INTRODUCTION to KNOTS & THEIR fundamental GROUPS

Def: A knot is a continuous embedding $S^1 \to \mathbb{R}^3$

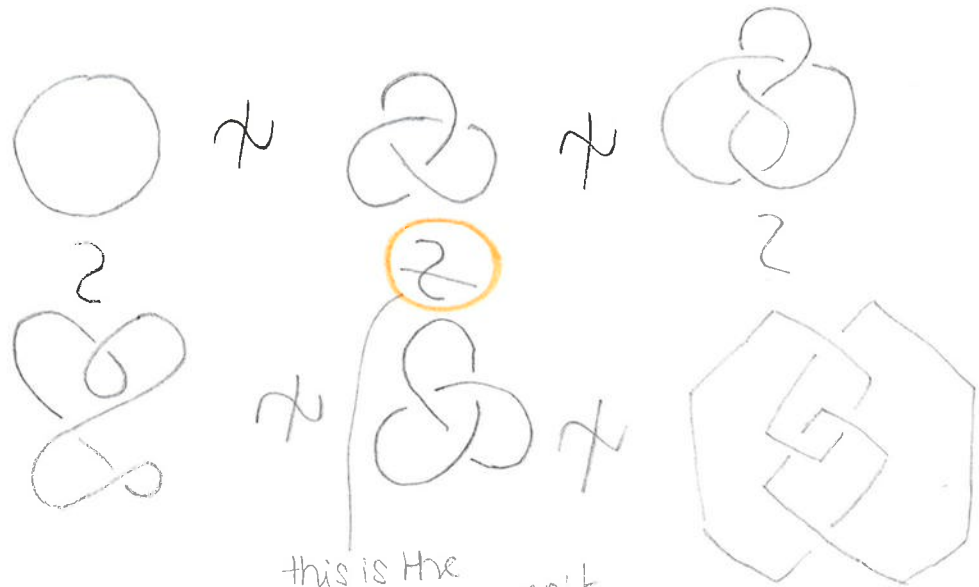
Rmk: K will denote a knot (sometimes its image)

Def: Two knots K, K' are equivalent if they are isotopic, i.e. $\exists$ a continuous map $F: [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ st

- $F(0,\cdot)$ is the identity map
- $F(t,\cdot)$ is a homeomorphism $\forall t$
- $F(1,\cdot) \circ K = K'$

Notation: We write $K \sim K'$ — Saying knot we will refer to its eq. class

Example:



this is the only one we won't be able to show at the end of today

Def: A polygonal knot is a knot whose image is a finite union of segments.
Knots equivalent to polygonal knots are called tame; the others are called wild.

From now on : ALL KNOTS will be TAME !

We now see how to draw a
KNOT DIAGRAM

Let K be a knot & denote $C = \operatorname{Im} K$

Def : Let $s : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear projection
A point $P \in s(C)$ is double if $\# s^{-1}(P) = 2$.

Def : A linear projection $s : \mathbb{R}^3 \to \mathbb{R}^2$ is regular (for K) if

1. $D = \bigcup_{\substack{P \in s(C) \\ \text{double}}} s^{-1}(P)$ is finite
2. $s : C \setminus D \to \mathbb{R}^2$ is injective
3. $\forall P$ double : $(s \circ K)^{-1}(P) = \{A, B\}$ and $s \circ K$ is differentiable at A, B st derivatives are linearly independent.

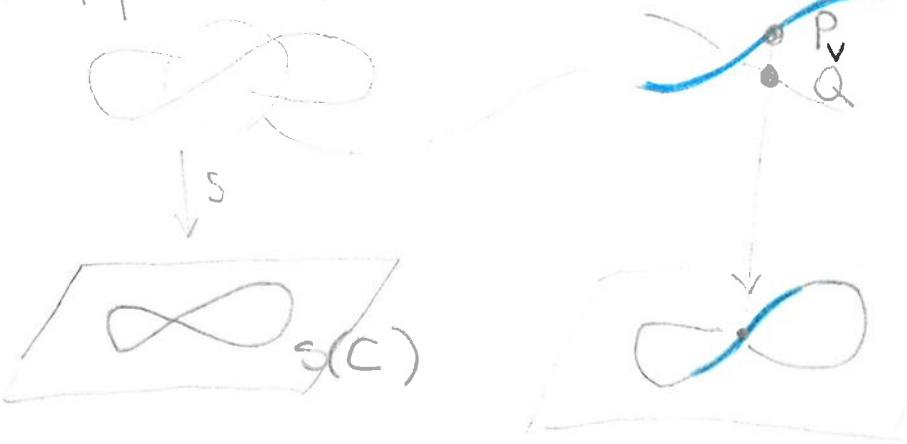
Let $s : \mathbb{R}^3 \to \mathbb{R}^2$ be regular.
Let now $\mathcal{B}$ be a positively oriented basis of $\mathbb{R}^2$.
Take then $\mathcal{C} = (u, v, w)$ a positively oriented basis of $\mathbb{R}^3$
st $s(u, v) = \mathcal{B}$ and $s(w) = 0$.
We get a total ordering $<$ on the double fibers of $s|_C$, that is :

if $P \neq Q \in C$ & $s(P) = s(Q)$, then
$P < Q$ if and only if $\pi_w(P) < \pi_w(Q)$.

Call such ordering an s-ordering on K & the pair $(s, <)$ a regular pair.

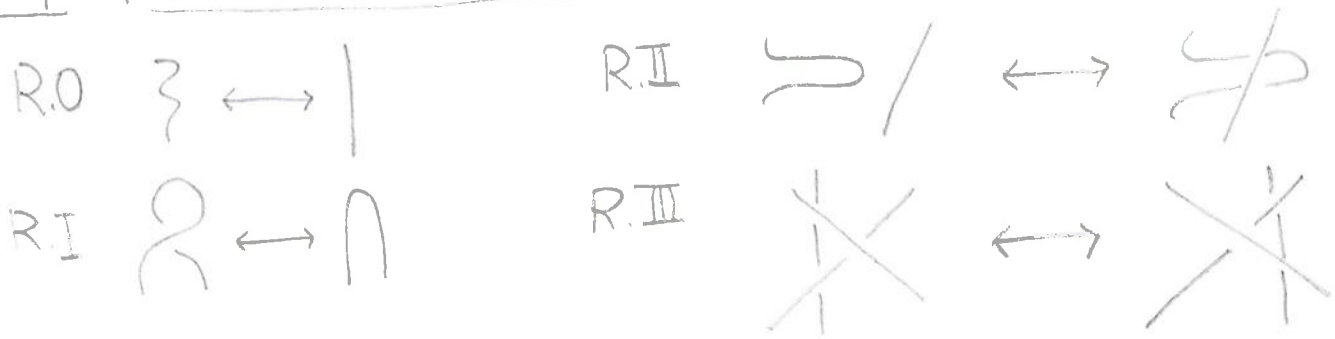
A knot diagram is a representation of K through a regular pair, in the following sense

Modify $s(C)$, to remember $<$



THE IDEA is: we want to be able to recover the eq. class of the knot from its knot diagram

Def: (Reidemeister moves)



fact: $K \sim K' \iff K \longleftrightarrow K'$ (knot diagrams!) via Reidemeister moves

QS: We know now how to show two knots are equivalent, but how to show they are not?

↓

Knot Invariants

- crossing / bridge number
- tricolorability
- knot group
- knot polynomials: Alexander (1928), Conway (1969), Jones (1984), Homfly ...
- knot genus

etc ...

THE KNOT GROUP

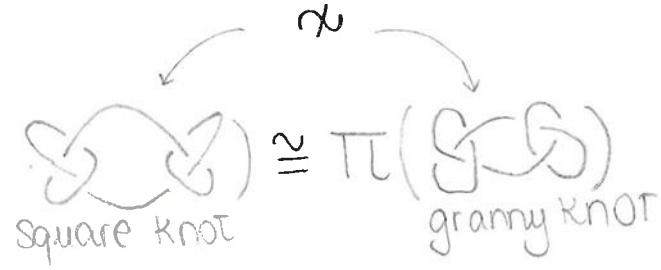
Obs: If K is a knot, then $\mathbb{R}^3 \setminus K$ is path-connected.
It follows that, $\forall P, Q \notin K$: $\pi_1(\mathbb{R}^3 \setminus K, P) \cong \pi_1(\mathbb{R}^3 \setminus K, Q)$.

Def: The knot group of K is $\pi(K) = \pi_1(\mathbb{R}^3 \setminus K, P)$, for $P \notin K$.

up to isomorphism!!

RMK: $K \sim K' \Rightarrow \pi(K) \cong \pi(K')$

the way back does not hold:



other ex: trefoil & its mirror image.

Very vague warning: the knot group does not determine the knot uniquely, but the "peripheral sbgp" does!
$\rightarrow$ Waldhausen ('60s)

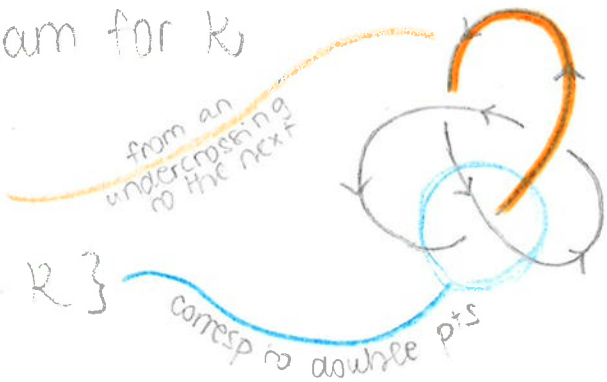
the WIRTINGER PRESENTATION

We see a way of computing a presentation of $\pi(K)$, the knot group of a knot K

⚠ This is not standard notation, some of it is inspired from Lipov's notes

We choose a knot diagram for K & an orientation

Call $A = \{\text{arcs of } K\}$
$C = \{\text{crossings of } K\}$



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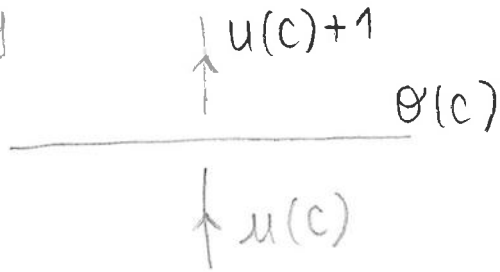
Define maps $\theta, u : C \longrightarrow A$ by



and call a crossing

- <u>right-handed</u> if
- <u>left-handed</u> if

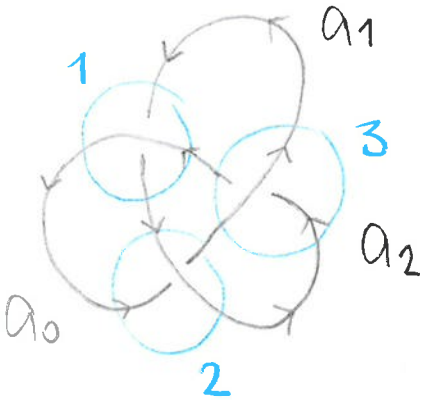
To conclude, define a map $h : C \longrightarrow \{\pm 1\}$

$$RH \longmapsto +1$$
$$LH \longmapsto -1$$

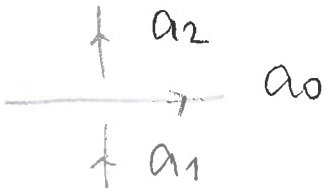
<u>Proposition</u>: Let K be a knot, then

$$\pi(K) = \langle a \in A \mid \forall c : u(c)+1 = \theta(c)^{h(c)} u(c)\, \theta(c)^{-h(c)} \rangle$$

<u>Example</u>:



all crossings are right!

<u>crossing</u> 1 : 

$$a_2 = a_0 a_1 a_0^{-1}$$

<u>crossing</u> 2 : $a_1 = a_2 a_0 a_2^{-1}$

<u>crossing</u> 3 : $a_0 = a_1 a_2 a_1^{-1}$

<u>Check</u>: one can derive 3 from 1 & 2 (& this will always be the case: we can always drop one relation)

+ <u>simplifying</u>:

$$G = \langle a, b \mid aba = bab \rangle \quad \left(\text{take } a = a_0,\ b = a_1\right)$$

$$\Big\downarrow \; x = ba,\ y = aba \qquad \Big\uparrow \; a = yx^{-1},\ b = x^2 y^{-1}$$

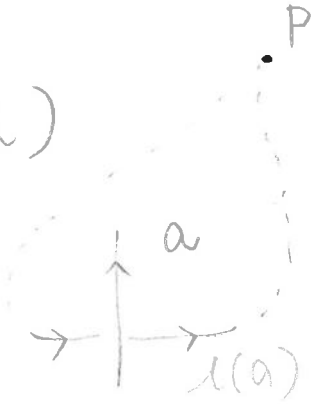
$$= \langle x, y \mid y^2 = x^3 \rangle \quad \leftsquigarrow \text{Torus knots!}$$

Sketch of Proof : (Proposition)

Define a map

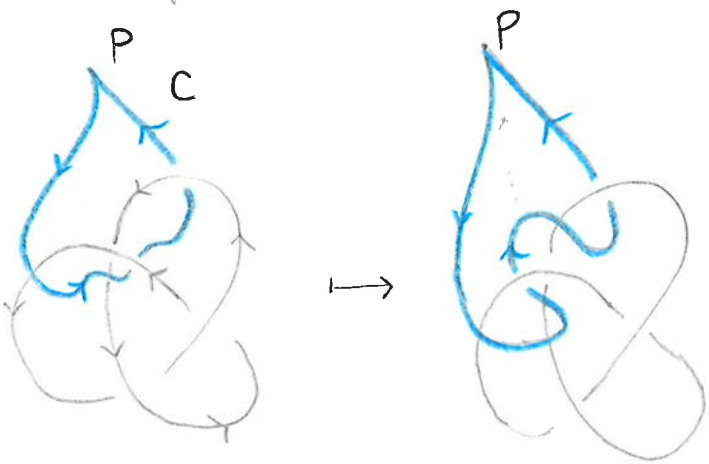
$$A \xrightarrow{l} \pi(K)$$
$$a \longmapsto l(a)$$

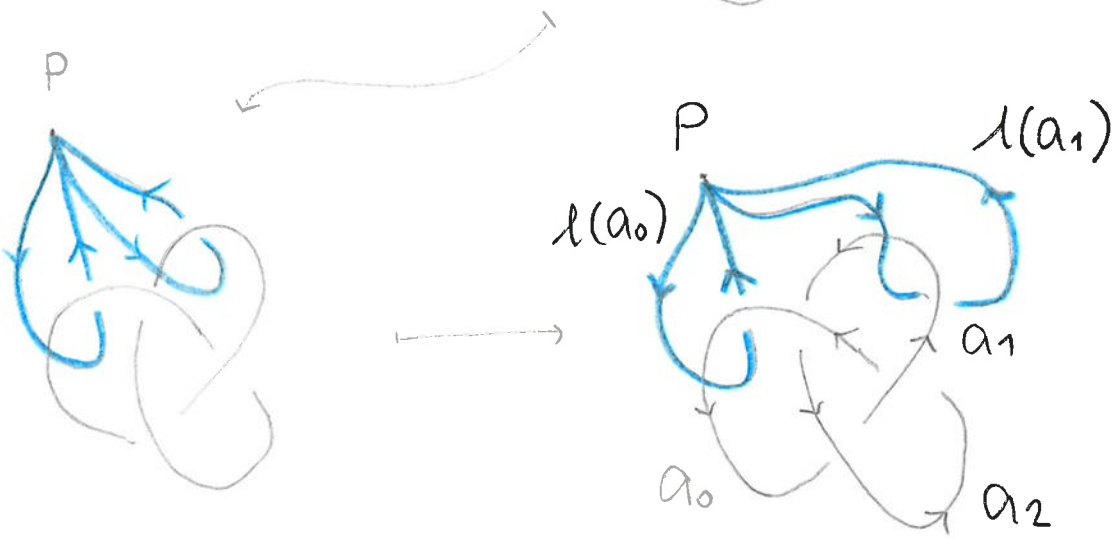
St



Obs ① : The $l(a)$'s generate $\pi(K)$

eg :





So $c = l(a_0) \cdot l(a_1)$

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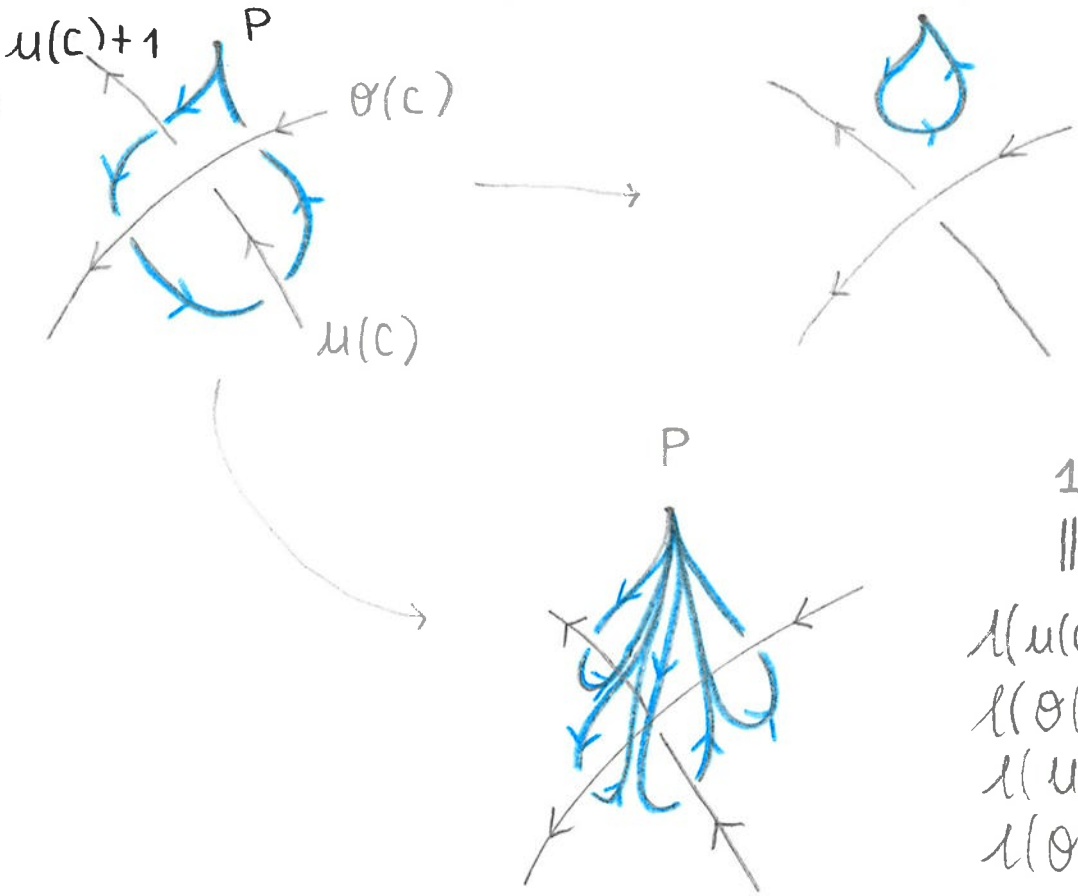
Obs ②: The relations come from

Left:



$$1 = l(u(c)+1)^{-1} \cdot l(\theta(c))^{-1} \cdot l(u(c)) \cdot l(\theta(c))$$

Same for right!

$\square$

Lemma: Let K be a knot & $G = \pi_1(K)$
Then $G^{ab} \cong \mathbb{Z}$.

Proof:

① G^{ab} is cyclic, since $\forall c \in C$:

$$u(c)+1 = [\theta(c)^{h(c)}, u(c)]\, u(c) \equiv u(c) \bmod [G,G]$$

So all $a \in A$ are sent to the same (generating!) element, say t.

② We show that t has ∞ order

Let F = free gp generated by A & let

$$\tilde{\varphi}: A \longrightarrow \mathbb{Z}$$
$$a \longmapsto 1$$

By the UP of free gps $\tilde{\varphi}$ extends to a homomorphism

$$\begin{tikzcd} \varphi : F \arrow[r] & \mathbb{Z} \\ A \arrow[u, hook] \arrow[ur, bend right, "\tilde{\varphi}"'] & \end{tikzcd}$$

and φ is surjective since $\mathbb{Z} = \langle 1 \rangle$.
Now, $\forall c \in C$: $u(c) + 1 = \theta(c)^{h(c)} u(c) h^{-1}(c)$ and so φ factors as

$$\begin{tikzcd} F \arrow[r, two heads] \arrow[d, two heads] & \mathbb{Z} \\ G \arrow[ur] \arrow[dr, two heads] & \\ & G^{ab} \arrow[uu, two heads] \end{tikzcd}$$

and $\mathbb{Z}$ being abelian

As G^{ab} surjects to an infinite cyclic group, t has infinite order & $G^{ab} \cong \mathbb{Z}$. $\square$

The last result is important for computing the Alexander Matrix of a knot.

THE ALEXANDER POLYNOMIAL

<u>Def</u>: A <u>derivative</u> on a group algebra $\mathbb{Z}G$ is a $\mathbb{Z}$-linear map $D: \mathbb{Z}G \longrightarrow \mathbb{Z}G$ such that $\forall x, y \in G$: $D(xy) = D(x) + xD(y)$.

<u>fact</u>: Let F be a free gp on X
Then $\forall x \in X$ $\exists$ a derivation $\partial/\partial x$ such that
$\forall y \in X$: $\dfrac{\partial y}{\partial x} = \delta_{xy}$

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Construction of the Alexander matrix

again up to some equivalence relation! (see Fox, Crowell for example)

Let K be a knot, $G = \langle x_j \in X \mid r_i = s_i \rangle$
F = free gp on X
then we have maps

$$\begin{tikzcd} \mathbb{Z}F \arrow[r, "\partial/\partial x_j"] & \mathbb{Z}F \arrow[r, two heads] \arrow[rr, bend left, "\pi"] & \mathbb{Z}G \arrow[r, two heads] & \mathbb{Z}G^{ab} = \mathbb{Z}\langle t \rangle \\ & & a_i \arrow[r, mapsto] & t \end{tikzcd}$$

<u>Def</u>: The <u>Alexander matrix</u> associated to the presentation $\langle x_j \mid r_i = s_i \rangle$ is

$$M = \left(\pi \left(\frac{\partial (r_i - s_i)}{\partial x_j} \right) \right)_{i,j}$$

<u>Example</u>: (Trefoil)
We had two presentations

$$G = \langle a, b \mid aba = bab \rangle \quad (W)$$
$$\overset{|}{=} \langle x, y \mid x^3 = y^2 \rangle \quad (T)$$

<u>matrix of (W)</u>: $r_1 - s_1 = aba - bab$

$$\frac{\partial (aba - bab)}{\partial a} = 1 + ab - b \overset{\pi}{\longmapsto} 1 + t^2 - t$$

$$\frac{\partial (aba - bab)}{\partial b} = a - 1 - ba \longmapsto t - 1 - t^2$$

So $M_W = (1 + t^2 - t \quad -1 - t^2 + t)$

<u>matrix of (T)</u>: $r_1 - s_1 = x^3 - y^2$

$$\frac{\partial (x^3 - y^2)}{\partial x} = 1 + x + x^2, \quad \frac{\partial (x^3 - y^2)}{\partial y} = -1 - y$$

Careful!: $x = ba \xmapsto{\pi} t^2$, $y = aba \xmapsto{\pi} t^3$

So $M_T = (1 + t^2 + t^4 \quad -1 - t^3)$

Def: Let K be a knot, $G = \pi_1(K) = \langle x_j \mid r_i - s_i \rangle$

$n = \#\{x_j\}$

The Alexander polynomial associated to the presentation is

$$\Delta_K(t) = \gcd\left\{ \begin{array}{l} \text{determinants of } (n-1)\times(n-1) \\ \text{submatrices of Alexander} \end{array} \right\}$$

rmk: $\Delta_0(t) = 1$

facts: ① Alexander polynomials associated to equivalent presentations differ by an element of $(\mathbb{Z}G^{ab})^\times = \pm\langle t \rangle$

② $K \sim K' \Rightarrow \Delta_K(t) \equiv \Delta_{K'}(t) \mod \pm\langle t \rangle$

Main reference: Crowell, Fox: "Introduction to Knot theory"

Many thanks to: all the people who contributed in adjusting the definitions of regular position & knot diagram.