20,2,38)$	159	$(20,40,6,54)$	160	$(40,40,22,58)$
161	$(20,20,8,32)$	162	$(40,40,28,52)$	163	$(20,20,14,26)$	164	$(20,40,18,42)$
165	$(40,40,34,46)$	166	$(20,40,12,48)$	167	$(24,24,4,44)$	168	$(36,36,16,56)$
169	$(30,30,12,48)$	170	$(30,30,24,36)$	171	$(30,30,20,40)$	172	$(30,30,10,50)$
173	$(20,40,15,45)$	174	$(20,40,10,50)$	175	$(20,20,10,30)$	176	$(40,40,30,50)$
177	$(20,20,5,35)$	178	$(40,40,25,55)$	179	$(15,45,10,50)$	180	$(15,15,5,25)$
181	$(45,45,35,55)$						

(b)

$$\begin{aligned}
\mathcal{C}(\mathbb{Q}(\sqrt{2})) &= \mathcal{C}_{48}(\mathbb{Q}(\sqrt{2})) \\
&= \big\{\tfrac{2+\sqrt{2}}{3}, 4-2\sqrt{2}, \tfrac{1+\sqrt{2}}{2}, -3+3\sqrt{2}, \tfrac{4}{3}, \sqrt{2}, \tfrac{3}{2}, \tfrac{2+\sqrt{2}}{2}, 2, 1+\sqrt{2}, \\
&\quad 3, 2+\sqrt{2}, 4, \tfrac{6+3\sqrt{2}}{2}, 3+2\sqrt{2}, 4+2\sqrt{2}, 4+3\sqrt{2}\big\}.
\end{aligned}$$

Moreover, all solutions of $f_m(d) \in \mathbb{Q}(\sqrt{2})$, where $m \geq 4$ and $d \in D_m$, are either of the form (xii) or (xiii) of Theorem 4.3 or are given, up to multiplication of m and d by the same factor, by $m = 48$ and d from the following list.

1	$(6,18,4,20)$	2	$(10,36,8,38)$	3	$(12,12,6,18)$	4	$(12,22,8,26)$
5	$(12,30,9,33)$	6	$(12,38,10,40)$	7	$(18,18,10,26)$	8	$(18,24,12,30)$
9	$(18,36,15,39)$	10	$(24,30,18,36)$	11	$(26,36,22,40)$	12	$(30,30,22,38)$
13	$(30,42,28,44)$	14	$(36,36,30,42)$	15	$(4,10,2,12)$	16	$(8,40,6,42)$
17	$(9,33,6,36)$	18	$(10,26,6,30)$	19	$(12,18,6,24)$	20	$(15,39,12,42)$
21	$(18,30,12,36)$	22	$(20,26,12,34)$	23	$(22,28,14,36)$	24	$(22,38,18,42)$
25	$(30,36,24,42)$	26	$(38,44,36,46)$	27	$(10,22,8,24)$	28	$(18,18,12,24)$
29	$(26,38,24,40)$	30	$(30,30,24,36)$	31	$(18,30,16,32)$	32	$(4,38,2,40)$
33	$(8,8,2,14)$	34	$(9,15,3,21)$	35	$(10,22,4,28)$	36	$(10,44,8,46)$
37	$(12,30,6,36)$	38	$(18,18,6,30)$	39	$(18,36,12,42)$	40	$(20,22,8,34)$
41	$(26,28,14,40)$	42	$(26,38,20,44)$	43	$(30,30,18,42)$	44	$(33,39,27,45)$
45	$(40,40,34,46)$	46	$(6,30,2,34)$	47	$(10,12,2,20)$	48	$(12,18,3,27)$
49	$(12,26,4,34)$	50	$(12,36,6,42)$	51	$(18,24,6,36)$	52	$(18,30,8,40)$
53	$(18,42,14,46)$	54	$(22,36,14,44)$	55	$(24,30,12,42)$	56	$(30,36,21,45)$
57	$(36,38,28,46)$	58	$(10,26,2,34)$	59	$(18,30,6,42)$	60	$(22,38,14,46)$
61	$(12,24,2,34)$	62	$(24,24,6,42)$	63	$(24,36,14,46)$	64	$(12,24,10,26)$
65	$(24,24,18,30)$	66	$(24,36,22,38)$	67	$(16,16,10,22)$	68	$(32,32,26,38)$
69	$(18,18,2,34)$	70	$(30,30,14,46)$	71	$(16,32,6,42)$	72	$(24,24,16,32)$
73	$(24,24,8,40)$	74	$(16,32,12,36)$	75	$(16,32,8,40)$	76	$(16,16,8,24)$
77	$(32,32,24,40)$	78	$(16,16,4,28)$	79	$(32,32,20,44)$	80	$(12,36,8,40)$
81	$(12,12,4,20)$	82	$(36,36,28,44)$				

(c)

$$\begin{aligned}
\mathcal{C}(\mathbb{Q}(\sqrt{3})) &= \mathcal{C}_{24}(\mathbb{Q}(\sqrt{3})) \\
&= \big\{8-4\sqrt{3}, \tfrac{3+2\sqrt{3}}{6}, \tfrac{-3+3\sqrt{3}}{2}, \tfrac{3+\sqrt{3}}{4}, \tfrac{2+\sqrt{3}}{3}, 3-\sqrt{3}, \tfrac{4}{3}, \tfrac{1+\sqrt{3}}{2}, -2+2\sqrt{3}, \\
&\quad \tfrac{3}{2}, \tfrac{3+\sqrt{3}}{3}, \sqrt{3}, \tfrac{2+\sqrt{3}}{2}, 2, \tfrac{3+2\sqrt{3}}{3}, \tfrac{3+\sqrt{3}}{2}, 1+\sqrt{3}, 3, \tfrac{6+2\sqrt{3}}{3}, 2+\sqrt{3}, 4, \\
&\quad 3+\sqrt{3}, \tfrac{5+3\sqrt{3}}{2}, 3+2\sqrt{3}, 4+2\sqrt{3}, 6+3\sqrt{3}, 7+4\sqrt{3}, 8+4\sqrt{3}\big\}.
\end{aligned}$$

Moreover, all solutions of $f_m(d) \in \mathbb{Q}(\sqrt{3})$, where $m \geq 4$ and $d \in D_m$, are either of the form (xii) or (xiii) of Theorem 4.3 or are given, up to multiplication of m and d by the same factor, by $m = 24$ and d from the *following list.*

1	$(4,8,2,10)$	2	$(6,16,4,18)$	3	$(8,10,4,14)$	4	$(8,14,5,17)$
5	$(8,18,6,20)$	6	$(10,16,7,19)$	7	$(14,16,10,20)$	8	$(16,20,14,22)$
9	$(3,15,2,16)$	10	$(4,6,2,8)$	11	$(6,14,4,16)$	12	$(9,21,8,22)$
13	$(10,12,6,16)$	14	$(10,18,8,20)$	15	$(12,14,8,18)$	16	$(18,20,16,22)$
17	$(10,14,8,16)$	18	$(4,4,2,6)$	19	$(4,14,3,15)$	20	$(5,7,3,9)$
21	$(5,17,4,18)$	22	$(6,10,4,12)$	23	$(7,19,6,20)$	24	$(8,14,6,16)$
25	$(10,10,6,14)$	26	$(10,16,8,18)$	27	$(10,20,9,21)$	28	$(14,14,10,18)$
29	$(14,18,12,20)$	30	$(17,19,15,21)$	31	$(20,20,18,22)$	32	$(4,10,2,12)$
33	$(10,14,6,18)$	34	$(14,20,12,22)$	35	$(4,16,2,18)$	36	$(6,8,2,12)$
37	$(8,10,3,15)$	38	$(8,14,4,18)$	39	$(8,20,6,22)$	40	$(10,16,6,20)$
41	$(14,16,9,21)$	42	$(16,18,12,22)$	43	$(8,10,6,12)$	44	$(14,16,12,18)$
45	$(3,9,1,11)$	46	$(4,18,2,20)$	47	$(6,10,2,14)$	48	$(6,20,4,22)$
49	$(10,12,4,18)$	50	$(12,14,6,20)$	51	$(14,18,10,22)$	52	$(15,21,13,23)$
53	$(4,10,1,13)$	54	$(4,20,2,22)$	55	$(5,7,1,11)$	56	$(5,17,2,20)$
57	$(6,14,2,18)$	58	$(7,19,4,22)$	59	$(8,10,2,16)$	60	$(10,14,4,20)$
61	$(10,18,6,22)$	62	$(14,16,8,22)$	63	$(14,20,11,23)$	64	$(17,19,13,23)$
65	$(6,8,1,13)$	66	$(6,16,2,20)$	67	$(8,12,2,18)$	68	$(8,18,4,22)$
69	$(12,16,6,22)$	70	$(16,18,11,23)$	71	$(6,8,4,10)$	72	$(6,16,5,17)$
73	$(8,12,6,14)$	74	$(8,18,7,19)$	75	$(12,16,10,18)$	76	$(16,18,14,20)$
77	$(6,12,4,14)$	78	$(12,18,10,20)$	79	$(6,18,2,22)$	80	$(10,12,2,20)$
81	$(12,14,4,22)$	82	$(8,14,2,20)$	83	$(10,16,4,22)$	84	$(8,16,2,22)$
85	$(10,14,2,22)$	86	$(12,12,2,22)$	87	$(12,12,10,14)$	88	$(4,14,2,16)$
89	$(10,10,4,16)$	90	$(10,20,8,22)$	91	$(14,14,8,20)$	92	$(6,12,2,16)$
93	$(12,18,8,22)$	94	$(8,8,6,10)$	95	$(16,16,14,18)$	96	$(10,10,2,18)$
97	$(14,14,6,22)$	98	$(10,10,8,12)$	99	$(14,14,12,16)$	100	$(12,12,8,16)$
101	$(12,12,4,20)$	102	$(8,16,6,18)$	103	$(8,16,4,20)$	104	$(8,8,4,12)$
105	$(16,16,12,20)$	106	$(8,8,2,14)$	107	$(16,16,10,22)$	108	$(6,18,4,20)$
109	$(6,6,2,10)$	110	$(18,18,14,22)$				

Proof. Applying Theorem 4.7 to the cases $n = 5, 8, 12$, the assertions follow from a direct computation. Note that in each case the last eleven entries of the lists above derive from (i)-(xi) of Theorem 4.3. □

5. Determination of convex subsets of algebraic Delone sets by X-rays

Definition 5.1. (a) Let F be a finite subset of $\mathbb{C}$, let $u \in \mathbb{S}^1$ be a direction, and let $\mathcal{L}_u$ be the set of lines in the complex plane in direction u. Then the *(discrete parallel) X-ray* of F *in direction* u is the function $X_uF : \mathcal{L}_u \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, defined by

$$X_uF(\ell) = \mathrm{card}(F \cap \ell\,)\,.$$

(b) Let $\mathcal{F}$ be a collection of finite subsets of $\mathbb{C}$ and let $U \subset \mathbb{S}^1$ be a finite set of directions. We say that the elements of $\mathcal{F}$ are *determined* by the X-rays in the directions of U if, for all $F, F' \in \mathcal{F}$, one has

$$(X_uF = X_uF' \;\; \forall u \in U) \;\Rightarrow\; F = F'\,.$$

The following negative result shows that, for algebraic Delone sets $\varLambda$, one has to impose some restriction on the finite subsets of $\varLambda$ to be determined. The proof only needs property (Hom).

Fact 5.2. [28, Proposition 3.1 and Remark 3.2] *Let $\varLambda$ be an algebraic Delone set and let $U \subset \mathbb{S}^1$ be a finite set of pairwise nonparallel $\varLambda$-directions. Then the finite subsets of $\varLambda$ are not determined by the X-rays in the directions of U.* □

Here, we shall focus on the convex subsets of algebraic Delone sets. One has the following fundamental result which even holds for Delone sets $\varLambda$ with property (Hom). See Figure 1 for an illustration of direction (i)⇒(ii).

Fact 5.3. [28, Proposition 4.6 and Lemma 4.5] *Let $\varLambda$ be an algebraic Delone set and let $U \subset \mathbb{S}^1$ be a set of two or more pairwise nonparallel $\varLambda$-directions. The following statements are equivalent:*

(i) *The convex subsets of $\varLambda$ are determined by the X-rays in the directions of U.*
(ii) *There is no U-polygon in $\varLambda$.*

In addition, if $\operatorname{card}(U) < 4$*, then there is a U-polygon in $\varLambda$.* □

The proof of the following central result uses Darboux's theorem on second midpoint polygons; see [12], [14, Chapter 1] or [18].

Fact 5.4. [15, Proposition 4.2] *Let $U \subset \mathbb{S}^1$ be a finite set of directions. Then there exists a U-polygon if and only if there is an affinely regular polygon such that each direction in U is parallel to one of its edges.* □

Remark 5.5. Clearly, U-polygons have an even number of vertices. Moreover, an affinely regular polygon with an even number of vertices is a U-polygon if and only if each direction of U is parallel to one of its edges. On the other hand, it is important to note that a U-polygon need not be affinely regular, even if it is a U-polygon in an algebraic Delone set. For example, there is a U-icosagon in the vertex set of the Tübingen triangle tiling of the plane (a 5-cyclotomic model set; see [27, Figure 1, Corollary 14 and Example 15]), which cannot be affinely regular since that restricts the number of vertices to 3, 4, 5, 6 or 10 by [25, Corollary 4.2]; see also [15, Example 4.3] for an example in the case of the square lattice. In general, there is an affinely regular polygon with $n \geq 3$ vertices in an algebraic Delone set $\varLambda$ if and only if $\mathbb{Q}(\zeta_n)^+ \subset K_{\varLambda}^+$, the latter being a relation which (due to property (Alg)) can only hold for finitely many values of n; cf. [25, Theorem 3.3].

We can now prove our main result on U-polygons which is an extension of [15, Theorem 4.5]. In fact, we use the same arguments as introduced by Gardner and Gritzmann in conjunction with Lemma 2.3 and Theorem 4.9. Note that the result even holds for arbitrary sets $\varLambda$ with property (Alg).

Theorem 5.6. *Let $\varLambda$ be an algebraic Delone set. Further, let $U \subset \mathbb{S}^1$ be a set of four or more pairwise nonparallel $\varLambda$-directions and suppose the existence of a U-polygon. Then the cross ratio of slopes of any four directions of U, arranged in order of increasing angle with the positive real axis, is an element of the set $\mathcal{C}(K_{\varLambda}^+)$. Moreover, $\mathcal{C}(K_{\varLambda}^+)$ is finite and* $\operatorname{card}(U)$ *is bounded above by a finite number $b_{\varLambda} \in \mathbb{N}$ that only depends on $\varLambda$.*

Proof. Let U be as in the assertion. By Fact 5.4, U consists of directions parallel to the edges of an affinely regular polygon. There is thus a linear automorphism Ψ of the complex plane such that

$$V = \left\{ \Psi(u)/|\Psi(u)| \,\middle|\, u \in U \right\}$$

is contained in a set of directions that are equally spaced in $\mathbb{S}^1$, i.e. the angle between each pair of adjacent directions is the same. Since the directions of U are pairwise nonparallel, we may assume that there is an $m \in \mathbb{N}$ with $m \geq 4$ such that each direction of V is given by $e^{h\pi i/m}$, where $h \in \mathbb{N}_0$ satisfies $h \leq m-1$. Let u_j,

$1 \leq j \leq 4$, be four directions of U, arranged in order of increasing angle with the positive real axis. By Lemma 2.3, one has

$$q = \big\langle \operatorname{sl}(u_1), \operatorname{sl}(u_2), \operatorname{sl}(u_3), \operatorname{sl}(u_4) \big\rangle \in K_\Lambda^+ .$$

We may assume that $\Psi(u_j)/|\Psi(u_j)| = e^{h_j \pi i/m}$, where $h_j \in \mathbb{N}_0$, $1 \leq j \leq 4$, and, $h_1 < h_2 < h_3 < h_4 \leq m-1$. Lemma 2.2 now implies

$$\begin{aligned} q &= \big\langle \operatorname{sl}(\Psi(u_1)), \operatorname{sl}(\Psi(u_2)), \operatorname{sl}(\Psi(u_3)), \operatorname{sl}(\Psi(u_4)) \big\rangle \\ &= \frac{(\tan(\frac{h_3\pi}{m}) - \tan(\frac{h_1\pi}{m}))(\tan(\frac{h_4\pi}{m}) - \tan(\frac{h_2\pi}{m}))}{(\tan(\frac{h_3\pi}{m}) - \tan(\frac{h_2\pi}{m}))(\tan(\frac{h_4\pi}{m}) - \tan(\frac{h_1\pi}{m}))} \\ &= \frac{\sin(\frac{(h_3-h_1)\pi}{m})\sin(\frac{(h_4-h_2)\pi}{m})}{\sin(\frac{(h_3-h_2)\pi}{m})\sin(\frac{(h_4-h_1)\pi}{m})} . \end{aligned}$$

Setting $k_1 = h_3 - h_1$, $k_2 = h_4 - h_2$, $k_3 = h_3 - h_2$ and $k_4 = h_4 - h_1$, one gets $1 \leq k_3 < k_1, k_2 < k_4 \leq m-1$ and $k_1+k_2 = k_3+k_4$. Using $\sin\theta = -e^{-i\theta}(1-e^{2i\theta})/2i$, one finally obtains

$$K_\Lambda^+ \ni q = \frac{(1-\zeta_m^{k_1})(1-\zeta_m^{k_2})}{(1-\zeta_m^{k_3})(1-\zeta_m^{k_4})} = f_m(d) ,$$

with $d = (k_1, k_2, k_3, k_4)$, as in (4.1). Then, $d \in D_m$ if its first two coordinates are interchanged, if necessary, to ensure that $k_1 \leq k_2$; note that this operation does not change the value of $f_m(d)$. This proves the first assertion.

Suppose that $\operatorname{card}(U) \geq 7$. Let U' consist of seven directions of U and let $V' = \{\Psi(u)/|\Psi(u)| \,|\, u \in U'\}$. We may assume that all the directions of V' are in the first two quadrants, so one of these quadrants, say the first, contains at least four directions of V'. Application of the above argument to these four directions gives integers h_j satisfying $0 \leq h_1 < h_2 < h_3 < h_4 \leq m/2$, where we may also assume, by rotating the directions of V' if necessary, that $h_1 = 0$. As above, we obtain a corresponding solution of $f_m(d) = q \in K_\Lambda^+$, where $d \in D_m$.

By property (Alg) and Theorem 4.9, the set $\mathcal{C}(K_\Lambda^+)$ is finite and there is a number $m_\Lambda \in \mathbb{N}$ such that all solutions of $f_m(d) \in K_\Lambda^+$, where $m \geq 4$ and $d \in D_m$, are either of the form (xii) or (xiii) of Theorem 4.3 or are given, up to multiplication of m and d by the same factor, by $m = m_\Lambda$ and d from a finite list. Without restriction, we may assume that m_Λ is even.

Suppose that the above solution is of the form (xii) or (xiii) of Theorem 4.3. Then using $h_1 = 0$, one obtains $h_4 = k_4 = k + s > m/2$, a contradiction. Thus, our solution derives from $m = m_\Lambda$ and finitely many values of $d \in D_m$. Since this applies to any four directions of V' lying in the first quadrant, all such directions correspond to angles with the positive real axis which are integer multiples of π/m_Λ.

We claim that all directions of V' have the latter property. To see this, suppose that there is a direction $v \in V'$ in the second quadrant, and consider a set of four directions v_j, $1 \leq j \leq 4$, in V', where $v_4 = v$ and v_j, $1 \leq j \leq 3$, lie in the first quadrant. Suppose that $v_j = e^{h_j\pi i/m}$, $1 \leq j \leq 4$. Then h_j is an integer multiple of m/m_Λ, for $1 \leq j \leq 3$. Again, we obtain a corresponding solution of $f_m(d) = q \in K_\Lambda^+$, where $d \in D_m$. If this solution derives from the finite list guaranteed by Theorem 4.9, then clearly h_4 is also an integer multiple of m/m_Λ. Otherwise, by Theorem 4.9, this solution is of the form (xii) or (xiii) of Theorem 4.3 and we can take $h_1 = 0$ as before, whence either $h_2 = k$, $h_3 = 2k$ and $h_4 = k + s$, $1 \leq k \leq s/2$, or $h_2 = s - k$, $h_3 = s$ and $h_4 = k + s$, $s/2 \leq k < s$, where $m = 2s$. Since $s = m/2 = (m_\Lambda/2)(m/m_\Lambda)$ is an integer multiple of m/m_Λ, we conclude in either case that k, and hence $h_4 = k + s$, is also an integer multiple of m/m_Λ. This proves the claim.

It thus remains to examine the case $m = m_\Lambda$ in more detail. Let h_j, $1 \le j \le 4$, correspond to the four directions of V' having the smallest angles with the positive real axis, so that $h_1 = 0$ and $h_j \le m/2$, $2 \le j \le 4$. We have already shown that the corresponding $d = (k_1, k_2, k_3, k_4)$ must occur in the finite list guaranteed by Theorem 4.9. Since $h_j \le m/2$, $1 \le j \le 4$, we also have $k_j \le m/2$, $1 \le j \le 4$. This yields only finitely many quadruples $(h_1, h_2, h_3, h_4) = (0, k_1 - k_3, k_1, k_4)$.

Suppose that h corresponds to any other direction of V' and replace (h_1, h_2, h_3, h_4) by (h_2, h_3, h_4, h). We obtain finitely many $d = (h_4 - h_2, h - h_3, h_4 - h_3, h - h_2) \in D_m$, which, by Theorem 4.9, either occur in (xii) or (xiii) of Theorem 4.3 with $m = m_\Lambda$ or occur in the finite list guaranteed by that result. This gives only finitely many possible finite sets of more than four directions, which implies that $\operatorname{card}(U)$ is bounded from above by a finite number that only depends on Λ (since the above analysis only depends on Λ). □

Similarly, the next result even holds for arbitrary sets Λ with property (n-Cyc), where $n \ge 3$.

Theorem 5.7. *Let $n \ge 3$ and let Λ be an n-cyclotomic Delone set. Further, let $U \subset \mathbb{S}^1$ be a set of four or more pairwise nonparallel Λ-directions and suppose the existence of a U-polygon. Then the cross ratio of slopes of any four directions of U, arranged in order of increasing angle with the positive real axis, is an element of the subset $\mathcal{C}(K_\Lambda^+)$ of $\mathcal{C}(\mathbb{Q}(\zeta_n)^+)$. Moreover*

$$\mathcal{C}(\mathbb{Q}(\zeta_n)^+) = \mathcal{C}_{\operatorname{lcm}(2n,12)}(\mathbb{Q}(\zeta_n)^+)$$

is finite and $\operatorname{card}(U)$ is bounded above by a finite number $b_n \in \mathbb{N}$ that only depends on n. In particular, one can choose $b_3 = b_4 = 6$, $b_5 = 10$, $b_8 = 8$ and $b_{12} = 12$.

Proof. Employing Theorem 4.7 together with the trivial observation that $K_\Lambda^+ \subset \mathbb{Q}(\zeta_n)^+$ for any n-cyclotomic Delone set, the general result follows from the same arguments as used in the proof of Theorem 5.6. The work of Gardner and Gritzmann shows that one can choose $b_3 = b_4 = 6$; cf. [15, Theorem 4.5]. The specific bounds b_n for $n = 5, 8, 12$ are obtained by following the proof of Theorem 5.6 and employing Corollary 4.10.

More precisely, let $n = 8$ (whence $\operatorname{lcm}(2n, 12) = 48$) and suppose that $\operatorname{card}(U) \ge 7$. Let U' consist of seven directions of U and let $V' = \{\Psi(u)/|\Psi(u)| \mid u \in U'\}$, with Ψ as described in the proof of Theorem 5.6. Then all directions of V' correspond to angles with the positive real axis which are integer multiples of $\pi/48$ and it suffices to examine the case $m = 48$ in more detail. Let h_j, $1 \le j \le 4$, correspond to the four directions of V' having the smallest angles with the positive real axis, so that $h_1 = 0$ and $h_j \le m/2 = 24$, $2 \le j \le 4$. The corresponding $d = (k_1, k_2, k_3, k_4)$ must occur in (1)-(82) of Corollary 4.10(b). Since $h_j \le 24$, $1 \le j \le 4$, we also have $k_j \le 24$, $1 \le j \le 4$. The only possibilities are (1), (3), (15), (19), (27), (28), (33), (34), (47), (67), (76) and (81) of Corollary 4.10(b). These yield

$$\begin{aligned}(h_1, h_2, h_3, h_4) \in \big\{&(0,2,6,20), (0,6,12,18), (0,2,4,12), (0,6,12,24),\\ &(0,2,10,24), (0,6,18,24), (0,6,8,14), (0,6,9,21),\\ &(0,8,10,20), (0,6,16,22), (0,8,16,24), (0,8,12,20)\big\}.\end{aligned}$$

Suppose that h corresponds to any other direction of V' and replace (h_1, h_2, h_3, h_4) by (h_2, h_3, h_4, h). The corresponding d either occur in (xii) or (xiii) of Theorem 4.3 with $m = 48$ or occur in (1)-(82) of Corollary 4.10(b). We obtain $(18, h-6, 14, h-2)$, $(12, h-12, 6, h-6)$, $(10, h-4, 8, h-2)$, $(18, h-12, 12, h-6)$, $(22, h-10, 14, h-2)$, $(18, h-18, 6, h-6)$, $(8, h-8, 6, h-6)$, $(15, h-9, 12, h-6)$, $(12, h-10, 10, h-8)$, $(16, h-16, 6, h-6)$, $(16, h-16, 8, h-8)$ and $(12, h-12, 8, h-8)$. The only possibilities are $h = 24, 30, 36, 42$ for $(12, h-12, 6, h-6)$, $h = 26, 40$ for $(10, h-4, 8, h-2)$,

$h = 30, 36, 42$ for $(18, h-12, 12, h-6)$, $h = 38, 46$ for $(22, h-10, 14, h-2)$, $h = 36, 42$ for $(18, h-18, 6, h-6)$, $h = 34$ for $(12, h-10, 10, h-8)$, $h = 32, 40$ for $(16, h-16, 8, h-8)$ and $h = 34$ for $(12, h-12, 8, h-8)$. It follows that the only possible sets of more than four directions only comprise directions of the form $e^{h\pi i/48}$ and are given by the ranges

$$\begin{gathered}\{0,8,16,24,32,40\}, \{0,8,12,20,34\}, \{0,6,12,18,24,30,36,42\},\\ \{0,2,4,12,26,40\}, \{0,6,12,24,30,36,42\}, \{0,2,10,24,38,46\},\\ \{0,6,18,24,36,42\}, \{0,8,10,20,34\}\end{gathered}$$

of h. In particular, $\mathrm{card}(U) \leq 8$.

With the help of Corollary 4.10, the cases $n = 5, 12$ can be treated analogously with the following results.

For $n = 12$, the only possible sets of more than four directions only comprise directions of the form $e^{h\pi i/24}$ and are given by the ranges

$$\begin{gathered}\{0,4,8,12,16,18,20,22\}, \{0,4,6,10,14,16,18,20,22\},\\ \{0,2,4,10,12,14,18,20,22\}, \{0,2,4,8,12,14,16,18,20,22\},\\ \{0,2,4,6,8,10,12,14,16,18,20,22\}, \{0,2,6,12,16,18,20,22\},\\ \{0,2,4,12,14,20,22\}, \{0,4,6,12,14,16,18,20,22\}, \{0,2,8,12,18,20,22\},\\ \{0,2,6,10,14,16,18,20,22\}, \{0,2,8,10,16,18,20,22\}, \{0,2,10,12,20,22\}\end{gathered}$$

of h, whence $\mathrm{card}(U) \leq 12$.

For $n = 5$, the only possible sets of more than four directions only comprise directions of the form $e^{h\pi i/60}$ and are given by the ranges

$$\begin{gathered}\{0,10,20,30,40,50\}, \{0,6,24,30,48,54\}, \{0,2,4,10,32,54\},\\ \{0,4,8,18,34,50\}, \{0,6,10,16,38\}, \{0,6,12,24,30,36,42,48,54\},\\ \{0,10,18,28,44\}, \{0,12,18,30,36,42,48,54\}, \{0,6,8,16,38\},\\ \{0,10,14,28,34\}, \{0,2,8,30,52,58\}, \{0,4,14,30,46,56\},\\ \{0,6,18,30,42,48,54\}, \{0,6,12,30,36,42,48,54\},\\ \{0,6,12,18,24,30,36,42,48,54\}, \{0,6,18,24,36,42,48,54\}\end{gathered}$$

of h, whence $\mathrm{card}(U) \leq 10$ in this case. □

Without further mention, the following result will be used in Remark 5.9 below.

Lemma 5.8. *Let Λ be a K-algebraic model set and let $U \subset \mathbb{S}^1$ be a finite set of directions. The following statements are equivalent:*

(i) *There is a U-polygon in Λ.*
(ii) *For any K-algebraic model set Λ', there is a U-polygon in Λ'.*

Proof. The assertion follows from Proposition 3.9 together with [28, Fact 4.4]. □

Remark 5.9. The work of Gardner and Gritzmann shows that $b_3 = b_4 = 6$ is best possible for any 3- or 4-cyclotomic model set; cf. [15, Example 4.3]. The U-icosagon in the vertex set of the Tübingen triangle tiling from Remark 5.5 has the property that $\mathrm{card}(U) = 10$; see [27, Figure 1]. This shows that, for any 5-cyclotomic model set, the number $b_5 = 10$ is best possible. Figure 1 shows a U-polygon with 24 vertices in the vertex set of the shield tiling with $\mathrm{card}(U) = 12$, hence $b_{12} = 12$ is best possible for any 12-cyclotomic model set. A similar example of a U-polygon with 16 vertices in the vertex set of the Ammann-Beenker tiling with $\mathrm{card}(U) = 8$ shows that $b_8 = 8$ is best possible for any 8-cyclotomic model set; cf. [28, Figure 2]. U-polygons of *class* $c \geq 4$ (i.e. U-polygons with 4 *consecutive edges* parallel to directions of U) in cyclotomic model sets were studied in [27]. By [27, Corollary

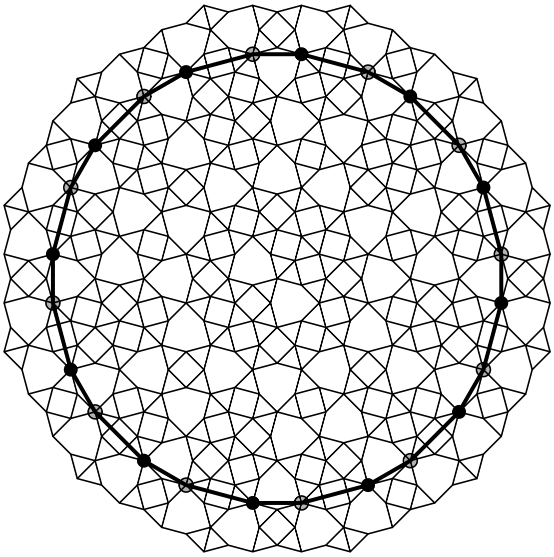

FIGURE 1. The boundary of a U-polygon in the vertex set $\varLambda$ of the twelvefold shield tiling, where U is the set of twelve pairwise nonparallel $\varLambda$-directions given by the edges and diagonals of the central regular dodecagon. The vertices of $\varLambda$ in the interior of the U-polygon together with the vertices indicated by the black and grey dots, respectively, give two different convex subsets of $\varLambda$ with the same X-rays in the directions of U.

14] (see also [13, Theorem 12]), the existence of a U-polygon of class $c \geq 4$ in an n-cyclotomic model set with $n \not\equiv 2 \pmod 4$ having the property that $\phi(n)/2$ is equal to one or a prime number implies that $\operatorname{card}(U) \leq a_n$, where $a_3 = a_4 = 6$, $a_8 = 8$, $a_{12} = 12$ and $a_n = 2n$ for all other such values of n. In particular, one observes the coincidence $b_n = a_n$ for $n = 3, 4, 5, 8, 12$; cf. Theorem 5.7. However, there does not seem to be a reason why the least possible numbers b_n in Theorem 5.7 may not be larger than a_n for other $n \geq 3$ having the above property.

Summing up, we finally obtain our main result on the determination of convex subsets of algebraic Delone sets; see [28, Theorem 4.21] for a weaker version.

Theorem 5.10. *Let $\varLambda$ be an algebraic Delone set.*

(a) *There are sets of four pairwise nonparallel $\varLambda$-directions such that the convex subsets of $\varLambda$ are determined by the corresponding X-rays. In addition, less than four pairwise nonparallel $\varLambda$-directions never suffice for this purpose.*
(b) *There is a finite number $c_{\varLambda} \in \mathbb{N}$ such that the convex subsets of $\varLambda$ are determined by the X-rays in any set of $c_{\varLambda}$ pairwise nonparallel $\varLambda$-directions.*

Proof. To prove (a), it suffices by Fact 5.3 and Theorem 5.6 to take any set of four pairwise nonparallel $\varLambda$-directions such that the cross ratio of their slopes, arranged in order of increasing angle with the positive real axis, is not an element of the finite set $\mathcal{C}(K_{\varLambda}^{+})$. Since $\varLambda$ is relatively dense, the set of $\varLambda$-directions is dense in $\mathbb{S}^1$. In particular, this shows that the set of slopes of $\varLambda$-directions is infinite. For example by fixing three pairwise nonparallel $\varLambda$-directions and letting the fourth one vary, one sees from this that the set of cross ratios of slopes of four pairwise nonparallel $\varLambda$-directions, arranged in order of increasing angle with the positive real axis, is infinite as well. The assertion follows. The additional statement follows immediately from Fact 5.3. Part (b) is a direct consequence of Fact 5.3 and Theorem 5.6. □

The following result improves [28, Theorem 4.33] and in particular solves Problem 4.34 of [28]; cf. Example 3.11 and compare [15, Theorem 5.7].

Theorem 5.11. *Let $n \geq 3$ and let Λ be an n-cyclotomic Delone set.*

(a) *There are sets of four pairwise nonparallel Λ-directions such that the convex subsets of Λ are determined by the corresponding X-rays. In addition, less than four pairwise nonparallel Λ-directions never suffice for this purpose.*
(b) *There is a finite number $c_n \in \mathbb{N}$ that only depends on n such that the convex subsets of Λ are determined by the X-rays in any set of c_n pairwise nonparallel Λ-directions. In particular, one can choose $c_3 = c_4 = 7$, $c_5 = 11$, $c_8 = 9$ and $c_{12} = 13$.*

Proof. Part (a) follows immediately from Theorem 5.10(a). Note that, by Fact 5.3 and Theorem 5.7, it suffices to take any set of four pairwise nonparallel Λ-directions such that the cross ratio of their slopes, arranged in order of increasing angle with the positive real axis, is not an element of the finite set $\mathcal{C}(\mathbb{Q}(\zeta_n)^+)$. Part (b) is a direct consequence of Fact 5.3 in conjunction with Theorem 5.7. □

Remark 5.12. Remark 5.9 shows that, for any n-cyclotomic model set with $n = 3, 4, 5, 8, 12$, the number c_n above is best possible with respect to the numbers of X-rays used. As already explained in the introduction, for practical applications, one additionally has to make sure that the Λ-directions used yield densely occupied lines in Λ. For the practically most relevant case of n-cyclotomic model sets with $n = 3, 4, 5, 8, 12$, this can actually be achieved; cf. [15, Remark 5.8] and [28, Section 4] for examples of suitable sets of four pairwise nonparallel Λ-directions in these cases. For the latter examples also recall that, for any n-cyclotomic model set Λ, the set of Λ-directions is precisely the set of $\mathbb{Z}[\zeta_n]$-directions; cf. Remark 3.10 and Example 3.11. It was shown in [26, Proposition 3.11] that *icosahedral model sets* $\Lambda \subset \mathbb{R}^3$ can be sliced orthogonal to a fivefold axis of their underlying $\mathbb{Z}$-module into 5-cyclotomic model sets. Applying Theorem 5.11 to each such slice, one sees that the convex subsets of Λ are determined by the X-rays in suitable four and any eleven pairwise nonparallel Λ-directions orthogonal to the slicing axis.

6. Determination of convex bodies by continuous X-rays

In [18], the following continuous version of Fact 5.3 was shown; compare Fact 5.4. Here, the *continuous X-ray* of a *convex body* $K \subset \mathbb{C}$ (i.e. K is convex and compact with nonempty interior) in direction $u \in \mathbb{S}^1$ gives the length of each chord of K parallel to u and the concept of determination is defined as in the discrete case; cf. [14], [18] for details.

Fact 6.1. *Let $U \subset \mathbb{S}^1$ be a set of two or more pairwise nonparallel directions. The following statements are equivalent:*

(i) *The convex bodies in $\mathbb{C}$ are determined by the continuous X-rays in the directions of U.*
(ii) *There is no U-polygon.*

In addition, if $\mathrm{card}(U) < 4$*, then there is a U-polygon.* □

Employing Fact 6.1 instead of Fact 5.3, the following result follows from the same arguments as used in the proofs of Theorems 5.10 and 5.11; compare [15, Theorem 6.2]. Note that neither the uniform discreteness of Λ nor property (Hom) are needed in the proof. More precisely, our proof of part (a) needs property (Alg) and the relative denseness of Λ, whereas part (b) and the additional statement hold for arbitrary sets Λ with property (Alg) and (n-Cyc) (where $n \geq 3$), respectively.

Theorem 6.2. *Let Λ be an algebraic Delone set.*

(a) *There are sets of four pairwise nonparallel Λ-directions such that the convex bodies in $\mathbb{C}$ are determined by the corresponding continuous X-rays. In addition, less than four pairwise nonparallel Λ-directions never suffice for this purpose.*

(b) *There is a finite number $c_\Lambda \in \mathbb{N}$ such that the convex bodies in $\mathbb{C}$ are determined by the continuous X-rays in any set of c_Λ pairwise nonparallel Λ-directions.*

Moreover, for any n-cyclotomic Delone set Λ, there is a finite number $c_n \in \mathbb{N}$ that only depends on n such that the convex bodies in $\mathbb{C}$ are determined by the continuous X-rays in any set of c_n pairwise nonparallel Λ-directions. In particular, one can choose $c_3 = c_4 = 7$, $c_5 = 11$, $c_8 = 9$ and $c_{12} = 13$. □

Remark 6.3. Employing the U-polygons from Remark 5.9, it is straightforward to show that the above numbers c_n, where $n = 3, 4, 5, 8, 12$, are best possible.

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