INTENSE AUTOMORPHISMS OF FINITE GROUPS

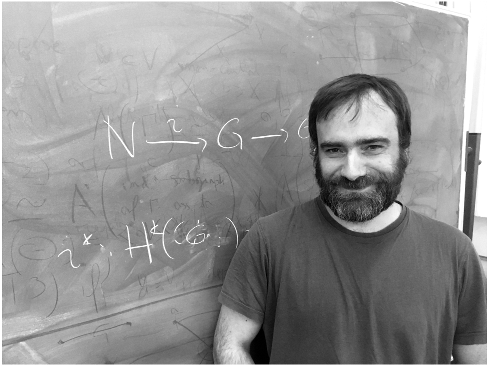
Mima Stanojkovski

Topics on Groups and their Representations
Gargnano sul Garda, 11th October 2017

Joint work with

Hendrik Lenstra (UL)

Andrea Lucchini (UniPD)



Jon González Sánchez (EHU)

Intense automorphisms of groups

Intense automorphisms

Let G be a finite group. An automorphism α of G is **intense** if for all $H \leq G$ there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Write $\alpha \in \mathrm{Int}(G)$.

Intense automorphisms

Let G be a finite group. An automorphism α of G is **intense** if for all $H \leq G$ there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Write $\alpha \in \mathrm{Int}(G)$.

Motivation: Intense automorphisms appear naturally as solutions to a certain cohomological problem. They (surprisingly!) give rise to a very rich theory.

Intense automorphisms

Let G be a finite group. An automorphism α of G is **intense** if for all $H \leq G$ there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Write $\alpha \in \mathrm{Int}(G)$.

Motivation: Intense automorphisms appear naturally as solutions to a certain cohomological problem. They (surprisingly!) give rise to a very rich theory.

Example:

- Every automorphism of a cyclic group is intense.
- Inner automorphisms are intense.
- Power automorphisms are intense.

Intensity

Let p be a prime number and let G be a finite p-group.

Intensity

Let p be a prime number and let G be a finite p-group. Then

$$\mathrm{Int}(G) \cong P \rtimes C$$

where

- P is a p-group.
- C is a subgroup of $\mathbb{F}_p^*$.

Intensity

Let p be a prime number and let G be a finite p-group. Then

$$\mathrm{Int}(G) \cong P \rtimes C$$

where

- P is a p-group.
- C is a subgroup of $\mathbb{F}_p^*$.

The **intensity** of G is $\mathrm{int}(G) = |C|$.

The problem

Can we classify all p-groups G satisfying $\mathrm{int}(G) > 1$?

The problem

Can we classify all p-groups G satisfying $\mathrm{int}(G) > 1$?

YES!

Strategy

Let p be a prime number and let G be a finite p-group. Let N be a normal subgroup.

Strategy

Let p be a prime number and let G be a finite p-group. Let N be a normal subgroup. Then we have natural maps

1. $\mathrm{Int}(G) \to \mathrm{Aut}(N)$,
2. $\mathrm{Int}(G) \to \mathrm{Int}(G/N)$,

Strategy

Let p be a prime number and let G be a finite p-group. Let N be a normal subgroup. Then we have natural maps

1. $\mathrm{Int}(G) \to \mathrm{Aut}(N)$,
2. $\mathrm{Int}(G) \to \mathrm{Int}(G/N)$,

and with a little extra work

3. if $N \neq G$, then $\mathrm{int}(G)$ divides $\mathrm{int}(G/N)$.

Strategy

Let p be a prime number and let G be a finite p-group. Let N be a normal subgroup. Then we have natural maps

1. $\mathrm{Int}(G) \to \mathrm{Aut}(N)$,
2. $\mathrm{Int}(G) \to \mathrm{Int}(G/N)$,

and with a little extra work

3. if $N \neq G$, then $\mathrm{int}(G)$ divides $\mathrm{int}(G/N)$.

Since we want G to have $\mathrm{int}(G) > 1$, we can forget about $p = 2$!!

Abelian groups

Let p be a prime number and let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let G be a finite abelian p-group.

Abelian groups

Let p be a prime number and let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let G be a finite abelian p-group. Then

$$\begin{tikzcd}
\mathbb{Z}_p^* \arrow[r] \arrow[d, dash] & \mathrm{Aut}(G) \arrow[d, dash] \\
\mathbb{F}_p^* \arrow[r] \arrow[ur] & \mathrm{Int}(G)
\end{tikzcd}$$

Abelian groups

Let p be a prime number and let $\mathbb{Z}_p$ denote the ring of p-adic integers. Let G be a finite abelian p-group. Then

$$\begin{tikzcd}
\mathbb{Z}_p^* \arrow[r] \arrow[d, dash] & \mathrm{Aut}(G) \arrow[d, dash] \\
\mathbb{F}_p^* \arrow[r] \arrow[ur] & \mathrm{Int}(G)
\end{tikzcd}$$

Theorem

Let p be a prime number and let $G \neq 1$ be a finite abelian p-group. Then $\mathrm{int}(G) = p - 1$.

Class 2

Let p be an odd prime and let G be an extraspecial group of exponent p.

Class 2

Let p be an odd prime and let G be an extraspecial group of exponent p. Then, for $\lambda \in \mathbb{Z}_p^*$, we have

$$\begin{tikzcd}
G/\gamma_2(G) \times G/\gamma_2(G) \arrow[r] \arrow[d, shift right=3, "\lambda"'] \arrow[d, shift left=3, "\lambda"] & \gamma_2(G) \arrow[d, "\lambda^2"] \\
G/\gamma_2(G) \times G/\gamma_2(G) \arrow[r] & \gamma_2(G)
\end{tikzcd}$$

Class 2

Let p be an odd prime and let G be an extraspecial group of exponent p. Then, for $\lambda \in \mathbb{Z}_p^*$, we have

$$\begin{tikzcd} G/\gamma_2(G) \times G/\gamma_2(G) \arrow[r] \arrow[d, shift right=3ex, "\lambda"'] \arrow[d, shift left=3ex, "\lambda"] & \gamma_2(G) \arrow[d, "\lambda^2"] \\ G/\gamma_2(G) \times G/\gamma_2(G) \arrow[r] & \gamma_2(G) \end{tikzcd}$$

Theorem

Let p be a prime number and let G be a finite p-group of class 2. Then $\mathrm{int}(G) > 1$ if and only if G is extraspecial of exponent p (in which case $\mathrm{int}(G) = p - 1$).

Class 3

Theorem

Let p be an odd prime and let G be a finite p-group of class 3. Then the following are equivalent.

1. *One has* $\mathrm{int}(G) > 1$.
2. *One has* $|G : \gamma_2(G)| = p^2$.

Class 3

Theorem

Let p be an odd prime and let G be a finite p-group of class 3. Then the following are equivalent.

1. *One has* $\mathrm{int}(G) > 1$.
2. *One has* $|G : \gamma_2(G)| = p^2$.
3. *One has* $\mathrm{int}(G) = 2$.

Class 3

Theorem

Let p be an odd prime and let G be a finite p-group of class 3. Then the following are equivalent.

1. *One has* $\mathrm{int}(G) > 1$.
2. *One has* $|G : \gamma_2(G)| = p^2$.
3. *One has* $\mathrm{int}(G) = 2$.

Corollary

Let p be a prime number and let $c \in \mathbb{Z}_{\geq 3}$. Then there exist, up to isomorphism, only finitely many finite p-groups of class c and intensity greater than 1.

Necessary conditions

Let p be a prime number and let G be a finite p-group of class $c \geq 3$. Define

$$w_i = \log_p |\gamma_i(G) : \gamma_{i+1}(G)|.$$

Necessary conditions

Let p be a prime number and let G be a finite p-group of class $c \geq 3$. Define

$$w_i = \log_p |\gamma_i(G) : \gamma_{i+1}(G)|.$$

If $1 \neq \alpha \in \mathrm{Int}(G)$ has order coprime to p, then the following hold.

Necessary conditions

Let p be a prime number and let G be a finite p-group of class $c \geq 3$. Define

$$w_i = \log_p |\gamma_i(G) : \gamma_{i+1}(G)|.$$

If $1 \neq \alpha \in \mathrm{Int}(G)$ has order coprime to p, then the following hold.

- $|\alpha| = 2$ and $\mathrm{int}(G) = 2$.

Necessary conditions

Let p be a prime number and let G be a finite p-group of class $c \geq 3$. Define

$$w_i = \log_p |\gamma_i(G) : \gamma_{i+1}(G)|.$$

If $1 \neq \alpha \in \mathrm{Int}(G)$ has order coprime to p, then the following hold.

- $|\alpha| = 2$ and $\mathrm{int}(G) = 2$.
- $\gamma_i(G)/\gamma_{i+1}(G)$ is elementary abelian and $\alpha \equiv (-1)^i$ on it.

Necessary conditions

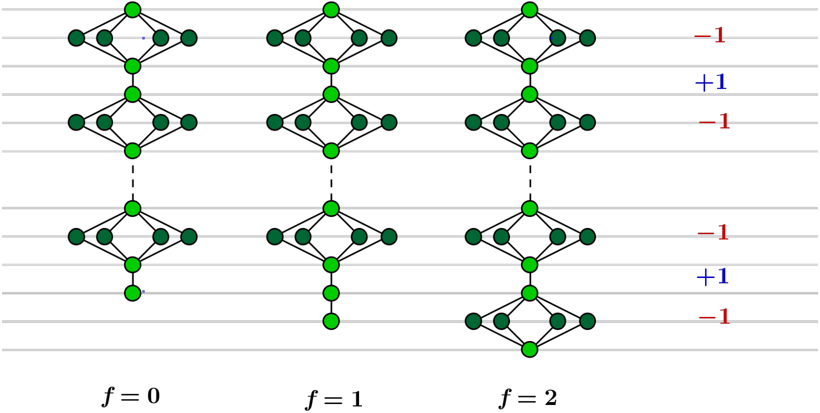
Let p be a prime number and let G be a finite p-group of class $c \geq 3$. Define

$$w_i = \log_p |\gamma_i(G) : \gamma_{i+1}(G)|.$$

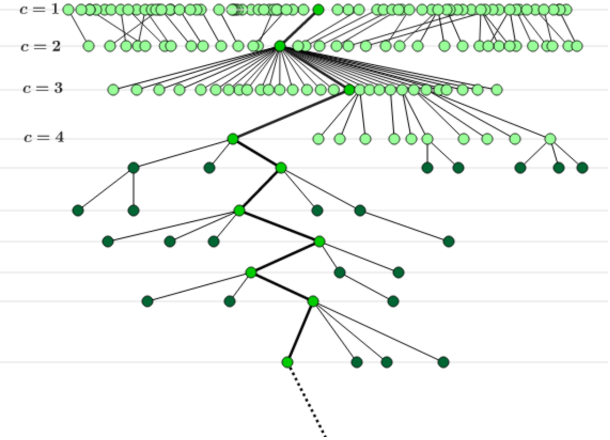
If $1 \neq \alpha \in \mathrm{Int}(G)$ has order coprime to p, then the following hold.

- $|\alpha| = 2$ and $\mathrm{int}(G) = 2$.
- $\gamma_i(G)/\gamma_{i+1}(G)$ is elementary abelian and $\alpha \equiv (-1)^i$ on it.
- $(w_i)_{i \geq 1} = (2, 1, 2, 1, \ldots, 2, 1, w, 0, 0, 0, \ldots)$ with $w \in \{0, 1, 2\}$.

Normal subgroups structure



Pro-p-help?



Pro-p-help

Let $p > 3$ be a prime number and let $t \in \mathbb{Z}_p$ satisfy $\left(\frac{t}{p}\right) = -1$. Set

$$\mathrm{A}_p = \mathbb{Z}_p + \mathbb{Z}_p\mathrm{i} + \mathbb{Z}_p\mathrm{j} + \mathbb{Z}_p\mathrm{ij}$$

with defining relations $\mathrm{i}^2 = t$, $\mathrm{j}^2 = p$, and $\mathrm{ji} = -\mathrm{ij}$. Then A_p is a non-commutative local ring such that $\mathrm{A}_p/\mathrm{j}\mathrm{A}_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : \mathrm{A}_p \to \mathrm{A}_p$ is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $G = \{a \in \mathrm{A}_p^* \mid a\overline{a} = 1 \text{ and } a \equiv 1 \bmod \mathrm{j}\mathrm{A}_p\}$ and, for all $a \in G$, define $\alpha(a) = \mathrm{i}a\mathrm{i}^{-1}$.

Pro-p-help

Let $p > 3$ be a prime number and let $t \in \mathbb{Z}_p$ satisfy $\left(\frac{t}{p}\right) = -1$. Set

$$\mathrm{A}_p = \mathbb{Z}_p + \mathbb{Z}_p\mathrm{i} + \mathbb{Z}_p\mathrm{j} + \mathbb{Z}_p\mathrm{ij}$$

with defining relations $\mathrm{i}^2 = t$, $\mathrm{j}^2 = p$, and $\mathrm{ji} = -\mathrm{ij}$. Then A_p is a non-commutative local ring such that $\mathrm{A}_p/\mathrm{j}\mathrm{A}_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : \mathrm{A}_p \to \mathrm{A}_p$ is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $G = \{a \in \mathrm{A}_p^* \mid a\overline{a} = 1 \text{ and } a \equiv 1 \bmod \mathrm{j}\mathrm{A}_p\}$ and, for all $a \in G$, define $\alpha(a) = \mathrm{i}a\mathrm{i}^{-1}$.

Theorem

G is a non-nilpotent pro-p-group and α induces an intense automorphism of order 2 on every non-trivial discrete quotient of G.

Pro-p-help

Let $p > 3$ be a prime number and let $t \in \mathbb{Z}_p$ satisfy $(\frac{t}{p}) = -1$. Set

$$\mathrm{A}_p = \mathbb{Z}_p + \mathbb{Z}_p\mathrm{i} + \mathbb{Z}_p\mathrm{j} + \mathbb{Z}_p\mathrm{ij}$$

with defining relations $\mathrm{i}^2 = t$, $\mathrm{j}^2 = p$, and $\mathrm{ji} = -\mathrm{ij}$. Then A_p is a non-commutative local ring such that $\mathrm{A}_p/\mathrm{j}\mathrm{A}_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : \mathrm{A}_p \to \mathrm{A}_p$ is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \ \mapsto \ \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $G = \{a \in \mathrm{A}_p^* \mid a\overline{a} = 1 \text{ and } a \equiv 1 \bmod \mathrm{j}\mathrm{A}_p\}$ and, for all $a \in G$, define $\alpha(a) = \mathrm{i}a\mathrm{i}^{-1}$.

Theorem

G is a non-nilpotent pro-p-group and α induces an intense automorphism of order 2 on every non-trivial discrete quotient of G. Moreover, G is "unique with this property".

Are 3-groups really special?

Are 3-groups really special?

Lemma

Let p be a prime number and let G be a finite p-group of class at least 4. If $\mathrm{int}(G) > 1$, then p-th powering induces a bijection $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

Are 3-groups really special?

Lemma

Let p be a prime number and let G be a finite p-group of class at least 4*. If* $\operatorname{int}(G) > 1$*, then p-th powering induces a bijection* $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

We define a κ-group to be a finite 3-group G with $|G : \gamma_2(G)| = 9$ such that cubing induces a bijection $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

Are 3-groups really special?

Lemma

Let p be a prime number and let G be a finite p-group of class at least 4. If $\operatorname{int}(G) > 1$, then p-th powering induces a bijection $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

We define a κ-group to be a finite 3-group G with $|G : \gamma_2(G)| = 9$ such that cubing induces a bijection $G/\gamma_2(G) \to \gamma_3(G)/\gamma_4(G)$.

Theorem

There is, up to isomorphism, a unique κ-group of class 3.

3-groups are really special

Let $R = \mathbb{F}_3[\epsilon]$ be of cardinality 9, with $\epsilon^2 = 0$. Set

$$\Delta = R + R\mathrm{i} + R\mathrm{j} + R\mathrm{ij}$$

with defining relations $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$ and $\mathrm{ji} = -\mathrm{ij}$. The standard involution is

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Write $\mathfrak{m} = \Delta\mathrm{i} + \Delta\mathrm{j}$ and define $\mathrm{MC}(3) = \{x \in 1 + \mathfrak{m} : \overline{x} = x^{-1}\}$.

3-groups are really special

Let $R = \mathbb{F}_3[\epsilon]$ be of cardinality 9, with $\epsilon^2 = 0$. Set

$$\Delta = R + R\mathrm{i} + R\mathrm{j} + R\mathrm{ij}$$

with defining relations $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$ and $\mathrm{ji} = -\mathrm{ij}$. The standard involution is

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Write $\mathfrak{m} = \Delta\mathrm{i} + \Delta\mathrm{j}$ and define $\mathrm{MC}(3) = \{x \in 1 + \mathfrak{m} : \overline{x} = x^{-1}\}$. The group $\mathrm{MC}(3)$ has order 729, class 4, and it is a κ-group.

3-groups are really special

Let $R = \mathbb{F}_3[\epsilon]$ be of cardinality 9, with $\epsilon^2 = 0$. Set

$$\Delta = R + R\mathrm{i} + R\mathrm{j} + R\mathrm{ij}$$

with defining relations $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$ and $\mathrm{ji} = -\mathrm{ij}$. The standard involution is

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

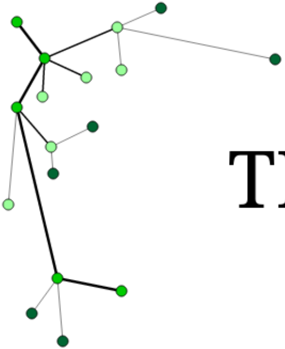
Write $\mathfrak{m} = \Delta\mathrm{i} + \Delta\mathrm{j}$ and define $\mathrm{MC}(3) = \{x \in 1 + \mathfrak{m} : \overline{x} = x^{-1}\}$. The group MC(3) has order 729, class 4, and it is a κ-group.

Theorem

Let G be a finite 3-group of class at least 4. Then $\mathrm{int}(G) > 1$ if and only if $G \cong \mathrm{MC}(3)$.

The actual classification

<table>
<tr><th colspan="4">Intensity</th></tr>
<tr><th>c \ p</th><th>2</th><th>3</th><th>≥ 5</th></tr>
<tr><td>0</td><td rowspan="6">1</td><td colspan="2">1</td></tr>
<tr><td>1</td><td colspan="2">$p-1$</td></tr>
<tr><td>2</td><td colspan="2">$p-1$ if G extraspecial of exponent p;
1 otherwise</td></tr>
<tr><td>3</td><td colspan="2">2 if $|G : G_2| = p^2$;
1 otherwise</td></tr>
<tr><td>4</td><td>2 if $G \cong \mathrm{MC}(3)$;
1 otherwise</td><td>2 if G is a p-obelisk with a concrete automorphism;
1 otherwise</td></tr>
<tr><td>≥ 5</td><td>1</td><td>2 if G is a p-obelisk with $|G_5| = p$, $\Phi(C_G(G_4)) = G_3$,
and G has a concrete automorphism;
2 if G is framed p-obelisk with $|G_5 : G_6| = p^2$
and G has a concrete automorphism;
1 in all other cases</td></tr>
</table>



THANK YOU